Modified Chapman-Enskog moment approach to diffusive phonon heat transport

Zbigniew Banach* and Wieslaw Larecki†

Institute of Fundamental Technological Research, Department of Theory of Continuous Media, Polish Academy of Sciences, Swietokrzyska 21, 00-049 Warsaw, Poland

(Received 6 October 2008; published 31 December 2008)

A detailed treatment of the Chapman-Enskog method for a phonon gas is given within the framework of an infinite system of moment equations obtained from Callaway's model of the Boltzmann-Peierls equation. Introducing no limitations on the magnitudes of the individual components of the drift velocity or the heat flux, this method is used to derive various systems of hydrodynamic equations for the energy density and the drift velocity. For one-dimensional flow problems, assuming that normal processes dominate over resistive ones, it is found that the first three levels of the expansion (i.e., the zeroth-, first-, and second-order approximations) yield the equations of hydrodynamics which are linearly stable at all wavelengths. This result can be achieved either by examining the dispersion relations for linear plane waves or by constructing the explicit quadratic Lyapunov entropy functionals for the linear perturbation equations. The next order in the Chapman-Enskog expansion leads to equations which are unstable to some perturbations. Precisely speaking, the linearized equations of motion that describe the propagation of small disturbances in the flow have unstable plane-wave solutions in the short-wavelength limit of the dispersion relations. This poses no problem if the equations are used in their proper range of validity.

DOI: 10.1103/PhysRevE.78.061137 PACS number(s): 05.90.+m, 51.10.+y, 47.45.Ab, 74.25.Kc

I. INTRODUCTION

In Ref. [1], we considered the phonon gas with nondispersion and isotropy in the frequency spectrum and derived the nonlinear second-order parabolic equations for the energy density and the drift velocity by means of the Chapman-Enskog method as applied to Callaway's model [2,3] of the Boltzmann-Peierls equation [4,5]. There, it was assumed that the effective relaxation time for normal processes (τ_N) is much smaller than the effective relaxation time for resistive processes (τ_R). In this regime, during the first time period, normal processes cause the phonon gas to approach the displaced Planck distribution (i.e., the distribution function with a drift velocity different from zero), and then during a longer time period, resistive processes return it to the equilibrium Planck distribution [6,7]. Consequently, instead of using the traditional Chapman-Enskog method of solution of phonon kinetic equations which is based on an expansion about an equilibrium Planck distribution, we developed systematically a modification of the traditional method so as to allow for an expansion about the displaced Planck distribution. This modification was effectively realized by expanding the phase density in powers of τ_N and $1/\tau_R$ and by including the spatial derivatives of the relevant hydrodynamic variables in the expansion. Since the displaced Planck distribution is a highly nonlinear function of the drift velocity [8,9], one obtains in this manner a system of parabolic equations which, unlike the usual set of parabolic equations for a phonon gas, does not restrict the magnitude of the individual components of the drift velocity or the heat flux in any way. This system is linearly stable at all wavelengths and is also fully consistent with the second law of thermodynamics in the sense that there exists a macroscopic entropy density which depends locally on the hydrodynamic variables and satisfies the balance equation with a non-negative entropy production due to both resistive and normal processes.

The method of Ref. [1], which consists in directly expanding the phase density about a displaced Planck distribution, was used to obtain and discuss the second-order parabolic equations for the energy density and the drift velocity. The analysis was based on a particularly simple kinetic theory of Callaway in order to allow for illustration of the general concepts without the technical complications involved in using the full Boltzmann-Peierls equation. Under the assumption that normal processes dominate over resistive ones, our aim here is to further study the Chapman-Enskog expansion for a phonon gas using the infinite system of moment equations derived from Callaway's model of the Boltzmann-Peierls equation. Since this system is formally equivalent to Callaway's equation, the Chapman-Enskog expansion of the moment equations must give the same results as the Chapman-Enskog expansion of Callaway's equation itself. Of course, any finite set of moment equations is not closed. However, the equations for the energy density, the heat flux, and the next two moments of the distribution function already contain enough elements of Callaway's model to reproduce correctly the first-order result of Ref. [1], namely, a closed-form expression for the deviatoric part of the flux of the heat flux in terms of the energy density, the drift velocity, and the first spatial derivatives of these variables. In this paper, we consider additional moment equations and continue the Chapman-Enskog expansion to higher orders. For the sake of simplicity, the expansion is presented only within the framework of one-dimensional flow problems.

The main new results can be stated as follows. The phonon dynamic equations derived from the second-order approximation are linearly stable at all wavelengths, so that higher-order Chapman-Enskog expansions do not necessarily lead to unstable equations. However, the next order in the

*Corresponding author. zbanach@ippt.gov.pl

†wlarecki@ippt.gov.pl

Chapman-Enskog expansion yields the equations of hydrodynamics which are unstable to some perturbations. Precisely speaking, the linearized equations of motion that describe the propagation of small disturbances in the flow have unstable plane-wave solutions in the short-wavelength limit of the dispersion relations. This poses no problem if the method is used in the proper range of its applicability, i.e., in the region where the spatial gradients of the energy density and the drift velocity are sufficiently small.

There are two basic methods to prove that solutions of the nonlinear equations of hydrodynamics are linearly stable, one via the examination of the dispersion relations for linear plane waves and one via the explicit construction of a quadratic Lyapunov entropy functional. Each comes with its own benefits and drawbacks. Both require dealing with linear equations governing the evolution of the perturbation fields. This paper discusses these two methods. The first method—and possibly the more common—is to verify the sign of the imaginary parts of the dispersion relations. The second method, which is the one we develop in Sec. VI, is based on defining the Lyapunov entropy functional for the linear perturbation equations which satisfies an H theorem (entropy inequality). There is no general procedure for finding the Lyapunov functionals for nonlinear partial differential systems, but for linear partial differential equations, the procedure partly reduces to the algebraic problem of choosing appropriate coefficients of the characteristic polynomial of a certain matrix [10]. However, for reasons to be explained in the discussion directly after Eq. (6.12), this very interesting procedure cannot be used here. Consequently, we propose a different algorithm for computing the Laypunov functional based on a suitably chosen entropy function.

Note that the idea of studying the detailed structure of the Chapman-Enskog method for the moment equations of a phonon gas is not entirely new. In Refs. [11–14], it was used to sum up the Chapman-Enskog series exactly, thereby providing for the first time the basis to avoid instabilities inherent in some low-order terms of the expansion and to compare various approximations in extending the hydrodynamic description beyond the first-order approximation. However, since the formalities are known to be rather awkward for the full moment system, the discussion was simplified in that the *linear* finite-moment equations were considered as the minimal kinetic models where the Chapman-Enskog series can be summed up exactly in a closed form. Mostly due to the linearizing assumption, the zeroth-order expansion terms[1] given in Refs. [11–14] differ from those deduced here. More explicitly, the former vanish identically because they correspond to the equilibrium Planck distribution, whereas the latter are nonlinear functions of the drift velocity because they are related to the displaced Planck distribution. As a consequence, although the present approach to the moment system has the disadvantage that it does not enable one to sum up the Chapman-Enskog series explicitly, the advantage of using this approach is that no limitations are introduced on the magnitudes of the drift velocity or the heat flux, i.e., these quantities are incorporated into the model in a nonperturbative manner.

One final word concerning the expository part of the paper: We follow the traditional introduction of the Chapman-Enskog expansion. This is mostly due to the fact that our aim is to study the evolution of a phonon gas in the setting of weakly nonlocal hydrodynamics. Governing equations of this hydrodynamics involve higher-order derivatives with respect to the position coordinate. However, the disadvantage of the traditional approach is that it remains not very clear what we are looking for, because the answer is not a solution of the initial kinetic equation or the equivalent set of moment equations, but generalized hydrodynamic equations for the reduced set of variables. Fortunately, after Fenichel and Jones published the papers in which the geometric singular perturbation theory took the definite form [15,16], a new approach was clearly formulated [17,18]: Singular perturbation series like the Hilbert and Chapman-Enskog expansions give us slow invariant manifolds for either kinetic or moment equations. In Sec. III B, we briefly explain why this modern geometric framework of the Chapman-Enskog method is more seminal than the traditional one.

The outline of this paper is as follows. Section II recalls the more relevant aspects of Callaway's model. Section III defines the moment system corresponding to Callaway's model and formulates the Chapman-Enskog expansion of this system. Section IV discusses the Chapman-Enskog method in further detail. Specifically, this section provides the hydrodynamic description beyond the zeroth-order level. Section V shows that the phonon dynamic equations obtained from the zeroth-, first-, and second-order approximations are linearly stable at all wavelengths. Section VI proves the existence of Lyapunov entropy functionals for the linear perturbation systems derived in Sec. V. Section VII analyzes the next order in the Chapman-Enskog expansion; it is demonstrated that the short-wavelength instability is present at the level of this approximation. Section VIII is for final remarks. The Appendix makes use of the results of Ref. [1] with the aim of verifying that the first-order hydrodynamic equations obtained in Sec. IV A can also be obtained from the first-order hydrodynamic equations for three-dimensional flow.

The units are defined by setting $\hbar = k_B = 1$. No distinction is made between longitudinal and transverse phonons. The dispersion relation has the form $\omega_D = c|\mathbf{p}|$, where c is the constant Debye speed and $\mathbf{p}$ is the momentum of a phonon particle. Since $\hbar = 1$, this momentum can also be interpreted as the phonon wave vector. The components of $\mathbf{p}$ range from $-\infty$ to $+\infty$. For the sake of simplicity, the effective relaxation times τ_R and τ_N are assumed to be constant quantities.

II. THE MODEL KINETIC EQUATION

For our purposes, we introduce the distribution function $f(t,\mathbf{x},\mathbf{p})$ defined in such a way that $f(t,\mathbf{x},\mathbf{p})d^3\mathbf{x}\, d^3\mathbf{p}$ is the number of phonons at time t in the volume element $d^3\mathbf{x}\, d^3\mathbf{p}$ around $(\mathbf{x},\mathbf{p})$. The evolution of this function is governed by a kinetic equation of the form

[1] These terms are the initial terms in the expansion of some suitably defined symmetric traceless moments of the distribution function. For further details, see our discussion in Secs. III A and III B.

$$\partial_t f + c g^i \partial_i f = J_R(f) + J_N(f), \tag{2.1}$$

where (g^i) are the components of $\mathbf{g} := \mathbf{p}/|\mathbf{p}|$, (∂_t, ∂_i) denote differentiation with respect to (t, x^i), and $J_R(f), J_N(f)$ stand for the collision terms due to resistive and normal processes, respectively. The actual construction of $J_R(f), J_N(f)$ is a matter of great difficulty, and in any event most expressions for $J_R(f), J_N(f)$ involve functional integrals of f itself, as for example in the original Boltzmann-Peierls equation [4]. In order to simplify the kinetic theory of a phonon gas, Callaway [2] has made use of the model collision terms

$$J_R(f) = \frac{1}{\tau_R}(F - f), \quad J_N(f) = \frac{1}{\tau_N}(F_d - f). \tag{2.2}$$

Here (τ_R, τ_N) are the effective relaxation times, which can depend on momentum if desired, but are often left constant (as we shall do), and (F, F_d) are the equilibrium and displaced Planck distributions defined by

$$F := \frac{y}{\exp(cp/T) - 1}, \quad F_d := \frac{y}{\exp[(cp/T_d)(1 - \mathbf{v}\cdot\mathbf{g})] - 1},$$

where

$$y := 3(2\pi)^{-3}, \quad p := |\mathbf{p}|, \quad \mathbf{g} := \mathbf{p}/|\mathbf{p}|, \quad |\mathbf{v}| < 1.$$

Note that y specifies the smallest element of the phase space that can accommodate a phonon. The scalar functions $T = T(t, \mathbf{x})$ and $T_d = T_d(t, \mathbf{x})$ represent two different temperature fields and the vector function $\mathbf{v} = \mathbf{v}(t, \mathbf{x})$ represents the drift velocity of a phonon gas. Precisely speaking, we refer to $\mathbf{v}$ as the dimensionless drift velocity. The physical drift velocity is obtained by multiplying $\mathbf{v}$ by c. Clearly, the temperature and velocity fields are not arbitrary. They are chosen so that relations (2.4) below hold true. We first observe that the general collision terms satisfy the following conditions:[2]

$$\int pJ_R(f) d^3\mathbf{p} = 0, \tag{2.3a}$$

$$\int pJ_N(f) d^3\mathbf{p} = 0, \quad \int \mathbf{p}J_N(f) d^3\mathbf{p} = \mathbf{0}. \tag{2.3b}$$

Any model collision terms must satisfy these conditions as well. For Callaway's model with effective relaxation times τ_R and τ_N independent of $\mathbf{p}$, we thus have

$$c\int pF\, d^3\mathbf{p} = c\int pF_d d^3\mathbf{p} = e := c\int pf\, d^3\mathbf{p}, \tag{2.4a}$$

$$c^2\int \mathbf{p}F_d d^3\mathbf{p} = \mathbf{q} := c^2\int \mathbf{p}f\, d^3\mathbf{p}, \tag{2.4b}$$

where e is the (actual) energy density and $\mathbf{q}$ is the (actual) heat flux. Direct evaluation of the integrals in (2.4) then yields an expression for T in terms of e and expressions for $(T_d, \mathbf{v})$ in terms of $(e, \mathbf{q})$. Conditions (2.4) are important because they ensure that Callaway's model satisfies the Boltzmann H theorem (for more details, see, e.g., Ref. [19], Sec. VI A)

We now simplify the discussion by assuming the one-dimensional, rotationally symmetric geometry. In this geometry, the distribution function depends only on time, a single spatial coordinate $x := x^1$, and the momentum in the x direction p_x:

$$f = f(t, x, p_x). \tag{2.5}$$

The displaced Planck distribution takes the form

$$F_d := \frac{y}{\exp[(cp/T_d)(1 - vg)] - 1},$$

where

$$v := v_x \in (-1, 1), \quad g := p_x/p \in [-1, 1].$$

Because of (2.5), the energy density is a function of t and x. The heat flux in the x direction $q(t, x)$ may vary; the heat fluxes in the orthogonal directions vanish. Equations (2.4) make it possible to relate T to e and $(T_d, \mathbf{v})$ to $(e, \mathbf{q})$. Explicitly, if the flow remains one-dimensional, these equations give

$$T = \left(\frac{e}{3\chi}\right)^{1/4}, \quad T_d = \left(\frac{e}{(3+u)\chi}\right)^{1/4}(1-u)^{3/4}, \tag{2.6a}$$

$$v = \frac{3q}{2ce + \sqrt{4c^2e^2 - 3q^2}}, \quad q = \frac{4cev}{3+u}, \tag{2.6b}$$

where

$$\chi := \pi^2/(30c^3), \quad u := v^2.$$

According to Eqs. (2.6), the temperature T_d depends on the energy density e and the heat flux q. Formally, this temperature is a crucial part of the definition of Callaway's model and Sec. IV A shows that a less formal explanation of the meaning of T_d is possible only within the framework of first-order hydrodynamic equations. In this case, we can provide a simple physical argument congruent with a current discussion on the concept of temperature for systems far from equilibrium (see, for example, Refs. [20,21]).

Observing that Eq. (2.5) can also be written as

$$f = f(t, x, pg),$$

we introduce the distribution function which depends on (t, x, g) and does not depend on p:

$$\phi := c\pi\int_0^\infty p^3 f(t, x, pg)dp.$$

This distribution function reduces to

$$\phi = \psi := \frac{3}{4}\chi T^4 = \frac{1}{4}e$$

if f equals F and to

[2]Equation (2.3a) shows that resistive processes conserve energy, while Eq. (2.3b) shows that normal processes conserve energy and momentum.

$$\phi = \psi_d := \frac{3\chi(T_d)^4}{4(1-vg)^4} = \frac{3e(1-u)^3}{4(3+u)(1-vg)^4} \tag{2.7}$$

if f equals F_d. Based on these preparatory definitions and assumptions, we may easily verify that Eqs. (2.1) and (2.2) yield the following model equation for ϕ:

$$\partial_t\phi + cg\partial_x\phi = \frac{1}{\tau_R}(\psi-\phi) + \frac{1}{\tau_N}(\psi_d-\phi). \tag{2.8}$$

Finally, in order to allow the effective relaxation time τ_N to become small and the effective relaxation time τ_R to become large, it will be convenient to take Eq. (2.8) with the formal small parameter ε inserted:

$$\partial_t\phi + cg\partial_x\phi = \frac{\varepsilon}{\tau_R}(\psi-\phi) + \frac{1}{\varepsilon\tau_N}(\psi_d-\phi). \tag{2.9}$$

With the help of ε, we can speak of the regime [7,22] where normal processes dominate over resistive ones. Equation (2.9) and the suitable moment system derived from it will be of interest to us subsequently.

III. CHAPMAN-ENSKOG EXPANSION

A. The moment system

With $w_0 := 1$ and $w_1 := g$, the first two moments of ϕ define the energy density and the heat flux:

$$\lambda_0 := e = 2\int_{-1}^{1} w_0\phi\, dg, \quad \lambda_1 := q = 2c\int_{-1}^{1} w_1\phi\, dg. \tag{3.1}$$

The higher moments of ϕ are given by

$$\lambda_n := \frac{2c^n n!}{(2n-1)!!}\int_{-1}^{1} w_n\phi\, dg,$$

where $n=2,3,4,\ldots,\infty$ and

$$n! := 1\times 2\times 3\times \cdots \times n,$$

$$(2n-1)!! := 1\times 3\times 5\times \cdots \times (2n-1), \tag{3.2}$$

$$w_n := \sum_{l=0}^{[n/2]} (-1)^l \frac{(2n-2l-1)!!}{2^l l!(n-2l)!} g^{n-2l}. \tag{3.3}$$

Here $[n/2]$ means the largest integer less than or equal to $n/2$. Let $M^{i_1 i_2 \cdots i_n}$, $n\geq 2$, be the symmetric traceless moments of ϕ defined by

$$M^{i_1 i_2\cdots i_n} := 2c^n\int_{-1}^{1} g^{\langle i_1} g^{i_2}\cdots g^{i_n\rangle}\phi\, dg, \tag{3.4}$$

where $g^{\langle i_1}g^{i_2}\cdots g^{i_n\rangle}$ is the traceless part of $g^{i_1}g^{i_2}\cdots g^{i_n}$. Then

$$\lambda_2 = M^{11}, \quad \lambda_3 = M^{111}, \quad \lambda_4 = M^{1111}, \quad \text{etc.} \tag{3.5}$$

From Eqs. (3.1), (3.4), and (3.5) it follows that $\lambda_n \neq 0$ if $\phi = \psi_d$ and that $\lambda_n = 0$ if $\phi=\psi$ and $n\geq 1$. The moments λ_n and $M^{i_1 i_2\cdots i_n}$ with $n\geq 2$ have no direct physical meaning, even though the moment λ_2 can be used as a basis for the representation of the deviatoric part of the flux of the heat flux:

$$\lambda_2 = M^{11} = -2M^{22} = -2M^{33}, \quad M^{ij} = 0 \quad (i\neq j).$$

Setting

$$\bar{\lambda}_n := \frac{2c^n n!}{(2n-1)!!}\int_{-1}^{1} w_n\psi_d dg \quad (n\geq 2) \tag{3.6}$$

and observing that

$$\int_{-1}^{1} w_0\psi_d dg = e/2, \quad \int_{-1}^{1} w_1\psi_d dg = q/(2c),$$

we find from

$$\int_{-1}^{1} w_0\psi\, dg = e/2, \quad \int_{-1}^{1} w_n\psi\, dg = 0 \quad (n\geq 1) \tag{3.7}$$

and (2.9) that the equations for (e,q) and $\lambda_n (n\geq 2)$ assume the forms

$$\partial_t e + \partial_x q = 0, \tag{3.8a}$$

$$\partial_t q + \partial_x\lambda_2 + \frac{c^2}{3}\partial_x e = -\frac{\varepsilon}{\tau_R}q, \tag{3.8b}$$

$$\partial_t\lambda_n + \partial_x\lambda_{n+1} + \frac{c^2 n^2}{4n^2-1}\partial_x\lambda_{n-1} = -\frac{\varepsilon}{\tau_R}\lambda_n - \frac{1}{\varepsilon\tau_N}(\lambda_n - \bar{\lambda}_n). \tag{3.8c}$$

Using (2.7), Eq. (3.6) becomes

$$\bar{\lambda}_n = \frac{3c^n n!}{2(2n-1)!!}\frac{e(1-u)^3}{3+u}\int_{-1}^{1}\frac{w_n}{(1-vg)^4}dg \quad (n\geq 2). \tag{3.9}$$

After specifying an integer $n\geq 2$, the integration over g can be carried out explicitly and then Eq. (3.9) gives $\bar{\lambda}_n$ as an explicit function of e and v. For example, if $n=2$, a short calculation leads to

$$\bar{\lambda}_2 = \frac{8c^2 eu}{3(3+u)}. \tag{3.10}$$

Recalling Eqs. (3.7) and the definition of ψ_d, one can also show that $\bar{\lambda}_n$ vanishes when $v=0$.

We mention that hierarchy (3.8) with $\varepsilon=1$ and $\bar{\lambda}_n=0$, as well as the use of a displaced Planck distribution to prove the Boltzmann H theorem for Callaway's model, is also found in Ref. [23] [see Eqs. (3.44) and (3.58)–(3.60)]. If $\bar{\lambda}_n=0$, the principle of superposition of solutions applies and then it is possible to use Laplace transforms analytically with a numerical evaluation of the inverse Laplace transformation. Technically, hierarchy (3.8) follows from a study of the one-dimensional, rotationally symmetric reduction of the three-dimensional hierarchy. See Ref. [24], Eqs. (4.4), for the definition of this more general hierarchy and Ref. [23] for the explicit description of a reduction process.

In view of (2.6b), it is possible to transform Eqs. (3.8a) and (3.8b) to a system of equations for (e,v). This system written in full is

$$\partial_t e + \frac{4cv}{3+u}\partial_x e + \frac{4ce(3-u)}{(3+u)^2}\partial_x v = 0, \tag{3.11a}$$

$$\partial_t v + \frac{3c(1-u)^2}{4e(3-u)}\partial_x e + \frac{8cuv}{9-u^2}\partial_x v + \frac{(3+u)^2}{4ce(3-u)}\partial_x N = -\frac{\varepsilon(3+u)v}{(3-u)\tau_R}, \tag{3.11b}$$

where

$$N := \lambda_2 - \bar{\lambda}_2. \tag{3.12}$$

Any formulation based on finitely many moment equations always results in an excess of unknowns compared to the number of equations, which denotes a closure problem. In this paper, we address the problem of closure of the system (3.11). Precisely speaking, we use the Chapman-Enskog method as applied to Eqs. (3.8c) and (3.11) in order to express the excessive unknown N in terms of the energy density, the dimensionless drift velocity, and the spatial derivatives of these variables.

B. General structure of the method

We start with the expansions

$$\lambda_n = \sum_{l=0}^{\infty} \varepsilon^l \lambda_{n|l} \quad (n \geq 2). \tag{3.13}$$

The expansion coefficients $\lambda_{n|l}$ are assumed to depend on (t,x) through the hydrodynamic variables (e,v) and their spatial derivatives up to order l:

$$\lambda_{n|l} = \lambda_{n|l}(e,\partial_x e,\partial_x^2 e,\ldots,\partial_x^l e; v,\partial_x v,\partial_x^2 v,\ldots,\partial_x^l v). \tag{3.14}$$

Using Eq. (3.13) with $n=2$ and recalling Eq. (3.12), we find that

$$N = \lambda_{2|0} - \bar{\lambda}_2 + \sum_{l=1}^{\infty} \varepsilon^l \lambda_{2|l}. \tag{3.15}$$

The time derivative is likewise expressed as a power series in ε. Precisely speaking, since Eqs. (3.11) link space and time derivatives, it is necessary to introduce an expansion of the time derivatives of (ε,v) as follows:

$$\partial_t e = \sum_{l=0}^{\infty} \varepsilon^l E_l, \quad \partial_t v = \sum_{l=0}^{\infty} \varepsilon^l V_l. \tag{3.16}$$

Substituting Eqs. (3.15) and (3.16) into Eqs. (3.11) and equating the terms that have like powers of ε yields

$$E_0 = -\frac{4cv}{3+u}\partial_x e - \frac{4ce(3-u)}{(3+u)^2}\partial_x v, \tag{3.17a}$$

$$E_l = 0 \quad (l \geq 1), \tag{3.17b}$$

$$V_0 = -\frac{3c(1-u)^2}{4e(3-u)}\partial_x e - \frac{8cuv}{9-u^2}\partial_x v - \frac{(3+u)^2}{4ce(3-u)}\partial_x(\lambda_{2|0} - \bar{\lambda}_2), \tag{3.17c}$$

$$V_1 = -\frac{(3+u)^2}{4ce(3-u)}\partial_x \lambda_{2|1} - \frac{(3+u)v}{(3-u)\tau_R}, \tag{3.17d}$$

$$V_l = -\frac{(3+u)^2}{4ce(3-u)}\partial_x \lambda_{2|l} \quad (l \geq 2). \tag{3.17e}$$

In view of (3.13), (3.14), and (3.16), and

$$\partial_t \partial_x^n e = \partial_x^n \partial_t e, \quad \partial_t \partial_x^n v = \partial_x^n \partial_t v,$$

we easily verify that

$$\partial_t \lambda_n = \sum_{l=0}^{\infty} \varepsilon^l \Lambda_{n|l} \quad (n \geq 2), \tag{3.18}$$

where

$$\Lambda_{n|l} := \sum_{m=0}^{l} \sum_{r=0}^{m} \left(\frac{\partial \lambda_{n|m}}{\partial(\partial_x^r e)} \partial_x^r E_{l-m} + \frac{\partial \lambda_{n|m}}{\partial(\partial_x^r v)} \partial_x^r V_{l-m} \right). \tag{3.19}$$

The precise meaning of the expansion coefficients in (3.13) is obtained by inserting Eqs. (3.13) and (3.18) into (3.8c) and equating like powers of ε. Given the auxiliary equation

$$\partial_x \lambda_1 = \partial_x q = \frac{4cv}{3+u}\partial_x e + \frac{4ce(3-u)}{(3+u)^2}\partial_x v,$$

which is a consequence of (2.6b) and (3.1), we arrive at

$$\lambda_{n|0} = \bar{\lambda}_n \quad (n \geq 2), \tag{3.20a}$$

$$\lambda_{2|1} = -\tau_N \left(\Lambda_{2|0} + \partial_x \lambda_{3|0} + \frac{16c^3 v}{15(3+u)}\partial_x e + \frac{16c^3(3-u)e}{15(3+u)^2}\partial_x v \right), \tag{3.20b}$$

$$\lambda_{2|l+1} = -\tau_N \left(\Lambda_{2|l} + \partial_x \lambda_{3|l} + \frac{1}{\tau_R}\lambda_{2|l-1} \right), \tag{3.20c}$$

$$\lambda_{n|1} = -\tau_N \left(\Lambda_{n|0} + \partial_x \lambda_{n+1|0} + \frac{c^2 n^2}{4n^2-1}\partial_x \lambda_{n-1|0} \right) \quad (n \geq 3), \tag{3.20d}$$

$$\lambda_{n|l+1} = -\tau_N \left(\Lambda_{n|l} + \partial_x \lambda_{n+1|l} + \frac{c^2 n^2}{4n^2-1}\partial_x \lambda_{n-1|l} + \frac{1}{\tau_R}\lambda_{n|l-1} \right) \quad (n \geq 3), \tag{3.20e}$$

where $l \geq 1$. This is a set of recurrent equations in which the explicit expressions for $\partial_x \lambda_{n|l}$ are provided by

$$\partial_x \lambda_{n|l} = \sum_{m=0}^{l} \left(\frac{\partial \lambda_{n|l}}{\partial(\partial_x^m e)} \partial_x^{m+1} e + \frac{\partial \lambda_{n|l}}{\partial(\partial_x^m v)} \partial_x^{m+1} v \right)$$

and in which $\lambda_{n|0}$ is identified with $\bar{\lambda}_n$. Because of (3.20a), expansion (3.15) becomes

$$N = \sum_{l=1}^{\infty} \varepsilon^l \lambda_{2|l} \tag{3.21}$$

and Eq. (3.17c) simplifies to

$$V_0 = -\frac{3c(1-u)^2}{4e(3-u)} \partial_x e - \frac{8cuv}{9-u^2} \partial_x v. \tag{3.22}$$

When we come to a study of system (3.20) in Sec. IV, we shall confirm that this system can indeed be used to calculate the basic excessive quantity N explicitly in various approximations.

All the above assumes, either explicitly or implicitly, the following picture. There exists a manifold of slow motion in the state space $\mathfrak{X}$ of a dynamical system. Here, the state space $\mathfrak{X}$ is the set of all possible states of system (3.8); each state of this system corresponds to a unique point in $\mathfrak{X}$ and is represented by a sequence $\lambda := \{e, v, \lambda_n | n \geqslant 2\}$ composed of continuous and differentiable functions of x. From the general initial condition $\lambda(t_0) \in \mathfrak{X}$ the system goes quickly into a small neighborhood of the manifold of slow motion, and after that moves slowly along this manifold (see, for example, Refs. [15–18,25]). The manifold of slow motion must be positively invariant: if the motion starts on the manifold at t_0, then it stays on the manifold at $t > t_0$. Under the assumption that system (3.8) has a positively invariant manifold $\mathfrak{M}_\varepsilon \subset \mathfrak{X}$ parametrized by the slow variables (e, v), this manifold can be realized as the graph of a function Ψ_ε defined by

$$\Psi_\varepsilon(e,v) := \left\{ \sum_{l=0}^{\infty} \varepsilon^l \lambda_{n|l}(e, \partial_x e, \partial_x^2 e, \ldots, \partial_x^l e; v, \partial_x v, \partial_x^2 v, \ldots, \partial_x^l v); \quad n \geqslant 2 \right\}.$$

Similarly, the graphs of the functions

$$\Psi_0(e,v) := \{\bar{\lambda}_n(e,v); \quad n \geqslant 2\}$$

and

$$\Psi_{\varepsilon,m}(e,v) := \left\{ \sum_{l=0}^{m} \varepsilon^l \lambda_{n|l}(e, \partial_x e, \partial_x^2 e, \ldots, \partial_x^l e; v, \partial_x v, \partial_x^2 v, \ldots, \partial_x^l v); \quad n \geqslant 2 \right\} \quad (m \geqslant 1).$$

can be used to define the initial manifold $\mathfrak{M}_0$ and the approximate invariant manifolds $\mathfrak{M}_{\varepsilon,m}$. We thus see that singular perturbation techniques like the Hilbert and Chapman-Enskog expansions were in essence developed for the construction of $\mathfrak{M}_\varepsilon$ and $\mathfrak{M}_{\varepsilon,m}$ for both kinetic and moment equations.

Write Eqs. (3.11) and (3.8c) for $\lambda := \{e, v, \lambda_n | n \geqslant 2\}$ in the abstract form

$$\partial_t \lambda = Q_\varepsilon(\lambda). \tag{3.23}$$

The manifold $\mathfrak{M}_\varepsilon$ is invariant with respect to system (3.23), which means that $Q_\varepsilon(\lambda) \in T_\lambda \mathfrak{M}_\varepsilon$ for all $\lambda \in \mathfrak{M}_\varepsilon$, where $T_\lambda \mathfrak{M}_\varepsilon$ is the tangent space at the point $\lambda \in \mathfrak{M}_\varepsilon$. The invariance condition for the manifold $\mathfrak{M}_\varepsilon$ also reads

$$P_\lambda(G_\varepsilon(\lambda)) - G_\varepsilon(\lambda) = 0, \tag{3.24}$$

where $P_\lambda(G_\varepsilon(\lambda))$ is the projection of $G_\varepsilon(\lambda)$ onto $T_\lambda \mathfrak{M}_\varepsilon$. This projection depends in a nontrivial way on $\lambda \in \mathfrak{M}_\varepsilon$ and is uniquely determined by the chosen parametrization of $\mathfrak{M}_\varepsilon$. In Ref. [17], condition (3.24) was considered as an equation for $\mathfrak{M}_\varepsilon$ that can be solved approximately, starting with the initial manifold $\mathfrak{M}_0$. The approximation was based on Newton's iteration rather than a series expansion in a smallness parameter. The Newton method provides a good estimation of $\mathfrak{M}_\varepsilon$ already after one iteration and is also convenient for obtaining the explicit formulas. We stress that, even if in the limit the Newton iteration method and the Chapman-Enskog expansion method lead to the same results, they can give different approximations on the way. As an illustration, using the first iteration of the Newton method, nonlocal hydrodynamic equations were derived from the classical Boltzmann equation [17]. These equations are linearly stable at all wavelengths. It may thus be important to understand the modern geometric framework of the Chapman-Enskog method, because this framework produces many new versions of asymptotic methods such as, e.g., the relaxation method based on a film extension of the original dynamical system [18,26], the Newton iteration method subject to incomplete linearization [17,25], and the method of natural projectors yielding new equations for the post-Navier-Stokes hydrodynamics [18].

Equations (3.11) are exact; we now analyze these equations to lowest order in ε. Expansion (3.21) implies that N is $O(\varepsilon^1)$. The left-hand side of (3.11b) is $O(\varepsilon^0)$, while the right-hand side is $O(\varepsilon^1)$. Hence to $O(\varepsilon^0)$ we may replace (3.11b) by

$$\partial_t v + \frac{3c(1-u)^2}{4e(3-u)}\partial_x e + \frac{8cuv}{9-u^2}\partial_x v = 0. \tag{3.25}$$

From (3.11a) and (3.25) we then obtain a system of hyperbolic differential equations for the energy density and the dimensionless drift velocity. This system is consistent with the nonlinear model of Nielsen and Shklovskii [22]. The only difference is that Eq. (3.25) does not contain the source term. However, this source term is recovered when the first-order approximation is concerned.

IV. HYDRODYNAMICS BEYOND THE ZEROTH-ORDER LEVEL

A. The first-order approximation

We may, to leading order in ε, replace expansion (3.21) by

$$N = \varepsilon \lambda_{2|1}. \tag{4.1}$$

The formal small parameter ε is set equal to one ($\varepsilon=1$) in the subsequent expressions for N. Let $n=2$ and $l=0$ in $\Lambda_{n|l}$. By (3.19) we have

$$\Lambda_{2|0} = \frac{\partial\bar{\lambda}_2}{\partial e}E_0 + \frac{\partial\bar{\lambda}_2}{\partial v}V_0, \tag{4.2}$$

where we have used $\lambda_{2|0}=\bar{\lambda}_2$. The next step is to substitute Eq. (4.2) into (3.20b). Use of Eqs. (3.10), (3.17a), (3.17c), (3.20a), and (4.1) then gives

$$N = \lambda_{2|1} = c^3\tau_N\left(\frac{4(1+3u)v}{5(9-u^2)}\partial_x e - \frac{48(1-4u-u^2)e}{5(3+u)(9-u^2)}\partial_x v - \partial_x\bar{\lambda}_3\right). \tag{4.3}$$

Here, we recall that $u := v^2$. The integral formula for $\bar{\lambda}_3$ can be obtained from Eqs. (3.9), (3.2), and (3.3) in the form

$$\bar{\lambda}_3 = \frac{3c^3 e(1-u)^3}{10(3+u)}\int_{-1}^{1}\frac{(5g^2-3)g}{(1-vg)^4}dg. \tag{4.4}$$

In order to help the calculation of the integral in (4.4), let us define A by

$$A := \frac{1}{u^2}\left[\frac{(1-u)^2}{2\sqrt{u}}\ln\left(\frac{1+\sqrt{u}}{1-\sqrt{u}}\right) + \frac{1}{3}(5u-3)\right]. \tag{4.5}$$

This quantity is well behaved as a function of u near $u=0$ and $u=1$:

$$\lim_{u\to 0_+} A = \frac{8}{15}, \quad \lim_{u\to 1_-} A = \frac{2}{3}.$$

After performing the integration over v, Eq. (4.4) reduces to

$$\bar{\lambda}_3 = \frac{c^3 e[8v - 15(1-u)(vA)]}{5(3+u)}. \tag{4.6}$$

To get an explicit expression for $\lambda_{2|1}$, it need only be noted that

$$\frac{\partial(vA)}{\partial v} = \frac{4(2-3A)}{3(1-u)}.$$

The substitution from Eq. (4.6) into (4.3) then yields

$$N = \lambda_{2|1} = -\frac{4c^3\tau_N H}{9-u^2}\left(\frac{3(1+u)e}{3+u}\partial_x v - \frac{1}{4}(1-u)v\partial_x e\right), \tag{4.7}$$

where

$$H := 3(3-u)A - 4. \tag{4.8}$$

Inspection shows that

$$\lim_{u\to 0_+} H = \frac{4}{5}, \quad \lim_{u\to 1_-} H = 0.$$

The integral expression for H is

$$H = \frac{3(1-u)^2}{2(3-u)}\int_{-1}^{1}\frac{[(3-u)g^2 - 4\sqrt{u}g + 3u - 1]^2}{(1-\sqrt{u}g)^5}dg.$$

Since $0 \leqslant u < 1$, it follows from this expression that H is a positive function of u:

$$H > 0. \tag{4.9}$$

Setting $\varepsilon=1$ in (3.11b), the system of equations for (ε, v) can be written as

$$\partial_t e + \frac{4cv}{3+u}\partial_x e + \frac{4ce(3-u)}{(3+u)^2}\partial_x v = 0, \tag{4.10a}$$

$$\partial_t v + \frac{3c(1-u)^2}{4e(3-u)}\partial_x e + \frac{8cuv}{9-u^2}\partial_x v + \frac{(3+u)^2}{4ce(3-u)}\partial_x N = -\frac{(3+u)v}{(3-u)\tau_R}, \tag{4.10b}$$

where, of course, N is given by (4.7). As demonstrated in Sec. VI, system (4.10) is entropy consistent in the sense that there exists a macroscopic entropy density s which depends locally on the hydrodynamic variables (e,q) and which satisfies the balance equation with a non-negative entropy production due to both resistive and normal processes. This entropy density is obtained by substituting the displaced Planck distribution F_d into the Boltzmann entropy functional given by Eq. (6.8). Now, according to the general ideas presented in Ref. [20], the thermodynamic temperature $\mathcal{T}$ for a phonon gas described by system (4.10) may by defined as

$$\mathcal{T} := \left(\frac{\partial s}{\partial e}\right)^{-1}.$$

We then obtain the result $\mathcal{T}=T_d$. This result shows us very clearly that the true thermodynamic temperature for system (4.10) is not the temperature appearing in the equilibrium Planck distribution, but the temperature appearing in the displaced Planck distribution.

It is significant to observe that system (4.10) involves two relaxation times. The first relaxation time (τ_R) appears on the right-hand side of Eq. (4.10b), whereas the second relaxation time (τ_N) appears on the left-hand side of this equation in the expression for N. This expression is linear in the energy density, nonlinear in the dimensionless drift velocity,[3] and linear in the spatial derivatives of these hydrodynamic variables. With the exception of the obvious inequalities $|v|<1$ and $|q|<ce$, there are effectively no limitations on the magnitudes of v and q, i.e., one can handle problems with large values of the drift velocity and the heat flux. This is a definite improvement over traditional approaches which only make allowances for small deviations in the drift velocity and the heat flux from zero. As demonstrated in Ref. [22], the conditions under which the values of $|v|$ and $|q/ce|$ are comparable to 1 may be attained experimentally.

A final remark should be made concerning the fact that the flow was assumed to be one dimensional. In Ref. [1], starting with Callaway's model and the Chapman-Enskog expansion for the distribution function, we presented a systematic derivation of the equations for the energy density and the dimensionless drift velocity without imposing any restrictions on the geometry of the flow. Consequently, it is important to verify that system (4.10) with the quantity N in the form (4.7) can also be obtained from the one-dimensional, rotationally symmetric reduction of the three-dimensional hydrodynamic equations. The Appendix is devoted to the study of this problem.

B. The second-order approximation

Neglecting the third- and higher-order terms in expansion (3.21), we obtain

$$N = \varepsilon\lambda_{2|1} + \varepsilon^2\lambda_{2|2}, \tag{4.11}$$

where $\lambda_{2|1}$ is given by (4.7) and $\lambda_{2|2}$ is calculated from

$$\lambda_{2|2} = -\tau_N\left(\partial_x\lambda_{3|1} + \frac{1}{\tau_R}\bar{\lambda}_2\right) - \tau_N\left(\frac{\partial\bar{\lambda}_2}{\partial v}V_1 + \frac{\partial\lambda_{2|1}}{\partial e}E_0 + \frac{\partial\lambda_{2|1}}{\partial v}V_0\right) - \tau_N\left(\frac{\partial\lambda_{2|1}}{\partial(\partial_x e)}\partial_x E_0 + \frac{\partial\lambda_{2|1}}{\partial(\partial_x v)}\partial_x V_0\right),$$

$$\lambda_{3|1} = -\tau_N\left(\frac{\partial\bar{\lambda}_3}{\partial e}E_0 + \frac{\partial\bar{\lambda}_3}{\partial v}V_0 + \frac{9c^2}{35}\partial_x\bar{\lambda}_2 + \partial_x\bar{\lambda}_4\right),$$

(3.10), (3.17a), (3.17d), (3.22), and (4.6)–(4.8),

$$\bar{\lambda}_4 = \frac{3c^4e(1-u)^3}{70(3+u)}\int_{-1}^{1}\frac{35g^4-30g^2+3}{(1-vg)^4}dg,$$

and

$$\int_{-1}^{1}\frac{35g^4-30g^2+3}{(1-vg)^4}dg = \frac{8[8(7-6u)-105(1-u)A]}{3(1-u)^3}.$$

Setting $\varepsilon=1$ in (4.11), it can be checked by straightforward if tedious working that

$$N = -\frac{4c^3\tau_N H}{9-u^2}\left(\frac{3(1+u)e}{3+u}\partial_x v - \frac{1}{4}(1-u)v\partial_x e\right) + \frac{2c^4\tau_N^2(1-u)}{9-u^2}\left(\frac{3(1-u)\alpha}{3-u}\partial_x^2 e + \frac{2e\beta v}{9-u^2}\partial_x^2 v\right) + \cdots, \tag{4.12}$$

where the coefficients α and β are defined by

$$\alpha := 2(1-u) - 3H, \quad \beta := \frac{1}{u}[(u^2+54u+45)H - 36(1-u^2)], \tag{4.13}$$

and the ellipsis stands for terms that play no role when a linear stability analysis is performed on the system comprised of Eqs. (4.10) and (4.12). Note that

$$18(1+u)\alpha + u\beta = -(9-u^2)H < 0 \tag{4.14}$$

and

$$\lim_{u\to 0_+}\alpha = -\frac{2}{5}, \quad \lim_{u\to 1_-}\alpha = 0, \quad \lim_{u\to 0_+}\beta = \frac{72}{35}, \quad \lim_{u\to 1_-}\beta = 0. \tag{4.15}$$

Because of (4.15), there is no true singularity in (4.13) and the coefficients α and β are regular, continuously differentiable functions of u.

V. LINEAR STABILITY ANALYSIS

In this section, assuming that N vanishes or is defined by either (4.7) or (4.12), we derive the linearized equations of motion for perturbations about an equilibrium state of the phonon gas. We then solve these equations for exponential plane waves on a constant background state. The dispersion relations for these waves are analyzed and are found to contain no growing modes.

The difference between the actual value of a field W at a given spacetime point and the value which W has in the background equilibrium state will be denoted by δW. The quantities δe and δv are the fields that describe the perturbations of a phonon gas about its equilibrium state. Fields that do not include the prefix δ will henceforth refer to the background configuration. This configuration is assumed to satisfy Eqs. (3.11), in addition to being constant. The equations of motion for the perturbation fields δe and δv are obtained by linearizing system (3.11) about the equilibrium state.

A. The case of the vanishing source term

If $\tau_R=\infty$, the background drift velocity is arbitrary and the equations for δe and δv can be written as

$$\partial_t\delta e + \frac{4cv}{3+u}\partial_x\delta e + \frac{4ce(3-u)}{(3+u)^2}\partial_x\delta v = 0, \tag{5.1a}$$

$$\partial_t\delta v + \frac{3c(1-u)^2}{4e(3-u)}\partial_x\delta e + \frac{8cuv}{9-u^2}\partial_x\delta v + \frac{(3+u)^2}{4ce(3-u)}\partial_x\delta N = 0. \tag{5.1b}$$

Here δN is defined by

[3]The dimensionless drift velocity is, in turn, a highly nonlinear function of the energy density and the heat flux; see Eq. (2.6b).

$$\delta N = -\frac{4c^3\tau_N\hat{H}}{9-u^2}\left(\frac{3(1+u)e}{3+u}\partial_x\delta v - \frac{1}{4}(1-u)v\partial_x\delta e\right) + \frac{2c^4\tau_N^2(1-u)}{9-u^2}\left(\frac{3(1-u)\hat{\alpha}}{3-u}\partial_x^2\delta e + \frac{2e\hat{\beta}v}{9-u^2}\partial_x^2\delta v\right), \tag{5.2}$$

where

$$(\hat{H},\hat{\alpha},\hat{\beta}) := \begin{cases}(0,0,0) & \text{in the calculation to order } \varepsilon^0,\\ (H,0,0) & \text{in the calculation to order } \varepsilon^1,\\ (H,\alpha,\beta) & \text{in the calculation to order } \varepsilon^2.\end{cases} \tag{5.3}$$

We look for exponential plane-wave solutions to the perturbation equations, i.e., we assume that

$$\delta e = \delta X \exp\left[\frac{\mathrm{i}}{\tau_N}\left(\omega t - \frac{k}{c}x\right)\right], \quad \delta v = \delta Y \exp\left[\frac{\mathrm{i}}{\tau_N}\left(\omega t - \frac{k}{c}x\right)\right], \tag{5.4}$$

where $(\delta X, \delta Y)$ is the complex constant amplitude of the wave, ω is its frequency, and k is its wave number. Substituting (5.4) into Eqs. (5.2) and (5.1), we obtain the system of equations for $(\delta X, \delta Y)$

$$\left(\omega - \frac{4vk}{3+u}\right)\delta X - \frac{4(3-u)ek}{(3+u)^2}\delta Y = 0, \tag{5.5a}$$

$$\frac{k}{4(3-u)}[3(1-u)^2 + (3+u)^2(\mathcal{P} - \mathrm{i}vk\mathcal{F})]\delta X - e\left(\omega - \frac{8uvk}{9-u^2} - \frac{(3+u)^2k}{4(3-u)}(v\mathcal{R} + \mathrm{i}k\mathcal{U})\right)\delta Y = 0, \tag{5.5b}$$

in which

$$\mathcal{P} := -\frac{6(1-u)^2\hat{\alpha}k^2}{(3-u)(9-u^2)}, \quad \mathcal{F} := \frac{(1-u)\hat{H}}{9-u^2}, \tag{5.6a}$$

$$\mathcal{R} := -\frac{4(1-u)\hat{\beta}k^2}{(9-u^2)^2}, \quad \mathcal{U} := \frac{12(1+u)\hat{H}}{(3+u)(9-u^2)}. \tag{5.6b}$$

This system can be expressed compactly in matrix form as

$$\begin{bmatrix} b_{11}(\omega,k) & b_{12}(\omega,k) \\ b_{21}(\omega,k) & b_{22}(\omega,k)\end{bmatrix}\begin{bmatrix}\delta X \\ \delta Y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}. \tag{5.7}$$

There will exist exponential plane-wave solutions of Eqs. (5.1) whenever ω and k have values that satisfy the condition

$$\begin{vmatrix} b_{11}(\omega,k) & b_{12}(\omega,k) \\ b_{21}(\omega,k) & b_{22}(\omega,k)\end{vmatrix} = 0. \tag{5.8}$$

The resulting relation between ω and k is called the dispersion relation. Given Eqs. (5.5) and (5.7), we easily verify that condition (5.8) is equivalent to the following quadratic equation for ω:

$$4(3-u)\omega^2 - [16v + (3+u)^2(v\mathcal{R} + \mathrm{i}k\mathcal{U})]k\omega + 4[(3+u)(u\mathcal{R} + \mathrm{i}vk\mathcal{U}) - (3-u)(\mathcal{P} - \mathrm{i}vk\mathcal{F}) - 1 + 3u]k^2 = 0. \tag{5.9}$$

The dispersion relations are the roots of this equation.

When $(\hat{H},\hat{\alpha},\hat{\beta}) = (0,0,0)$, then $\mathcal{F} = \mathcal{P} = \mathcal{R} = \mathcal{U} = 0$ and Eq. (5.9) becomes

$$4(3-u)\omega^2 - 16vk\omega - 4(1-3u)k^2 = 0.$$

This yields the dispersion relations for linear plane waves that travel in the x direction without damping:

$$\omega = \frac{[2v \pm \sqrt{3}(1-u)]k}{3-u}.$$

When $(\hat{H},\hat{\alpha},\hat{\beta}) = (H,0,0)$ or $(\hat{H},\hat{\alpha},\hat{\beta}) = (H,\alpha,\beta)$, from Eq. (5.9) we find for the real and imaginary parts of ω

$$\mathrm{Re}(\omega) = \frac{k}{8(3-u)}\left(16v + (3+u)^2v\mathcal{R} \pm \sqrt{2\sqrt{\mathbb{A}^2 + \mathbb{B}^2} - 2\mathbb{B}}\right), \tag{5.10a}$$

$$\mathrm{Im}(\omega) = \frac{k}{8(3-u)}\left((3+u)^2k\mathcal{U} \mp \mathrm{sgn}(\mathbb{A})\sqrt{2\sqrt{\mathbb{A}^2 + \mathbb{B}^2} + 2\mathbb{B}}\right), \tag{5.10b}$$

where

$$\mathbb{A} := 16(3-u)^2\mathcal{F}vk + (3+u)\left(24(1-u) - \frac{1}{2}(3+u)^3\mathcal{R}\right)\mathcal{U}vk, \tag{5.11a}$$

$$\mathbb{B} := \frac{1}{4}(3+u)^4k^2\mathcal{U}^2 - \frac{1}{4}(3+u)^4u\mathcal{R}^2 - 16(3-u)^2\mathcal{P} + 24(1-u)[(3+u)u\mathcal{R} - 2(1-u)], \tag{5.11b}$$

and $\mathrm{sgn}(\mathbb{A})$ is the function of $\mathbb{A}$ defined formally as

$$\mathrm{sgn}(\mathbb{A}) := \begin{cases} 1 & \text{if } \mathbb{A} \geqslant 0,\\ -1 & \text{if } \mathbb{A} < 0.\end{cases}$$

Stability requires that $\mathrm{Im}(\omega) \geqslant 0$, i.e.,

$$\Omega_1 := (3+u)^2|k|\mathcal{U} - \sqrt{2\sqrt{\mathbb{A}^2 + \mathbb{B}^2} + 2\mathbb{B}} \geqslant 0.$$

Note that Ω_1 is a continuous function of u and $|k|$ since $\mathcal{U}$, $\mathbb{A}^2$, and $\mathbb{B}$ are continuous functions of u and $|k|$. In view of

$$\Omega_1 = \Omega_2 := (3+u)^2|k|\mathcal{U} - \frac{2|\mathbb{A}|}{\sqrt{2\sqrt{\mathbb{A}^2 + \mathbb{B}^2} - 2\mathbb{B}}} \quad (\mathbb{A} \neq 0), \tag{5.12a}$$

an equivalent statement to $\mathbb{A} \neq 0$, $\Omega_1 \geqslant 0$, is $\mathbb{A} \neq 0$, $\Omega_2 \geqslant 0$. The insertion of $\hat{H} = H$ into the second equation in (5.6b) gives, with (4.9), the inequality

$$\mathcal{U} = \frac{12(1+u)H}{(3+u)(9-u^2)} > 0. \tag{5.12b}$$

Let

$$Z := (3+u)^8(\mathbb{A}^2+\mathbb{B}^2)k^4\mathcal{U}^4 - [2\mathbb{A}^2+(3+u)^4\mathbb{B}k^2\mathcal{U}^2]^2. \tag{5.13}$$

It is evident from (5.12a) and (5.12b) and the aforementioned continuity property of Ω_1 that $\Omega_1 \geqslant 0$ if $Z \geqslant 0$. The key to the proof of $Z \geqslant 0$ consists of substituting Eqs. (5.11) into (5.13). A little algebra, aided by Eqs. (5.6) with $\hat{H}=H$, shows that

$$Z = Z_1 Z_2,$$

where

$$Z_1 := \frac{[16(1-u)Hk]^4[2(9+u)(3-u)^2+3(1+u)\hat{\beta}k^2]^2 u}{(3+u)(3-u)^8},$$

$$Z_2 := (3+u)(3-u)^3 - 3(1+u)[18(1+u)\hat{\alpha}+u\hat{\beta}]k^2.$$

If $18(1+u)\alpha+u\beta$ is expressed in the form (4.14), then

$$Z_2 = \begin{cases} (3+u)(3-u)^3 & \text{for } (\hat{\alpha},\hat{\beta})=(0,0), \\ (9-u^2)[(3-u)^2+3(1+u)Hk^2] & \text{for } (\hat{\alpha},\hat{\beta})=(\alpha,\beta), \end{cases}$$

which yields $Z_2>0$. Also, the inequality $Z \geqslant 0$ is implied, since $Z=Z_1Z_2$ and $Z_1 \geqslant 0$. This completes the proof that the dispersion relations defined by (5.10) contain no growing modes.

B. The case of a nonzero source term

If $\tau_R \neq \infty$, the background drift velocity is zero and the equations for δe and δv are

$$\partial_t \delta e + \frac{4}{3} c e \partial_x \delta v = 0, \tag{5.14a}$$

$$\partial_t \delta v + \frac{c}{4e}\partial_x \delta e + \frac{3}{4ce}\partial_x \delta N = -\frac{1}{\tau_R}\gamma \delta v, \tag{5.14b}$$

where[4]

$$\delta N = -\frac{4c^3\tau_N}{45}(4\gamma e \partial_x \delta v + \xi c \tau_N \partial_x^2 \delta e), \tag{5.15}$$

with

$$(\gamma,\xi) := \begin{cases} (0,0) & \text{in the calculation to order } \varepsilon^0, \\ (1,0) & \text{in the calculation to order } \varepsilon^1, \\ (1,1) & \text{in the calculation to order } \varepsilon^2. \end{cases}$$

Inserting (5.4) into Eqs. (5.15) and (5.14), we obtain

$$\omega \delta X - \frac{4}{3} e k \delta Y = 0,$$

$$\frac{1}{4}k\left(1+\frac{4}{15}\xi k^2\right)\delta X - e\left(\omega - \mathrm{i}\gamma\eta - \frac{4}{15}\gamma k^2 \mathrm{i}\right)\delta Y = 0,$$

where $\eta := \tau_N/\tau_R$. The dispersion relation appears in the form

$$\begin{vmatrix} \omega & -\frac{4}{3}ek \\ \frac{1}{4}k\left(1+\frac{4}{15}\xi k^2\right) & -e\left(\omega - \mathrm{i}\gamma\eta - \frac{4}{15}\gamma k^2 \mathrm{i}\right) \end{vmatrix} = 0;$$

that is,

$$3\omega^2 - 3\left(\gamma\eta + \frac{4}{15}\gamma k^2\right)\omega \mathrm{i} - k^2\left(1+\frac{4}{15}\xi k^2\right) = 0. \tag{5.16}$$

Equation (5.16) is a quadratic equation for ω, yielding two values that may be regarded, for prescribed (γ,ξ,η), as functions of k:

$$\omega = \omega_{(\pm)} := \frac{\mathrm{i}}{2}(\mathcal{W} \pm \sqrt{\mathcal{W}^2 - \mathcal{Z}}), \tag{5.17}$$

where

$$\mathcal{W} := \gamma\eta + \frac{4}{15}\gamma k^2, \quad \mathcal{Z} := \frac{4}{3}k^2\left(1+\frac{4}{15}\xi k^2\right).$$

It follows from this that

$$\mathrm{Im}(\omega_{(+)}) \geqslant \frac{1}{2}\left(\gamma\eta + \frac{4}{15}\gamma k^2\right) \geqslant 0,$$

$$0 \leqslant \mathrm{Im}(\omega_{(-)}) \leqslant \frac{1}{2}\left(\gamma\eta + \frac{4}{15}\gamma k^2\right).$$

Hence, the modes represented by Eqs. (5.17) are found to be stable for all wave numbers.

VI. H THEOREM AND ITS RELATION TO STABILITY

In order to list the explicit form of an H theorem for the system made up of Eqs. (5.1) and (5.2), we require some preliminary definitions:

$$\hat{\vartheta} := -\frac{18(1+u)\hat{\alpha}+u\hat{\beta}}{9-u^2},$$

$$\Theta := 12e(1+u)\delta v - (1-u)(3+u)v\,\delta e,$$

$$\Gamma := (1-u)(3+u)[4e(15-u)\delta v - 3\Theta]\delta e - 16e^2(3-14u-u^2)v(\delta v)^2,$$

$$\Pi := 72e(1-u^2)(3+u)\hat{\alpha}[\delta v\,\partial_x^2\delta e - (\partial_x\delta e)(\partial_x\delta v)] + 24e^2(1+u)\hat{\beta}v[2\delta v\,\partial_x^2\delta v - (\partial_x\delta v)^2] - 4e(1-u)(3+u)u\hat{\beta}\delta e\,\partial_x^2\delta v - 3(1-u)^2(3+u)^2\hat{\alpha}v[2\delta e\,\partial_x^2\delta e - (\partial_x\delta e)^2] + 2(1-u)(3+u)^3\hat{\vartheta}v(\partial_x\delta e)^2.$$

[4]In the case when v approaches zero and e and the first and second spatial derivatives of δe and δv are arbitrarily fixed, Eq. (5.2) simplifies to (5.15).

Given the definition of $\hat{\vartheta}$, it follows from Eqs. (5.3), (4.14), and (4.9) that

$$\hat{\vartheta}=\begin{cases}0 & \text{in the calculation to order } \varepsilon^0,\\ 0 & \text{in the calculation to order } \varepsilon^1,\\ H>0 & \text{in the calculation to order } \varepsilon^2.\end{cases}$$

Now, we introduce the non-negative and strictly convex entropy density

$$\mathcal{H}:=\frac{3(1-u)^2(3+u)^2}{8e(3-u)}(\delta e)^2-(1-u)(3+u)v\,\delta e\,\delta v + 6e(1+u)(\delta v)^2+\frac{c^2\tau_N^2(1-u)^2(3+u)^4\hat{\vartheta}}{8e(3-u)^3}(\partial_x\delta e)^2, \tag{6.1}$$

take its time derivative, and replace the time derivatives of δe, δv, and $\partial_x\delta e$ by means of Eqs. (5.1) to obtain the balance law for entropy—the second law—as

$$\partial_t\mathcal{H}+\partial_x\Phi=\Sigma, \tag{6.2}$$

with the entropy flux

$$\Phi:=\frac{c}{8e(3-u)}\left(\Gamma-\frac{c\tau_N}{3-u}\hat{H}\partial_x\Theta^2+\frac{2c^2\tau_N^2(1-u)}{(3-u)^2}\Pi\right)$$

and the bulk entropy generation rate

$$\Sigma:=-\frac{c^2\tau_N}{4e(3-u)^2}\hat{H}(\partial_x\Theta)^2,$$

which is either zero (then $\hat{H}=0$ or $\partial_x\Theta=0$) or strictly negative (then $\hat{H}=H$ and $\partial_x\Theta\neq 0$).

As regards Eqs. (5.14) and (5.15), these equations also ensure consistency with the entropy balance law in the form (6.2) provided that $\mathcal{H}$, Φ, and Σ are given by

$$\mathcal{H}:=\frac{9}{8e}(\delta e)^2+6e(\delta v)^2+\frac{3}{10e}\xi c^2\tau_N^2(\partial_x\delta e)^2, \tag{6.3a}$$

$$\Phi:=3c\,\delta e\,\delta v-\frac{8}{5}\gamma e c^2\tau_N\partial_x(\delta v)^2-\frac{4}{5}\xi c^3\tau_N^2[\delta v\,\partial_x^2\delta e-(\partial_x\delta e)\times(\partial_x\delta v)], \tag{6.3b}$$

$$\Sigma:=-\frac{12}{\tau_R}\gamma e(\delta v)^2-\frac{16}{5}\gamma e c^2\tau_N(\partial_x\delta v)^2. \tag{6.3c}$$

Let $[a,b]$, $a<b$, be a closed bounded interval in $\mathbb{R}$ and suppose that $\delta e(t,x)$, $\delta v(t,x)$ is a nonzero solution of system (5.1) or system (5.14) such that $\Phi(t,a)=\Phi(t,b)=0$ for $t\in[0,\infty)$. Then we arrive at the H theorem

$$\frac{d}{dt}\int_a^b\mathcal{H}(t,x)dx=\int_a^b\Sigma(t,x)dx\leqslant 0, \tag{6.4}$$

which shows the stability of systems (5.1) and (5.14) if $(\hat{H},\gamma)=(0,0)$,

$$\int_a^b\Sigma(t,x)dx=0,$$

and asymptotic stability of these systems if $(\hat{H},\gamma)=(H,1)$,

$$\int_a^b\Sigma(t,x)dx<0.$$

Moreover, using the inner product

$$\langle\phi,\psi\rangle:=\int_a^b\phi(x)\psi(x)dx$$

and the norm $\|\phi\|:=\sqrt{\langle\phi,\phi\rangle}$, we see from Eqs. (6.1), (6.3a), and (6.4) that

$$\mathcal{L}:=\frac{3(1-u)^2(3+u)^2}{8e(3-u)}\|\delta e\|^2-(1-u)(3+u)v\langle\delta e,\delta v\rangle + 6e(1+u)\|\delta v\|^2+\frac{c^2\tau_N^2(1-u)^2(3+u)^4\hat{\vartheta}}{8e(3-u)^3}\|\partial_x\delta e\|^2 \tag{6.5}$$

is the Lyapunov entropy functional for system (5.1) defined on $(0,\infty)\times(a,b)$ and

$$\mathcal{L}:=\frac{9}{8e}\|\delta e\|^2+6e\|\delta v\|^2+\frac{3}{10e}\xi c^2\tau_N^2\|\partial_x\delta e\|^2 \tag{6.6}$$

is the Lyapunov entropy functional for system (5.14) defined on $(0,\infty)\times(a,b)$.

At first sight, it may seem that the quantity $\mathcal{H}$ and the corresponding Lyapunov entropy functional $\mathcal{L}$ are postulated *ad hoc*. However, at least in the cases $(\hat{H},\hat{\alpha},\hat{\beta},\hat{\vartheta})=(H,0,0,0)$ and $(\gamma,\xi)=(1,0)$, it is possible to relate $\mathcal{H}$ directly to the second differential of the true entropy density. Namely, it was shown in Ref. [1] that the nonlinear system comprised of Eqs. (A1) and (A3) has an entropy function

$$s:=\frac{4(1-u)}{3+u}\frac{e}{T_d} \tag{6.7}$$

that satisfies the balance law with a non-negative entropy production. This entropy function is identical with the entropy of the Nielsen-Shklovskii model [22,27] and is obtained by substituting $f=F_d$ into the following well-known kinetic-theory expression for the entropy functional [5]:

$$s(f):=y\int\left[\left(1+\frac{f}{y}\right)\ln\left(1+\frac{f}{y}\right)-\frac{f}{y}\ln\left(\frac{f}{y}\right)\right]d^3\mathbf{p}. \tag{6.8}$$

Equation (6.7) also defines the entropy function for Eqs. (4.7) and (4.10) because these equations can be derived from the one-dimensional, rotationally symmetric reduction of Eqs. (A1) and (A3). In view of (2.6), we may evaluate s explicitly as a function of either (e,q) or (e,v). Let δs and $\delta^2 s$ be the first and second differentials of the entropy function $\bar{s}:=s(e+\delta e,q+\delta q)$ evaluated at (e,q):

$$\delta s:=\frac{\partial s}{\partial e}\delta e+\frac{\partial s}{\partial q}\delta q=\frac{\partial s}{\partial e}\delta e+\frac{\partial s}{\partial v}\delta v,$$

$$\delta^2 s := \frac{\partial^2 s}{\partial e^2}(\delta e)^2 + 2\frac{\partial^2 s}{\partial e \partial q}\delta e\, \delta q + \frac{\partial^2 s}{\partial q^2}(\delta q)^2.$$

If the perturbation of the background state (e,q) is small, then the entropy function $\bar{s}$ can be approximated as

$$\bar{s} \cong \tilde{s} := s + \delta s + \delta^2 s. \tag{6.9}$$

The equations

$$\frac{\partial^2 s}{\partial^2 e} = -\frac{(1+5u)(3+u)^2 s}{16(3-u)(1-u)^2 e^2}, \quad \frac{\partial^2 s}{\partial e \partial q} = \frac{(5+u)(3+u)^2 s v}{16(3-u)(1-u)^2 c e^2},$$

$$\frac{\partial^2 s}{\partial^2 q} = -\frac{3(1+u)(3+u)^2 s}{16(3-u)(1-u)^2 c^2 e^2},$$

$$\delta q = \frac{4cv}{3+u}\delta e + \frac{4ce(3-u)}{(3+u)^2}\delta v,$$

and system (5.1) imply that the second differential of $\bar{s}$ and the time derivative of $s+\delta s$ take the forms

$$\delta^2 s = -\mathcal{C}\mathcal{H}, \quad \partial_t(s+\delta s) = -\partial_x \Phi_0, \tag{6.10}$$

where $\mathcal{C}$

$$\mathcal{C} := \frac{(3-u)s}{2(1-u)^2(3+u)^2 e}, \quad \Phi_0 := \frac{\partial s}{\partial e}\delta q + \frac{\partial s}{\partial q}\left(\frac{c^2}{3}\delta e + \delta N\right).$$

Note that $\mathcal{C}$ is a positive constant uniquely determined by the background quantities e and $u := v^2$. Using (6.9) and (6.10), the formal entropy balance law (6.2) can now be reformulated in terms of the true approximate entropy density $\tilde{s}$ as

$$\partial_t \tilde{s} + \partial_x(\Phi_0 - \mathcal{C}\Phi) = -\mathcal{C}\Sigma \geqslant 0.$$

Moreover, the Laypunov entropy functional $\mathcal{L}$ may be put in the form

$$\mathcal{L} = \int_a^b \mathcal{H}\, dx = -\frac{1}{\mathcal{C}}\int_a^b \delta^2 s\, dx,$$

from which it is seen to be exactly proportional to the integral of $\delta^2 s$ over $[a,b]$.

For a general system of linear partial differential equations, say

$$\partial_t w_\mu = \sum_{\nu=1}^{k} p_{\mu\nu}(D_x) w_\nu \quad (\mu = 1, \ldots, k), \tag{6.11}$$

where $D_x := (\partial/\partial x_1, \ldots, \partial/\partial x_m)$ and $p_{\mu\nu}(\zeta)$ are polynomials in $\zeta := (\zeta_1 \ldots, \zeta_m)$ with real constant coefficients, it is not easy to construct or find a Lyapunov candidate functional which proves the stability of an equilibrium state, and the inability to find a Lyapunov functional is inconclusive with respect to stability, which means that not finding a Lyapunov functional does not mean that the system is unstable. However, for system (6.11) with

$$p_{\mu\nu}(D_x) = A_{\mu\nu}\nabla^2,$$

where ∇^2 denotes the Laplacian operator on $\mathbb{R}^m$ and $A_{\mu\nu}$ are real constants, Zhang [10] has derived formulas of explicit quadratic Lyapunov functionals by choosing appropriate coefficients of the characteristic polynomial of a matrix $(A_{\mu\nu})_{k\times k}$. If $(\hat{H}, \hat{\alpha}, \hat{\beta}) = (H,0,0)$ and $(\gamma, \xi) = (1,0)$, we obtain the differential equations in the form

$$\partial_t w_\mu = \sum_{\nu=1}^{2} (A_{\mu\nu}\nabla^2 w_\nu + B_{\mu\nu}\nabla w_\nu + C_{\mu\nu} w_\nu) \quad (\mu = 1,2), \tag{6.12}$$

where $\nabla = \partial_x$. Since the matrix $(A_{\mu\nu})_{2\times 2}$ derived from system (5.1) or system (5.14) is degenerate in that it has only one strictly positive eigenvalue, the method of Zhang cannot be directly used. In this case, as shown above, the alternative method based on the entropy function (6.7) provides a more successful algorithm for computing the Lyapunov functional. If $(\hat{H}, \hat{\alpha}, \hat{\beta}) = (H, \alpha, \beta)$ and $(\gamma, \xi) = (1,1)$, we obtain linear partial differential equations for $(\delta e, \delta v)$ which are third order with respect to x. Then the method of Zhang also cannot be used. The same observation concerns the method based on Eq. (6.7). Consequently, in this case, finding Eqs. (6.5) and (6.6) with $(\hat{\vartheta}, \xi) = (H,1)$ might be a matter of luck, i.e., trial and error is one of the methods to use when testing various Lyapunov candidate functionals on some equilibrium solutions.

Analogous problems are encountered when considering the Chapman-Enskog method as applied to classical gases described by the Boltzmann equation. There, the results of the computations made for Maxwellian molecules and rigid spheres show that the solutions of the conventional Burnett equations [28] are unstable with respect to short-wavelength perturbations [29,30] and violate the second law of thermodynamics [31], while the Euler and Navier-Stokes equations are linearly stable at all wavelengths and provide entropy consistent results. Moreover, it was demonstrated by Bobylev [32] that the linear version of the so-called hyperbolic Burnett equations satisfies an H theorem and possesses an explicit quadratic Lyapunov functional for the determination of asymptotic stability of solutions. The hyperbolic Burnett equations are not to be confused with the conventional Burnett equations [28] or the augmented Burnett equations [33]. Due to the instability paradox, in the terminology of Jin and Slemrod [34], the classical Chapman-Enskog procedure does not work at the level of conventional Burnett equations (the next step after the Navier-Stokes equations). The hyperbolic Burnett equations were derived from the Boltzmann equation by modifying the classical Chapman-Enskog procedure [32]. Here, we do not modify the method because we use the Callaway model. Then the instability paradox does not appear at the level corresponding to the Burnett level of approximation. Surprisingly enough, the Burnett equations of the standard Bhatnagar-Gross-Krook (BGK) model ([35], Pr=1) are linearly stable at all wavelengths [28,36], indicating that the stability of the BGK-Burnett equations is related to the value of the Prandtl number Pr.

VII. THE THIRD-ORDER APPROXIMATION

A. Instability of the solutions of linearized equations

When $\tau_R = \infty$, the investigation of the stability problem can proceed as in Sec. V A in that Eqs. (5.1) still apply. Now, however, from

$$N = \varepsilon\lambda_{2|1} + \varepsilon^2\lambda_{2|2} + \varepsilon^3\lambda_{2|3} \tag{7.1}$$

and (3.20),

$$\begin{aligned}\delta N = &-\frac{4c^3\tau_N H}{9-u^2}\left(\frac{3(1+u)e}{3+u}\partial_x\delta v - \frac{1}{4}(1-u)v\partial_x\delta e\right)\\ &+\frac{2c^4\tau_N^2(1-u)}{9-u^2}\left(\frac{3(1-u)\alpha}{3-u}\partial_x^2\delta e + \frac{2e\beta v}{9-u^2}\partial_x^2\delta v\right)\\ &+\frac{6c^5\tau_N^3}{(3-u)(9-u^2)}\left(\frac{(1-u)\sigma v}{2(3-u)}\partial_x^3\delta e - \frac{2e\theta}{9-u^2}\partial_x^3\delta v\right),\end{aligned} \tag{7.2}$$

where[5]

$$\sigma := \frac{1}{u}[4(17u-15)(1-u)^2 + (75-u)(1-u)^2H - 2u(1+u)H^2],$$

$$\theta := \frac{1}{u}[4(2u^3+49u^2-12u-27)(1-u)^2 + (7u^2+222u+135)(1-u)^2H - 6u(1+u)^2H^2];$$

so the introduction of the notation

$$\mathcal{P} := -\frac{6(1-u)^2\alpha k^2}{(3-u)(9-u^2)}, \quad \mathcal{F} := \frac{1-u}{9-u^2}\left(H - \frac{3\sigma k^2}{(3-u)^2}\right), \tag{7.3a}$$

$$\mathcal{R} := -\frac{4(1-u)\beta k^2}{(9-u^2)^2},$$

$$\mathcal{U} := \frac{12}{(3+u)(9-u^2)}\left((1+u)H - \frac{\theta k^2}{(3-u)^2}\right) \tag{7.3b}$$

enables the equations for δX and δY to be written as Eqs. (5.5), which clearly yield (5.10) if $\mathbb{A}$ and $\mathbb{B}$ are defined by (5.11a) and (5.11b), respectively.

Since the dispersion relations are rather complicated, it is informative to examine their limiting forms in order to obtain some insight into these modes. We first consider the $k\to 0$ limit. This limit corresponds to the regime in which the perturbations vary slowly in space. Expanding Eqs. (5.10) about $k=0$ in a Taylor series in k, we find that the leading terms in the expansion of the dispersion relations are given by

$$\omega \cong \frac{[2v \pm \sqrt{3}(1-u)]k}{3-u} + \frac{\sqrt{3}(\sqrt{3}\mp v)Hk^2\mathrm{i}}{6(\sqrt{3}\pm v)^2}.$$

The imaginary part of ω is strictly positive, so that the modes are damped at lowest order in k.

Another way to see that the third-order approximation predicts inhibition of the instability at sufficiently small wave numbers is to apply the following inequalities to Eqs. (7.3):

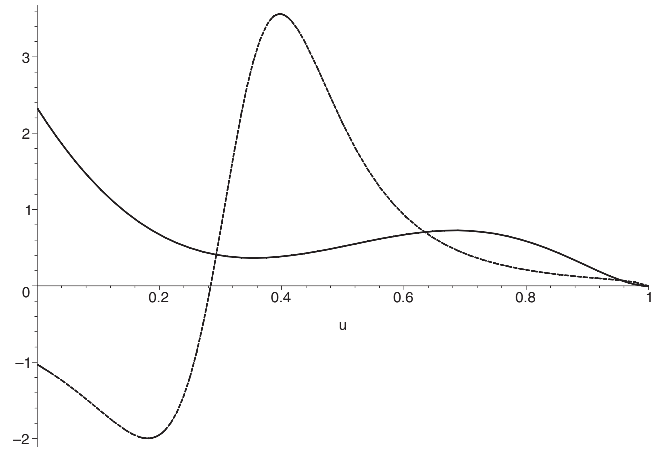

FIG. 1. θ and $(6\alpha\theta+u\beta\sigma)/\theta^2$ as functions of u. The solid curve refers to θ, while the dashed curve corresponds to $(6\alpha\theta+u\beta\sigma)/\theta^2$.

$$H(1-k^2) \leqslant H - \frac{3\sigma k^2}{(3-u)^2} \leqslant H\left(1 + \frac{116}{105}k^2\right),$$

$$(1+u)H(1-k^2) \leqslant (1+u)H - \frac{\theta k^2}{(3-u)^2} \leqslant (1+u)H\left(1 - \frac{1}{20}k^2\right).$$

Evidently, when k^2 can be neglected compared with unity, these equations are much the same as Eqs. (5.6) with $(\hat{H},\hat{\alpha},\hat{\beta})=(H,\alpha,\beta)$, and instability is absent.

Next, we consider the short-wavelength ($|k|\to\infty$) limit of the dispersion relations, even though this limit is not relevant for a hydrodynamic theory because the derivation of Eq. (7.2) from (3.20) and (7.1) is generally not valid when the perturbation fields δe and δv are assumed to vary rapidly in space. As a matter of fact, to lowest order in an expansion in powers of k^{-2}, we may replace Eqs. (5.10) by

$$\omega_{(1)} = k^3[\varpi_{r(1)} + O(k^{-2})] - \mathrm{i}k^4[\varpi_{i(1)} + O(k^{-2})],$$

$$\omega_{(2)} = k[\varpi_{r(2)} + O(k^{-2})] + \mathrm{i}[\varpi_{i(2)} + O(k^{-2})],$$

where

$$\varpi_{r(1)} := -\frac{(1-u)\beta v}{(3-u)^3}, \quad \varpi_{r(2)} := \frac{(1-u)[6\theta+(3-u)^2\sigma]v}{(9-u^2)\theta},$$

$$\varpi_{i(1)} := \frac{3\theta}{(3-u)^4}, \quad \varpi_{i(2)} := \frac{(1-u)^2(3-u)^2(6\alpha\theta+u\beta\sigma)}{3(3+u)\theta^2}.$$

Let us look at some of the implications of these equations. It is clear that the sign of $\varpi_{i(1)}$ is the same as that of θ, and that the sign of $\varpi_{i(2)}$ is the same as that of $(6\alpha\theta+u\beta\sigma)/\theta^2$. Hence, if k^2 is sufficiently large, $\mathrm{Im}(\omega_{(1)})$ is positive or negative according as θ is negative or positive, and $\mathrm{Im}(\omega_{(2)})$ is positive or negative according as $(6\alpha\theta+u\beta\sigma)/\theta^2$ is positive or negative. Figure 1 shows θ and $(6\alpha\theta+u\beta\sigma)/\theta^2$ plotted against u. Since $\theta>0$ for all values of u, it follows that the short-wavelength ($|k|\to\infty$) mode represented by $\varpi_{r(1)}$ and

[5]In the limit $u\to 0_+$, we have $\sigma=-464/175$ and $\theta=408/175$. In the limit $u\to 1_-$, we obtain $\sigma=\theta=0$.

$\varpi_{i(1)}$ grows exponentially with time. Consequently, this mode is an unstable mode. It also follows that the second mode is unstable if u is smaller than $u_c \cong 0.2838$.

We finally consider the case where $\tau_R \neq \infty$, so that $v=0$. With Eq. (5.15) changed to

$$\delta N = -\frac{4c^3\tau_N}{45}\left[c\tau_N\partial_x^2\delta e + 4e\left(\partial_x\delta v + \frac{34c^2\tau_N^2}{105}\partial_x^3\delta v\right)\right] \tag{7.4}$$

the system of equations for δX and δY is seen to be

$$\omega\delta X - \frac{4}{3}ek\delta Y = 0,$$

$$\frac{1}{4}k\left(1+\frac{4}{15}k^2\right)\delta X - e\left[\omega - \mathrm{i}\eta - \frac{4}{15}k^2\left(1-\frac{34}{105}k^2\right)\mathrm{i}\right]\delta Y = 0,$$

where $\eta := \tau_N/\tau_R$. Thus,

$$\omega = \omega_{(\pm)} := \frac{\mathrm{i}}{2}(\mathcal{W} \pm \sqrt{\mathcal{W}^2 - \mathcal{Z}}), \tag{7.5}$$

where

$$\mathcal{W} := \eta + \frac{4}{15}k^2 - \frac{136}{1575}k^4, \quad \mathcal{Z} := \frac{4}{3}k^2\left(1+\frac{4}{15}k^2\right).$$

This gives

$$\lim_{k\to 0}\omega_{(+)} = \eta\mathrm{i}, \quad \lim_{k\to\pm\infty}\omega_{(+)} = -\frac{35}{34}\mathrm{i},$$

$$\lim_{k\to 0}\left(\frac{1}{k^2}\omega_{(-)}\right) = \frac{1}{3\eta}\mathrm{i}, \quad \lim_{k\to\pm\infty}\left(\frac{1}{k^4}\omega_{(-)}\right) = -\frac{136}{1575}\mathrm{i},$$

implying the existence of an instability in the large-wave-number limit.

The third-order equations yield the hydrodynamic model at the level corresponding to the super-Burnett level of approximation. What are the implications of the instability of the third-order equations for the use of these equations to describe the heat transport in nanosystems? Let l be the mean free path ($l := c\tau_N \ll c\tau_R$) and assume that the characteristic length is defined by

$$L := \frac{1}{2}\left(\left|\frac{1}{\delta e}\partial_x\delta e\right|^{-1} + \left|\frac{1}{\delta v}\partial_x\delta v\right|^{-1}\right). \tag{7.6}$$

The heat transfer in nanosystems is significantly different from that in macrosystems. In particular, the ratio between the mean free path and the characteristic length, $\mathcal{K} := l/L$, becomes comparable to or higher than 1. Substituting Eqs. (5.4) into Eq. (7.6), we obtain $L = l/|k|$. Then the quantity $\mathcal{K}$ is effectively the dimensionless wave number: $\mathcal{K} = |k|$. The simplest third-order system consists of Eqs. (5.14) and (7.4), where $\gamma = 1$. For this system, assuming that the wave number is such that $\mathcal{W} \geq 0$, the dispersion relation (7.5) gives $\mathrm{Im}(\omega) \geq 0$. Hence, we conclude that the instability is absent when $\mathcal{K}$ is smaller than

$$\mathcal{K}_c := \sqrt{\frac{105}{68}\left(1 + \sqrt{\frac{7\tau_R + 34\tau_N}{7\tau_R}}\right)}.$$

If, say, $\tau_N = \tau_R/10$, then $\mathcal{K}_c \cong 1.8510$. The stable region with $\mathcal{K} \geq 1$ is defined by

$$1 \leq \mathcal{K} < \mathcal{K}_c.$$

Whether or not the third-order equations can be applied to the description of a phonon system in this region is an open problem that remains to be seen. The instability paradox has led some workers in the field to use a higher number of equations in the hierarchy, instead of truncating the expansion of the system (see, e.g., Ref. [37]). Other authors have proposed the regularization of the Chapman-Enskog expansion [38] or introduced the partial summation technique [11–14].

B. More on the cause of instability

We first present *an ad hoc* method to avoid instability. In Eqs. (7.2) and (7.3) we make the replacement

$$\sigma \to -\Xi, \quad \theta \to -3(1+u)\Xi, \tag{7.7}$$

assuming that Ξ is an arbitrary non-negative function of u. This replacement can be seen to give

$$\mathcal{U} = \frac{12(1+u)}{(3+u)(9-u^2)}\left(H + \frac{3\Xi k^2}{(3-u)^2}\right) \tag{7.8}$$

and

$$F = \frac{(1-u)(3+u)\mathcal{U}}{12(1+u)}. \tag{7.9}$$

We now check that the quantities $\mathcal{U}$ and Z defined, respectively, by (7.8) and (5.13), satisfy the inequalities $\mathcal{U} > 0$ and $Z \geq 0$, since the stability of solutions of system (5.5) is a direct consequence of these inequalities:

$$(\mathcal{U} > 0, \quad Z \geq 0) \Rightarrow \mathrm{Im}(\omega) \geq 0. \tag{7.10}$$

The inequality $\mathcal{U} > 0$ follows trivially from (4.9), $\Xi \geq 0$, and (7.8). In order to show that $Z \geq 0$, we substitute (5.11) into (5.13). Then Z can be evaluated with the aid of (7.3), (7.9), and (4.14); we obtain

$$Z = \frac{[4(1-u)(3+u)^2\mathcal{U}k]^4[(3-u)^2 + 3(1+u)Hk^2]}{81(3-u)^3(1+u)^4} \times [2(9+u)(3-u)^2 + 3(1+u)\beta k^2]^2 u \geq 0.$$

This result, in conjuction with $\mathcal{U} > 0$, implies that the imaginary part of ω is non-negative.

The replacement (7.7) can be generalized to a replacement of the form

$$\sigma \to -\Xi + H\Upsilon, \quad \theta \to -3(1+u)\Xi,$$

where Ξ and Υ are functions of u with the properties to be specified below. The explicit expression for Z is now

$$Z = 16u(3-u)(1-u)^2k^4(a_0 + a_1k^2 + a_2k^4 + a_3k^6) \times [(3+u)^4\mathcal{R}\mathcal{U} - 48(1-u)(3+u)\mathcal{U} - 32(3-u)^2\mathcal{F}]^2, \quad (7.11)$$

where

$$a_0 := 16H^2, \quad (7.12)$$

$$a_1 := \frac{96u(9+u)H^2\Upsilon}{(3+u)(3-u)^3} + \frac{48H[(1+u)H^2 + 2\Xi]}{(3-u)^2}, \quad (7.13)$$

$$a_2 := \frac{144u\mu_0H\Upsilon}{(3+u)^2(3-u)^5} + \frac{144\Xi[2(1+u)H^2 + \Xi]}{(3-u)^4}, \quad (7.14)$$

$$a_3 := \frac{432(1+u)\mu_1H\Xi}{(3+u)(3-u)^7}, \quad (7.15)$$

with

$$\mu_0 := 2(3+u)(9+u)\Xi + (1+u)(3+u)H\beta - (3-u)^2H\Upsilon,$$

$$\mu_1 := (9-u^2)\Xi + u\beta\Upsilon.$$

Then, choosing the functions Ξ and Υ such that

$$\mu_0 \geqslant 0, \quad \mu_1 \geqslant 0, \quad (7.16)$$

imposing the additional conditions

$$\Xi \geqslant 0, \quad \Upsilon \geqslant 0, \quad (7.17)$$

and also using Eqs. (7.8) and (7.11)–(7.15), we obtain the inequalities

$$\mathcal{U} > 0, \quad Z \geqslant 0,$$

from which, again with the help of (7.10), system (5.5) can be shown to have no unstable solutions.

Our conclusions are simply these. The Callaway model of the collision integrals and the first four terms of the Chapman-Enskog expansion of the distribution function form the basis of the third-order system of equations for the energy density and the drift velocity. The cause that this system is linearly unstable can be attributed to the fact that the functions Ξ and Υ defined by

$$\Xi := -\frac{\theta}{3(1+u)}, \quad \Upsilon := \frac{1}{H}\left[\sigma - \frac{\theta}{3(1+u)}\right]$$

do not satisfy conditions (7.16) and (7.17). Specifically, these functions are strictly negative for all values of u. Among other things, this implies that $\mathcal{U}<0$ and

$$(3+u)^2|k|\mathcal{U} - \sqrt{2\sqrt{\mathbb{A}^2 + \mathbb{B}^2} + 2\mathbb{B}} < 0$$

when

$$|k| > k_c := \sqrt{\frac{(3-u)^2H}{3|\Xi|}}.$$

As a result, in the case $|k| > k_c$, there exists at least one root of the quadratic equation (5.9) which violates the stability condition

$$\mathrm{Im}(\omega) = \frac{k}{8(3-u)}((3+u)^2k\mathcal{U} \mp \mathrm{sgn}(\mathbb{A})\sqrt{2\sqrt{\mathbb{A}^2 + \mathbb{B}^2} + 2\mathbb{B}}) \geqslant 0.$$

VIII. FINAL REMARKS

The traditional way of deriving the equations of phonon hydrodynamics is based on the use of a small parameter, which is inserted into Callaway's model as follows [39]:

$$\partial_t f + cg^i\partial_i f = \frac{1}{\varepsilon\tau_R}(F - f) + \frac{1}{\varepsilon\tau_N}(F_d - f).$$

The zeroth-order distribution function is obtained by considering the first term of the Chapman-Enskog expansion

$$f = \sum_{l=0}^{\infty} \varepsilon^l f_l = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \cdots \quad (8.1)$$

and by solving the equation

$$\frac{1}{\tau_R}(F - f_0) + \frac{1}{\tau_N}(F_d - f_0) = 0 \quad (8.2a)$$

and conditions (2.4) with $f=f_0$:

$$\int pF\, d^3\mathbf{p} = \int pF_d d^3\mathbf{p} = \int pf_0 d^3\mathbf{p},$$

$$\int \mathbf{p}F_d d^3\mathbf{p} = \int \mathbf{p}f_0 d^3\mathbf{p}. \quad (8.2b)$$

Since the solution of (8.2a) and (8.2b) leads to $f_0=F$, we see that such a Chapman-Enskog technique develops the distribution function as a perturbative expansion about the equilibrium Planck distribution. Beyond the first order, expansion (8.1) then augments the heat flux expression in the diffusive equation with terms that contain higher-order spatial derivatives of the energy density and products of these derivatives. It therefore appears that, while the traditional Chapman-Enskog method uses the equilibrium Planck distribution and aims to derive the higher-order equations governing the evolution of the energy density, the present modification of this method is based on a perturbative expansion of the distribution function about the displaced Planck distribution and aims to derive the equations for the energy density and the drift velocity (or the energy density and the heat flux).

Some interesting descriptions of the Chapman-Enskog method valid in the case when resistive bulk processes may be neglected have previously appeared, particularly those of Sussmann and Thellung [6] and Nielsen and Shklovskii [7] (see also the book by Gurevich [5]). The approach of Sussmann and Thellung cannot be compared directly with ours because these authors do not assume the validity of the so-called compatibility conditions [1],

$$\int p f_l d^3\mathbf{p} = 0, \qquad \int \mathbf{p} f_l d^3\mathbf{p} = \mathbf{0} \quad (l \geqslant 1),$$

which guarantee that each order of the expansion of the distribution function yields the same energy density and heat flux. Concerning the work of Nielsen and Shklovskii, the key difference boils down to expanding the displaced Planck distribution in powers of the drift velocity and truncating the expansion by retaining only the first three terms [see [7] Eq. (4)]. Consequently, the Nielsen-Shklovskii method of deriving the hydrodynamic equations for the energy density and the drift velocity requires that the phonon gas be close to local equilibrium.

Instead of using Eq. (2.9) or Eq. (A2), we can also use the equation

$$\partial_t f + c g^i \partial_i f = \frac{1}{\tau_R}(F - f) + \frac{1}{\varepsilon \tau_N}(F_d - f) \tag{8.3}$$

to begin a perturbative derivation of closed systems of equations for the energy density and the drift velocity. Another possible way to start the Chapman-Enskog expansion of Callaway's model is to insert the two small parameters into this model:

$$\partial_t f + c g^i \partial_i f = \frac{\varepsilon_1}{\tau_R}(F - f) + \frac{1}{\varepsilon_2 \tau_N}(F_d - f). \tag{8.4}$$

If we apply the Chapman-Enskog procedure to Eq. (8.3), then we will find that the first-order distribution function, namely, the function

$$f = f_0 + \varepsilon f_1,$$

yields the hydrodynamic equations violating the second law of thermodynamics. In order to overcome this problem, we have developed in Ref. [1] an alternative strategy for modifying the Chapman-Enskog technique based on model (A2). The use of the first-order distribution function then results in the equations consistent with the second law of thermodynamics. Moreover, the next order in the Chapman-Enskog expansion leads to the equations which are linearly stable at all wavelength.

In the case of model (8.4), the Chapman-Enskog procedure begins by expanding the distribution function in powers of both ε_1 and ε_2:

$$f = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \varepsilon_1^l \varepsilon_2^m f_{l,m}. \tag{8.5}$$

After identifying $f_{0,0}$ with the displaced Planck distribution, the next step is to look for expansion (8.5) such that the first- and higher-order terms are chosen so they have no contribution to the moments expressed in Eqs. (2.4):

$$\int p f_{l,m} d^3\mathbf{p} = 0, \qquad \int \mathbf{p} f_{l,m} d^3\mathbf{p} = \mathbf{0} \quad (l + m \geqslant 1).$$

The first-order truncation formally requires a knowledge of the functions $f_{1,0}$ and $f_{0,1}$,

$$f = F_d + \varepsilon_1 f_{1,0} + \varepsilon_2 f_{0,1},$$

and can be shown to give Eqs. (A1) and (A3) when the fact that $f_{1,0}$ equals zero is used. The second-order distribution function, which is defined by

$$f = F_d + \varepsilon_2 f_{0,1} + \varepsilon_1^2 f_{2,0} + \varepsilon_1 \varepsilon_2 f_{1,0} + \varepsilon_2^2 f_{0,2},$$

gives, in turn, the results of Sec. IV B when the equations for three-dimensional flow are specialized to the one-dimensional, rotationally symmetric geometry. The conclusion then is that, although model (8.4) more clearly emphasizes the difference between resistive and normal processes, it yields results which in practice do not differ from those obtained by inserting only one small parameter into Callaway's equation.

We finally mention that the Chapman-Enskog expansion method, which we employ in this paper, simultaneously requires both sufficiently rare resistive (momentum-destroying) phonon collisions and sufficiently frequent normal (momentum-conserving) ones. Recently, Koreeda *et al.* [40] considered the temperature dependence of τ_N and τ_R in $SrTiO_3$. They demonstrated that several temperature ranges may be defined according to the relation between τ_N and τ_R. Below approximately 50 K, τ_N gradually deviates from τ_R, seemingly suggesting the existence of a temperature region where the present Chapman-Enskog expansion method can eventually be used.

ACKNOWLEDGMENTS

This investigation was conducted within the framework of a scientific research project N N501 0074 33, financed by the Polish Ministry of Science and Higher Education, Contract No. 0074/B/T02/2007/33. The authors are thankful to the referees for their suggestions and efforts to improve the quality of this paper.

APPENDIX: FIRST-ORDER HYDRODYNAMIC EQUATIONS FOR THREE-DIMENSIONAL FLOW

Using the Callaway model in the form (2.1), it is possible to derive the following equations for the energy density e and the dimensionless drift velocity $\mathbf{v}$:

$$\partial_t e + \frac{4c}{3+u} v^j \partial_j e + \frac{4ce}{3+u} \partial_j v^j - \frac{4ce}{(3+u)^2} v^j \partial_j u = 0, \tag{A1a}$$

$$\partial_t v^i + \frac{c(1-u)}{4e} \delta^{ij} \partial_j e - \frac{c(1-u)}{2e(3-u)} v^i v^j \partial_j e - \frac{c(1-u)}{3-u} v^i \partial_j v^j + c v^j \partial_j v^i - \frac{c}{3+u} \delta^{ij} \partial_j u + \frac{2c}{9-u^2} v^i v^j \partial_j u + \frac{3+u}{4ce} \partial_j N^{ij} + \frac{3+u}{2ce(3-u)} v^i v_j \partial_k N^{jk} = -\frac{(3+u) v^i}{(3-u)\tau_R}, \tag{A1b}$$

where δ^{ij} is the Kronecker delta, $u := |\mathbf{v}|^2$, and

$$N^{ij} := c^3 \int p g^{\langle i} g^{j\rangle}(f - F_d) d^3\mathbf{p}.$$

In Eqs. (A1), the Einstein summation rule, according to which a repeated index implies summation over all values of that index, is used. Angular brackets denote the symmetric traceless part, e.g.,

$$g^{\langle i} g^{j\rangle} := g^i g^j - \frac{1}{3}\delta^{ij},$$

$$g^{\langle i} g^j g^{k\rangle} := g^i g^j g^k - \frac{1}{5}(g^i \delta^{jk} + g^j \delta^{ik} + g^k \delta^{ij}).$$

We also adopt the convention, whereby the indices are lowered and raised with the Kronecker δ's e.g., $v_i := \delta_{ij} v^j$. If the model kinetic equation, namely,

$$\partial_t f + c g^i \partial_i f = \frac{\varepsilon}{\tau_R}(F - f) + \frac{1}{\varepsilon \tau_N}(F_d - f), \tag{A2}$$

is solved in the first-order Chapman-Enskog approximation, then we find [1]

$$N^{ij} = -\frac{3c^3 e \tau_N}{3+u}(A\mu^{ij} + B v_k \mu^{k\langle i} v^{j\rangle} + C\mu_{kl} v^k v^l v^{\langle i} v^{j\rangle}), \tag{A3}$$

where A is defined by (4.5) and

$$\mu_{ij} := (1-u)\left(\partial_{\langle i} v_{j\rangle} - \frac{1}{4e} v_{\langle i} \partial_{j\rangle} e\right) + \frac{5+u}{2(3+u)} v_{\langle i} \partial_{j\rangle} u,$$

$$B := \frac{2}{u}\left(\frac{8}{3} - 5A\right), \quad C := \frac{8(4u-21) + 105(3-u)A}{6u^2(3-u)}.$$

In the case when $|\mathbf{v}|$ approaches zero and the values of e, $\partial_i e$, and $\partial_i v$ are arbitrarily fixed, Eq. (A3) simplifies to

$$N^{ij} = -\frac{8}{15} c^3 e \tau_N \partial_{\langle i} v_{j\rangle}.$$

Putting together Eqs. (A1) and (A3), we get a system of linearly stable entropy-consistent first-order hydrodynamic equations for a phonon gas where there are effectively no limitations on the magnitude of the individual components of the drift velocity. For more details, see Ref. [1].

We are now ready to present a study of the one-dimensional, rotationally symmetric reduction of Eqs. (A1) and (A3). We first observe that

$$\mu^{kl} v_k v_l = -\frac{1}{6e}(1-u) u v^k \partial_k e - \frac{1}{3}(1-u) u \partial_k v^k + \frac{9+4u-u^2}{3(3+u)} v^k v_l \partial_k v^l, \tag{A4a}$$

$$N^{kl} v_k v_l = -\frac{3c^3 e \tau_N}{3+u}\left(A + \frac{2}{3}uB + \frac{2}{3}u^2 C\right)\mu^{kl} v_k v_l = -\frac{6c^3 e \tau_N H}{9-u^2}\mu^{kl} v_k v_l, \tag{A4b}$$

where H is defined by Eq. (4.8). In the one-dimensional, rotationally symmetric geometry, all variables are functions of time and a single spatial coordinate x. Moreover, we have

$$v := v^1 \neq 0, \quad v^2 = v^3 = 0, \tag{A5a}$$

$$N := N^{11} = -2N^{22} = -2N^{33} \neq 0, \quad N^{ij} = 0 \quad (i \neq j). \tag{A5b}$$

Substitution from (A5) into (A4) gives

$$N = -\frac{4c^3 \tau_N H}{9-u^2}\left(\frac{3(1+u)e}{3+u}\partial_x v - \frac{1}{4}(1-u) v \partial_x e\right),$$

which is identical with (4.7); and the substitution of (A5) in (A1) gives equations for (e, v) which are identical with Eqs. (4.10).

[1] Z. Banach and W. Larecki, J. Phys. A: Math. Theor. **41**, 375502 (2008).
[2] J. Callaway, Phys. Rev. **113**, 1046 (1959).
[3] J. Callaway, *Quantum Theory of the Solid State* (Academic Press, Boston, 1991).
[4] R. E. Peierls, *Quantum Theory of Solids* (Oxford University Press, London, 1995).
[5] V. L. Gurevich, *Kinetika Fononnykh Sistem* (Nauka, Moscow, 1980).
[6] J. A. Sussmann and A. Thellung, Proc. Phys. Soc. London **81**, 1122 (1963).
[7] H. Nielsen and B. I. Shklovskii, Fiz. Tverd. Tela (Leningrad) **10**, 3602 (1968).
[8] B. Alascio, H. Nazareno, and G. Carabelli, Nuovo Cimento B **55**, 362 (1968).
[9] H. Beck, in *Dynamical Properties of Solids*, edited by G. K. Horton and A. A. Maradudin (North-Holland, Amsterdam, 1975), pp. 205–284.
[10] B. Zhang, J. Math. Anal. Appl. **325**, 1442 (2007).
[11] I. V. Karlin, G. Dukek, and T. F. Nonnenmacher, Phys. Rev. B **55**, 6324 (1997).
[12] I. V. Karlin and A. N. Gorban, Ann. Phys. **11**, 783 (2002).
[13] P. A. Zakharchenko and E. V. Radkevich, Dokl. Phys. **49**, 451 (2004).
[14] V. V. Palin and E. V. Radkevich, J. Math. Sci. (N.Y.) **135**, 2721 (2006).
[15] N. Fenichel, J. Differ. Equations **31**, 53 (1979).
[16] C. K. R. T. Jones, in *Dynamical Systems*, edited by L. Arnold, Lecture Notes in Mathematics Vol. 1609 (Springer-Verlag, Berlin, 1995), pp. 44–118.
[17] A. N. Gorban and I. V. Karlin, Transp. Theory Stat. Phys. **23**, 559 (1994).

[18] A. N. Gorban, I. V. Karlin, and A. Yu. Zinovyev, Phys. Rep. **396**, 197 (2004).
[19] Z. Banach and S. Piekarski, J. Math. Phys. **30**, 1826 (1989).
[20] J. Casas-Vázquez and D. Jou, Rep. Prog. Phys. **66**, 1937 (2003).
[21] M. Criado-Sancho, D. Jou, and J. Casas-Vázquez, Phys. Lett. A **350**, 339 (2006).
[22] H. Nielsen and B. I. Shklovskii, Zh. Eksp. Teor. Fiz. **56**, 709 (1969).
[23] W. Dreyer and S. Struchtrup, Continuum Mech. Thermodyn. **5**, 3 (1993).
[24] Z. Banach and S. Piekarski, J. Math. Phys. **30**, 1816 (1989).
[25] M. R. Roussel and S. J. Fraser, J. Chem. Phys. **93**, 1072 (1990).
[26] C. Foias, M. S. Jolly, I. G. Kevrekidis, G. R. Sell, and E. S. Titi, Phys. Lett. A **131**, 433 (1988).
[27] W. Larecki, Nuovo Cimento Soc. Ital. Fis., D **14**, 141 (1992).
[28] R. K. Agarwal, K-Y Yun, and R. Balakrishnan, Phys. Fluids **13**, 3061 (2001).
[29] A. V. Bobylev, Sov. Phys. Dokl. **27**, 29 (1982).
[30] F. J. Uribe, R. M. Velasco, and L. S. Garciá-Colińń, Phys. Rev. E **62**, 5835 (2000).
[31] K. A. Comeaux, D. R. Chapman, and R. W. MacCormack, Analysis of the Burnett Equations Based on the Second Law of Thermodynamics, AIAA Paper No. 95–0415, 1995, http://www.aiaa.org/tc/tp/awards/paper.html
[32] A. V. Bobylev, J. Stat. Phys. **124**, 371 (2006).
[33] X. Zhong, Ph.D. thesis, Stanford University, Stanford, CA, 1991.
[34] S. Jin and M. Slemrod, J. Stat. Phys. **103**, 1009 (2001).
[35] P. L. Bhatnagar, E. P. Gross, and M. Krook, Phys. Rev. **94**, 511 (1954).
[36] R. Balakrishnan, J. Fluid Mech. **503**, 201 (2004).
[37] D. Jou, J. Casas-Vázqueza, G. Lebon, and M. Grmela, Appl. Math. Lett. **18**, 963 (2005).
[38] P. Rosenau, Phys. Rev. A **40**, 7193 (1989).
[39] W. S. Jiaung and J. R. Ho, Phys. Rev. E **77**, 066710 (2008).
[40] A. Koreeda, R. Takanao, and S. Saikan, Phys. Rev. Lett. **99**, 265502 (2007).