Nonuniversal dependence of spatiotemporal regularity on randomness in coupling connections

Zahera Jabeen and Sudeshna Sinha
Institute of Mathematical Sciences, Chennai, India

(Received 4 April 2008; revised manuscript received 6 October 2008; published 31 December 2008)

We investigate the spatiotemporal dynamics of a network of coupled nonlinear oscillators, modeled by sine-circle maps, with varying degrees of randomness in coupling connections. We show that the change in the basin of attraction of the spatiotemporal fixed point due to varying fraction of random links, p, is crucially related to the nature of the local dynamics. Even the *qualitative* dependence of the spatiotemporal regularity on p changes drastically as the angular frequency of the oscillators changes, ranging from a monotonic increase or monotonic decrease to nonmonotonic variation. Thus it is evident here that the influence of random coupling connections on spatiotemporal order is highly nonuniversal and depends very strongly on the nodal dynamics.

DOI: 10.1103/PhysRevE.78.066120 PACS number(s): 89.75.−k, 05.45.−a

I. INTRODUCTION

The dynamics of spatially extended systems has been a focus of intense research activity in the past two decades. In recent years it has become evident that modeling large interactive systems by finite-dimensional lattices on the one hand and fully random networks on the other is inadequate, as various networks, ranging from collaborations of scientists to metabolic networks, do not fit in either paradigm [1,2]. Some alternate scenarios have been suggested, and one of the most popular ones is the small-world network [3]. Here one starts with a structure on a lattice—for instance, regular nearest-neighbor connections. Then each link from a site to its nearest neighbor is rewired randomly with probability p, i.e., the site is connected to another randomly chosen lattice site. This model is proposed to mimic real-life situations in which nonlocal connections exist along with predominantly local connections.

There is much evidence that random nonlocal connections, even in a small fraction, significantly affect geometrical properties, such as the characteristic path length [3]. However, their *implications for dynamical characteristics are still unclear and even conflicting*. While the dynamics of coupled oscillators and coupled maps on regular lattices [known as "coupled-map lattices" (CMLs)] has been extensively investigated [4], there have been far fewer studies on the spatiotemporal dynamics of nonlinear elements on networks of different topologies [5]. Most studies so far have indicated that the regularity of systems increases monotonically with p [6].

In this paper we will provide evidence of a system where the dependence of spatiotemporal regularity on the degree of randomness in coupling connections is highly *nonuniversal*. We will show how this dependence ranges from *monotonically increasing* to *monotonically decreasing* via *nonmonotonic variation* as the local dynamics changes. Thus we will demonstrate that the interplay between local dynamics and connectivity acts in nontrivial and nonintuitive ways, and so even the qualitative effect of random links on spatiotemporal regularity can be completely reversed by changing the nodal dynamics.

II. MODEL

Here we consider nonlinear oscillators coupled to nearest neighbors on a regular ring, with some fraction p of the regular links rewired randomly. The individual sites (nodes) are modeled by sine-circle maps, which have widespread relevance for oscillatory phenomena [7] and are given as

$$f(x) = x + \Omega - \frac{K}{2\pi}\sin(2\pi x),$$

where K measures the strength of the nonlinear term and Ω represents the natural frequency of the map in the absence of nonlinearity (i.e., when $K=0$). We restrict our studies to the parameter region $0 \leqslant \Omega \leqslant \frac{1}{2\pi}$ and $K=1$. In this region, the single sine-circle map settles down to the spatiotemporal fixed point $x^{\star} = \frac{1}{2\pi}\sin^{-1}(\frac{2\pi\Omega}{K})$.

Under diffusive coupling, such a coupled sine-circle map lattice is given as

$$x_{n+1}(i) = (1-\epsilon)f(x_n(i)) + \frac{\epsilon}{2}[f(x_n(i-1)) + f(x_n(i+1))] \pmod 1, \quad (1)$$

where $i=1,\ldots,L$ denotes the site index, n denotes the time index, and ϵ represents the coupling strength between the sites ($0 \leqslant \epsilon \leqslant 1$). Periodic boundary conditions have been used; namely, one has a ring of oscillators.

Earlier studies of this coupled-map lattice have been carried out for various types of initial conditions [8]. In particular, the evolution of this coupled-map lattice with random initial conditions shows interesting spatiotemporal dynamics including spatiotemporal fixed points, spatial and spatiotemporal intermittency, and spatiotemporal chaos [9].

Now we introduce randomness in this regular lattice by rewiring the nearest-neighbor links with a probability p to randomly chosen sites on the lattice. Namely, the connectivity matrix of the system, on average, has a fraction p of random connections replacing the regular links. So each site connects to two other sites, and with probability p these are random sites in the system and with probability $(1-p)$ they are the regular nearest neighbors. The case of $p=0$ corresponds to the completely regular lattice, and $p=1$ corresponds to a completely random network of coupled maps. Such a model of connectivity has seen much research focus

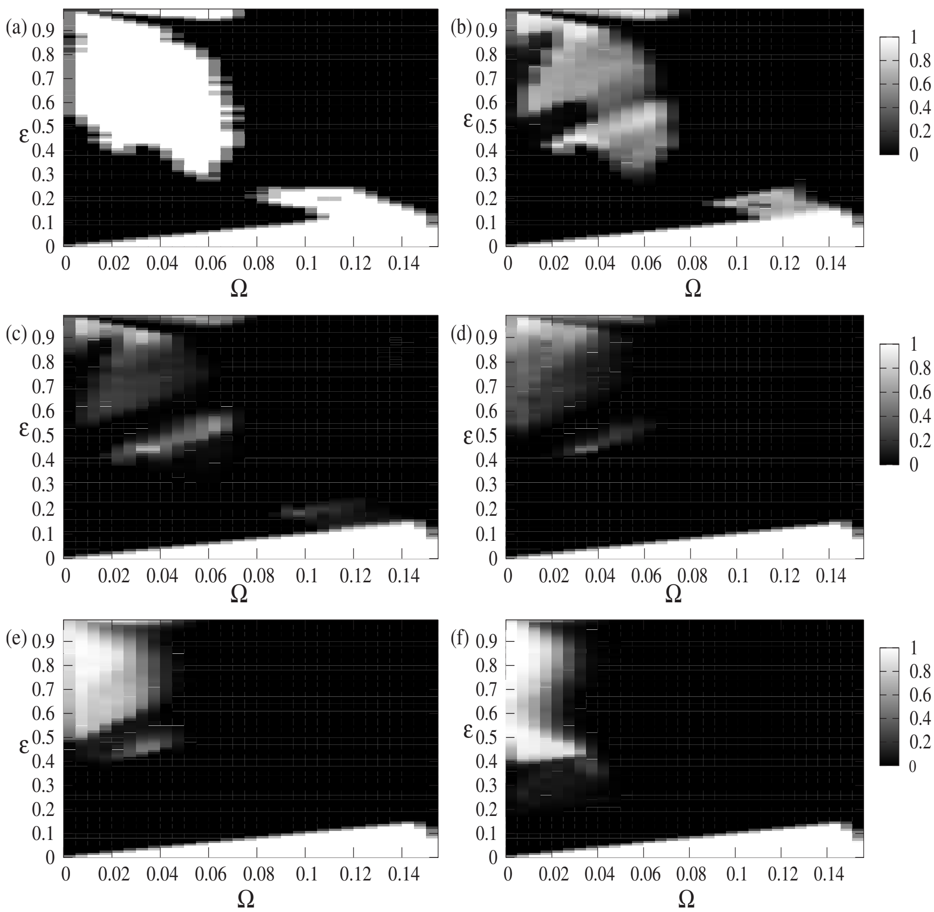

FIG. 1. Basin of attraction of the spatiotemporal fixed point $x^{\star}$ obtained for different random rewiring probabilities: (a) $p=0.0$, (b) $p=0.04$, (c) $p=0.1$, (d) $p=0.3$, (e) $p=0.5$, and (f) $p=0.8$. These plots have been obtained for 50 rewiring configurations for a lattice of size $L=200$ after discarding 5000 transients.

in recent years, as many real networks, ranging from biological to human engineered, have been found to fall in this class [1].

Also note that here we consider the behavior of the system under *static rewiring*; namely, the randomness in spatial coupling is quenched or "frozen" in time. Ensembles of such randomly rewired systems are studied.

III. RESULTS

A. Basin of attraction for the spatiotemporal fixed point

We study the spatiotemporal dynamics of this system, starting from random initial conditions under varying rewiring probabilities p $(0 \leqslant p \leqslant 1)$, for different realizations of static rewiring.

Specifically we obtain the basin of attraction for the spatiotemporal fixed point, B, by calculating the fraction of rewiring configurations which leads to a spatiotemporally steady state for different (Ω, ϵ) values. Figure 1 shows a gray-scale plot of B in a large region of parameter space for different values of p. The white areas indicate the parameter regions, where all initial coupling configurations lead to a spatiotemporal fixed point; namely, all the sites in the system relax to the fixed point $x^{\star}$ such that $x(i)=x^{\star}$ for all $i=1,\ldots,N$ and for all time n. The black regions indicate parameter regions where none of the coupling configurations yield a spatiotemporal fixed point. The gray areas indicate parameter regimes where $0<B<1$, i.e., the spatiotemporal fixed point coexists with other dynamical behaviors, and the spatiotemporal fixed point is not an attractor of the dynamics for all coupling configurations.

Figure 1(a) shows the basin of attraction when the rewiring fraction is equal to zero or, in other words, the ring has only regular-nearest neighbor connections. As the rewiring fraction p is varied, the spatiotemporal fixed-point regions also show a change. Figures 1(b)–1(f) display the basin of attraction of the spatiotemporal fixed point for $p=0.04$, 0.1, 0.3, 0.5, and 0.8. We see that the dependence of this basin of attraction on the degree of random rewiring is qualitatively very different for different values of Ω and ϵ. Interestingly this variation ranges from a monotonic increase to a monotonic decrease, as well as nonmonotonic behavior, along different "cuts" in (Ω, ϵ) space.

Further, the dependence of spatiotemporal order on the rewiring probability, averaged over a large parameter range, varies nonmonotonically with p. This is clear from the fact that the extent of the spatiotemporal fixed-point basin at intermediate p $(p \sim 0.1-0.3)$ in Figs. 1(c) and 1(d) is much smaller than that for low and high p. So the gray-scale basin plots in Figs. 1(c) and 1(d) appear far less "white" in general, across large parameter regimes, as compared to Figs. 1(a) and 1(f).

Figure 2 shows the variation of the basin of attraction, B, with rewiring fraction p at $\epsilon=0.5$ and for $\Omega=0.0$, 0.02, 0.04, and 0.06. These plots have been obtained for 50 rewiring configurations for a lattice of size $L=200$ after discarding 10 000 transients. When the natural frequency of the circle

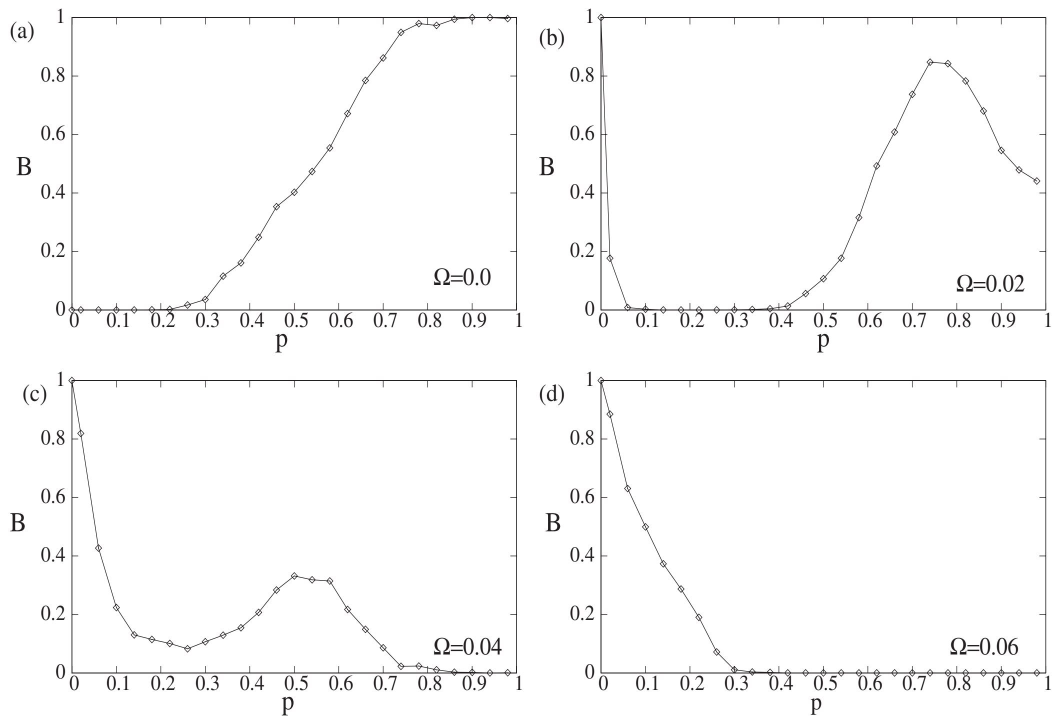

FIG. 2. Variation of the basin of attraction, B, with the rewiring fraction p plotted for the circle map frequencies $\Omega = 0, 0.02, 0.04, 0.06$ and coupling strength $\epsilon = 0.5$.

map is equal to zero ($\Omega = 0$), the ring does not yield a spatiotemporal fixed point when coupling connections are completely regular. However, the regularity of the system increases as the rewiring fraction p is increased. This can be seen in Fig. 2(a), in which a global spatiotemporal fixed-point attractor is obtained for values of the rewiring fraction $p > 0.6$.

The bifurcation diagram in Fig. 3(a), showing the spatiotemporal dynamics of the system with respect to the fraction of random links p, further underscores this feature. Here the system has a complex spatial pattern for lower values of the random rewiring probability. However, it settles down to the spatiotemporal fixed point ($x^{\star} = 0$, in this case) as the fraction of random links, p, approaches 1.

In contrast, the variation of the basin of attraction, B, in the case where the frequency Ω is equal to 0.02 is shown in Fig. 2(b). Here, we see that the system yields a spatiotemporal fixed point with probability 1 for zero rewiring fraction p, but shows a nonmonotonic variation as the rewiring fraction p is changed. We see that although the basin of attraction decreases to zero in the interval $p \sim 0.1-0.4$, it gradually increases for rewiring fractions, $p > 0.4$, until it again registers a decrease in the large-p limit [10]. Hence, the basin of attraction, B, shows a *nonmonotonic* variation with change in p. A similar nonmonotonic variation is seen in Fig. 2(c), where $\Omega = 0.04$.

In the case of $\Omega = 0.06$, as displayed in Fig. 2(d), the basin of attraction *decreases* to zero as the rewiring fraction is increased. In this case, the system settles to the spatiotemporal fixed point $x^{\star}$ for smaller rewiring fractions, but exhibits a complex spatial pattern when the degree of rewiring in the system is increased. This is further illustrated in the bifurcation diagram of the system shown in Fig. 3(b). This is exactly the opposite trend to that observed in the case of $\Omega = 0$. So, as the local frequency of the nonlinear oscillator changes, the effect of random rewiring on spatiotemporal properties is completely reversed.

Hence, we see that for the same coupling strength ϵ and for the same set of rewired configurations, the basin of attraction of the spatiotemporal fixed point shows a very strong dependence on the local dynamics—namely, on the frequency Ω of the nonlinear oscillators. So it is evident that the spatiotemporal regularity depends crucially, not just quantitatively, but also qualitatively, on the nodal dynamics.

Similarly, when the nodal dynamics is fixed and the coupling strength ϵ is varied, we see that the basin of attraction shows a nonmonotonic variation with change in rewiring fraction p. This is illustrated in Fig. 4 where the basin of attraction has been plotted for $\Omega = 0.01$ and for various representative values of the coupling strength ϵ.

Hence, the spatiotemporal regularity of the dynamics on a network depends quite crucially on the interplay between the nodal dynamics and the network topology. That is, coupling configurations with the same degree of randomness may enhance or inhibit spatiotemporal order depending on the properties of the local oscillators.

Additionally we have checked the robustness of our observations for K close to 1. Qualitatively similar behavior is observed for a band of K around 1.

B. Analysis

The coupled sine-circle map lattice can be written in vector form as

$$\mathbf{x}_{n+1} = \mathbf{H} \cdot f(\mathbf{x}_n), \tag{2}$$

where n is the discrete time index, $\mathbf{x}$ is an N-dimensional vector, $f(x)$ is the sine-circle map, and $\mathbf{H}$ is the connectivity

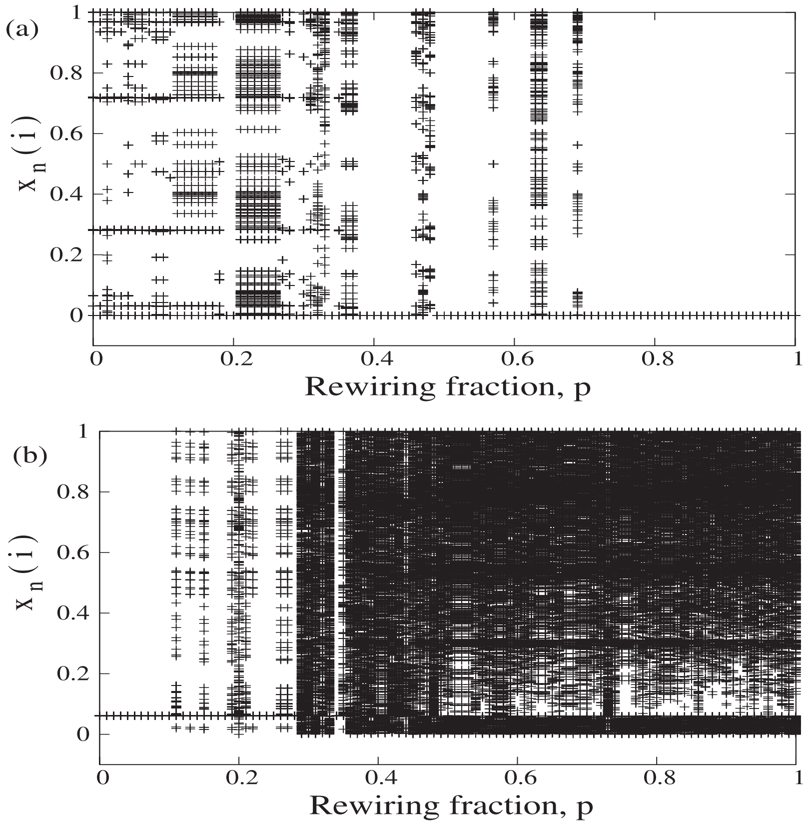

FIG. 3. Bifurcation diagram in which the state variables $x_n(i)$ $(i=1,\ldots,100)$ have been plotted as a function of the fraction of random links, p, for a representative coupling configuration, for the parameter values (a) $\Omega=0.0$ and (b) $\Omega=0.06$ at $\epsilon=0.5$. Here, $n=1,\ldots,5$ iterations have been plotted after discarding 5000 transients. Note that for regions where $0<B<1$, there are rewiring configurations that lead to the spatiotemporal fixed point, coexisting with rewiring configurations that yield spatiotemporal chaos.

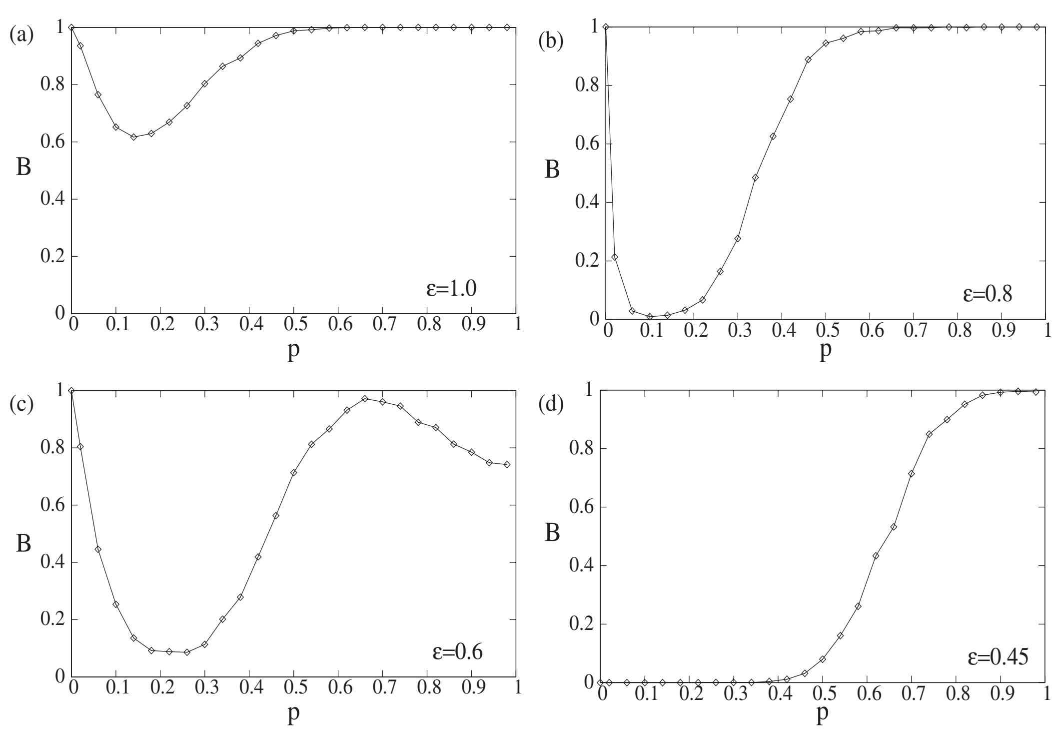

FIG. 4. Variation of the basin of attraction, B, with the rewiring fraction p for the circle map frequency, $\Omega=0.01$ and coupling strengths $\epsilon=1.0$, 0.8, 0.6, and 0.45.

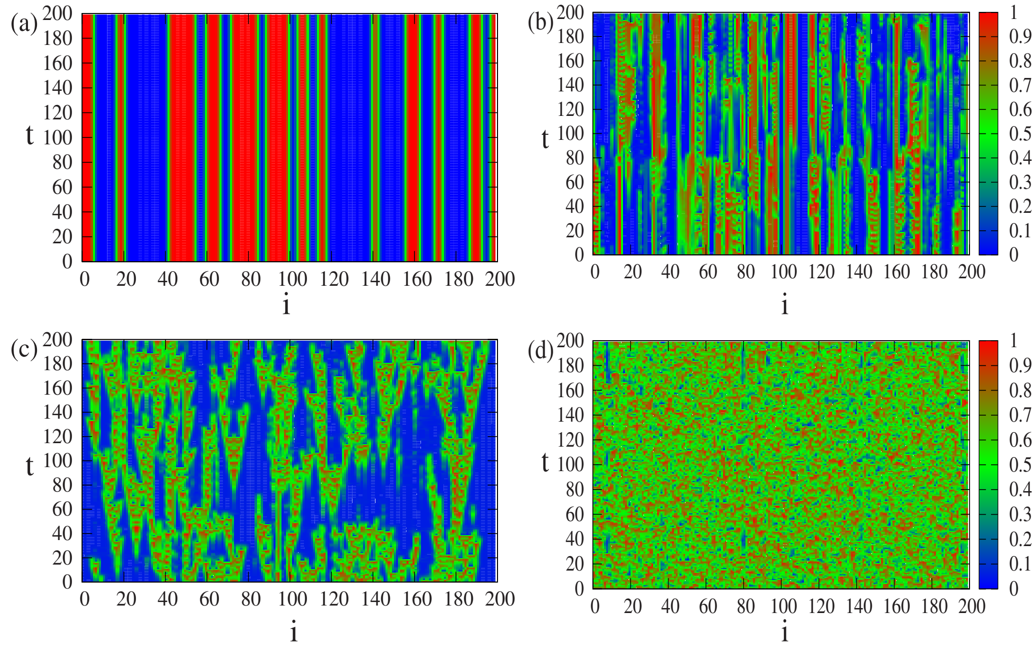

FIG. 5. (Color online) Space-time plots obtained for various values of the circle map frequencies Ω and rewiring probabilities p with the coupling strength fixed at $\epsilon=0.5$. (a) $\Omega=0.0$, $p=0.0$, where the basin of attraction of the spatiotemporal fixed point $B\sim 0$; (b) $\Omega=0.04$, $p=0.5$, where the basin of attraction of the spatiotemporal fixed point $B\sim 0.3$; (c) $\Omega=0.06$, $p=0.2$, where the basin of attraction of the spatiotemporal fixed point $B\sim 0.2$; and (d) $\Omega=0.06$, $p=0.8$, where the basin of attraction of the spatiotemporal fixed point $B\sim 0$.

matrix. For the case of regular rewiring $p=0$, $\mathbf{H}$ is defined as

$$\mathbf{H}=\begin{pmatrix} (1-\epsilon) & \epsilon/2 & 0 & \cdots & 0 & \epsilon/2 \\ \epsilon/2 & (1-\epsilon) & \epsilon/2 & 0 & \cdots & 0 \\ 0 & \epsilon/2 & (1-\epsilon) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \epsilon/2 & (1-\epsilon) & \epsilon/2 \\ \epsilon/2 & 0 & \cdots & 0 & \epsilon/2 & (1-\epsilon) \end{pmatrix}.$$

The Jacobian of the coupled-map lattice obtained after linearizing around the fixed-point solution $x^{\star}=\frac{1}{2\pi}\sin^{-1}(\frac{2\pi\Omega}{K})$ is then

$$\mathbf{J}=\mathbf{D}(\mathbf{H})\cdot f'(\mathbf{x}^{\star}).$$

Since $\mathbf{H}$ is a circulant matrix [11], its eigenvalues are given by

$$\lambda_l=(1-\epsilon)+\frac{\epsilon}{2}(\omega_l+\omega_l^{-1}),$$

where

$$\omega_l=\exp\left(\frac{2\pi il}{N}\right),\quad l=0,\ldots,N-1.$$

Hence, the eigenvalues of the Jacobian $\mathbf{J}$ are as follows:

$$\Lambda_l=(1-\epsilon)f'(x^{\star})+\epsilon\cos\left(\frac{2\pi l}{N}\right)f'(x^{\star}).$$

The largest eigenvalue corresponds to the $l=0$ term—i.e., $\Lambda_{max}=f'(x^{\star})$.

Now the single sine-circle map is stable in our interval of interest, namely, $0\leqslant\Omega\leqslant\frac{1}{2\pi}$ and $K=1.0$, as

$$f'(x^{\star})=1-K\cos(2\pi x^{\star})=1-K\sqrt{1-4\pi^2\Omega^2}\leqslant 1.$$

So for all our cases, the analysis above indicates that the synchronized fixed point is stable to small perturbations.

For the case of random rewiring—namely, $p\neq 0$—one can examine the linear stability of the spatiotemporal fixed point by considering suitable connectivity matrices in Eq. (2) [6]. For instance, for the case of fully random static connections $p=1$, one has $\mathbf{H}=(1-\epsilon)\mathbf{I}+\frac{\epsilon}{2}\mathbf{C}$, where $\mathbf{I}$ is the identity matrix and $\mathbf{C}$ is an $N\times N$ sparse nonsymmetric matrix with two random entries of 1 on each row. Now the real part of the eigenvalues of different realizations of the connectivity matrix $\mathbf{C}$ is bounded between -2 and 2. So the eigenvalues of the connectivity matrix $\mathbf{H}$ are bounded between -1 and 1 for $0\leqslant\epsilon<1$. So again, as for $p=0$, linear stability indicates that the spatiotemporal fixed point in this system is stable to small perturbations.

However, importantly, information on the size of the basin of attraction of the spatiotemporal fixed point *cannot* be obtained from linear stability analysis in this system. Strictly speaking, asymptotic analysis is justified *only* in a small neighborhood of the attractor. Namely, the stability analysis is relevant only to initial states very close to the spatiotemporal fixed point—i.e., with initial $x(i)$ close to $x^{\star}$ for all i. Such initial states are, of course, of measure zero in the space of all initial conditions in such large multivariable systems. Typically, then, stability analysis is too local a criterion to indicate any global properties, such as the size or boundaries of the basin of attraction of different spatiotemporal states here.

The inability of linear stability analysis to predict global dynamical features holds true in many systems of coupled nonlinear elements, as there typically exist multiple coexisting attractors in such large interactive systems, often with complicated (perhaps fractal) basin boundaries. Indeed the dynamical states one obtains, apart from the spatiotemporal fixed point, have very varied spatiotemporal patterns. Figure 5 shows some representative space-time plots of the rich variety of coexisting spatiotemporal attractors in the parameter regions where $B<1$. Some of the dynamical states competing with the spatiotemporal fixed point are reminiscent of spatial intermittency [Fig. 5(a)] and spatiotemporal intermittency [Fig. 5(c)].

Last, we reiterate that it is important to know the basin of attraction of a state, as in most practical experimental situations the system cannot be prepared in definite random rewiring configurations, but evolves from some random configuration. So the probability of obtaining a particular state from a typical configuration (as reflected from numerics on ensembles of rewiring configurations, such as we have presented) is most useful in order to make reasonable contact with theoretical prediction.

IV. CONCLUSIONS

In summary, we have investigated the spatiotemporal dynamics of a network of coupled nonlinear oscillators, modeled by sine-circle maps, with varying degrees of randomness in coupling connections. We showed that the variation of the basin of attraction of the spatiotemporal fixed point, with increasing fraction of random links p, crucially depends on the nature of the local dynamics. Even the qualitative relationship between spatiotemporal regularity and p changes drastically as the angular frequency of the oscillators changes, ranging from a monotonic increase or decrease to nonmonotonic variation. Thus it is evident that the influence of random coupling connections on spatiotemporal order is highly nonuniversal here and depends strongly on the angular frequency of the nodal oscillators. This implies that the delicate interplay between local dynamics and connectivity is crucial in determining the emergence of spatiotemporal order in complex networks of dynamical elements.

ACKNOWLEDGMENT

We would like to thank Professor N. Gupte for useful discussions.

[1] R. Albert and A.-L. Barabasi, Rev. Mod. Phys. **74**, 47 (2002), and references therein.
[2] A.-L. Barabasi, *Linked: The New Science of Networks* (Persus, Boston, 2002); T. Gross and B. Blasius, J. R. Soc., Interface **5**, 259 (2008).
[3] D. J. Watts and S. H. Strogatz, Nature (London) **393**, 440 (1998).
[4] K. Kaneko, *Theory and Applications of Coupled Map Lattices* (Wiley, New York, 1993).
[5] Z. Gao, B. Hu, and G. Hu, Phys. Rev. E **65**, 016209 (2001); M. Barahona and L. M. Pecora, Phys. Rev. Lett. **89**, 054101 (2002); J. Jost and M. P. Joy, Phys. Rev. E **65**, 016201 (2001); M. G. Cosenza and K. Tucci, *ibid.* **65**, 036223 (2002); P. R. A. Campos, V. M. de Oliveira, and F. G. Brady Moreira, *ibid.* **67**, 026104 (2003); P. M. Gade and S. Sinha, Int. J. Bifurcation Chaos Appl. Sci. Eng. **16**, 2767 (2006); Soon-Hyung Yook and H. Meyer-Ortmanns, Physica A **371**, 781 (2006); C. Y. Yin, B. H. Wang, W. X. Wang, and G. R. Chen, Phys. Rev. E **77**, 027102 (2008).
[6] P. M. Gade and C.-K. Hu, Phys. Rev. E **62**, 6409 (2000); S. Sinha, *ibid.* **66**, 016209 (2002); S. Rajesh, S. Sinha, and S. Sinha, *ibid.* **75**, 011906 (2007); Maruthi Pradeep Kanth Jampa, A. R. Sonawane, P. M. Gade, and S. Sinha, *ibid.* **75**, 026215 (2007).
[7] M. H. Jensen, P. Bak, and T. Bohr, Phys. Rev. A **30**, 1960 (1984); T. Bohr, P. Bak, and M. H. Jensen, *ibid.* **30**, 1970 (1984).
[8] G. R. Pradhan, N. Chatterjee, and N. Gupte, Phys. Rev. E **65**, 046227 (2002); N. Chatterjee and N. Gupte, *ibid.* **53**, 4457 (1996); Physica A **224**, 422 (1996).
[9] Z. Jabeen and N. Gupte, Phys. Rev. E **74**, 016210 (2006); **72**, 016202 (2005); T. M. Janaki, S. Sinha, and N. Gupte, *ibid.* **67**, 056218 (2003).
[10] Note that the average time taken by the system to relax to the spatiotemporal fixed point in the large-p regime ($p \sim 0.8-1.0$) grows exponentially with p and the configurations eventually relax to the spatiotemporal fixed point with probability 1, but only at asymptotically long times. This is in contrast to the region of rewiring fractions $p \sim 0.1-0.4$, where most initial states do not appear to relax to the spatiotemporal fixed-point state, even after very long transience.
[11] P. J. Davis, *Circulant Matrices* (Wiley, New York, 1979).