Persistent random walk on a one-dimensional lattice with random asymmetric transmittances

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We study the persistent random walk of photons on a one-dimensional lattice of random asymmetric transmittances. Each site is characterized by its intensity transmittance t ($t' \neq t$) for photons moving to the right (left) direction. Transmittances at different sites are assumed independent, distributed according to a given probability density $\mathcal{F}(t,t')$. We use the effective medium approximation and identify two classes of $\mathcal{F}(t,t')$ which lead to the normal diffusion of photons. Monte Carlo simulations confirm our predictions. We mention that the metamaterial introduced by Fedetov *et al.* [Nano Lett. **7**, 1996 (2007)] can be used to realize a lattice of random asymmetric transmittances.

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I. INTRODUCTION

Random walks in random environments is a field of continuous research [1–4]. Hopping conduction of classical particles or excitations [5], transport in porous and fractured rocks [6], and diffusive transport of light in disordered media [7] are a few examples.

Random walks with correlated displacements figure in a multitude of different problems. Among the correlated walks, the persistent random walk introduced by Fürth [8] and Taylor [9], is possibly the simplest one to incorporate a form of momentum in addition to random motion. In its basic realization on a one-dimensional lattice, a persistent random walker possesses constant probabilities for either taking a step in the same direction as the immediately preceding one or for reversing its motion [1,2,4]. Generalized persistent random walk models are utilized in the description of polymers [10], chemotaxis [11], general transport mechanisms [12,13], Landauer diffusion coefficient for a one-dimensional solid [14], etc.

Recently, the persistent random walk model was used to study the role of liquid films for diffusive transport of light in foams [15,16]. Diffusing-wave spectroscopy experiments have confirmed the photon diffusion in foams [17]. A relatively dry foam consists of cells separated by thin liquid films [18]. Cells in a foam are much larger than the wavelength of light, thus one can employ ray optics and follow a light beam or photon as it is transmitted through the liquid films with a probability t called the intensity transmittance. This naturally leads to a persistent random walk of the photons. In the ordered honeycomb (Kelvin) foams, the one-dimensional persistent walk arises when the photons move perpendicular to a cell edge (face). Thin-film transmittance depends on the film thickness. Films are not expected to have the same thickness. These observations motivated us to consider persistent random walk on a one-dimensional lattice of random transmittances [16]. We assumed that transmittances at different sites are independent random variables, distributed according to a given probability density $f(t)$. Assuming that $\langle 1/t \rangle = \int_0^1 f(t)/t dt$ is finite, we validated the classical persistent random walk with an effective transmittance t_{eff}, where $1/t_{\text{eff}} = \langle 1/t \rangle$. We also investigated the transport on a line with infinite $\langle 1/t \rangle$. We showed that if $f(t) \to f(0)$ as $t \to 0$, the mean square displacement after n steps is proportional to $n/\ln(n)$. If $f(t) \sim f_\alpha t^{-\alpha}$ $(0 < \alpha < 1)$ as $t \to 0$, we found that the mean square displacement is proportional to $n^{(2-2\alpha)/(2-\alpha)}$. Quite interesting, we found that anomalous diffusion of persistent walkers and hopping particles on a site-disordered lattice [5,19] are similar. To observe photon subdiffusion experimentally, we suggested a dielectric film stack for realization of a distribution $f(t)$ [16].

In the realm of diffusion on one-dimensional lattices with random hopping rates $w_{j,j'}$ from site $\boldsymbol{j'}$ to site $\boldsymbol{j}$, the asymmetric hopping model with $w_{j,j+1} \neq w_{j+1,j}$ has gained much attention [20–26]: At variance with the symmetric case, the asymmetric model can display anomalous diffusion behavior without broad distribution of hopping rates. The asymmetric hopping model has been used to discuss hopping conductivity in presence of an external electric field, molecular motors [27], evolution of a domain wall in a one-dimensional random field Ising model [3], helix-coil transition of heteropolymers [3,28], etc. These points suggest us to investigate persistent random walk of photons on a one-dimensional lattice with random asymmetric transmittances. For a given dielectric stack, the transmittance for incidence on the right side, is equal to that for incidence on the left side [29]. However, optical elements with different transmission in the forward and backward directions, and even optical diodes which allow unidirectional propagation, are realized [30]. For example, Fedetov *et al.* [31] showed that asymmetric transmission through a planar metal nanostructure consisting of twisted elements can be observed in the optical part of the spectrum. For a normally incident circularly polarized light of wavelength 630 nm, this metamaterial is 1.3 times more transparent from one side than from the other. There is a good reason to believe that the experimental observation of photons' persistent random walk is not out of reach: Barthelemy, Bertolotti, and Wiersma have recently verified Lévy flight of photons in their synthesized Lévy glass [32].

Apart from interest in the optics of random media, our work has been motivated by the Lorentz gas model introduced to describe the diffusion of conduction electrons in metals [33,34]. One-dimensional persistent random walk and

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stochastic Lorentz gas are intimately related [35]. Stochastic Lorentz model consists of fixed scatterers on a lattice and one moving light particle. The light particle runs at velocity c or $-c$, and when collides with a scatterer it is transmitted with a site-dependent probability t or reflected with a probability $1-t$. In other words, the light particle performs a persistent random walk. Thus here we are investigating a variant of the one-dimensional Lorentz gas, where each scatterer is characterized by a random asymmetric transmission coefficient.

In this paper we consider the persistent random walk of photons on a one-dimensional lattice of random asymmetric transmittances. Each site is characterized by its intensity transmittance t (t') for photons moving to the right (left) direction. Transmittances at different sites are assumed independent, distributed according to a given probability density $\mathcal{F}(t,t')$. We generalize a variant of the effective medium approximation introduced by Sahimi, Hughes, Scriven, and Davis [19,36] to identify two classes of $\mathcal{F}(t,t')$ which lead to the normal diffusion of photons: (i) $\langle 1/t\rangle$ is finite and $\langle t'/t\rangle$ is less than 1 and (ii) $\langle 1/t'\rangle$ is finite and $\langle t/t'\rangle$ is less than 1. Here $\langle\cdots\rangle$ denotes averaging with respect to the distribution $\mathcal{F}(t,t')$.

Our paper is organized as follows. In Sec. II we introduce the model. In Sec. III we present the effective medium approach to the problem. The numerical treatment and its results are reported in Sec. IV. Section V is devoted to a discussion of our results.

II. MODEL

We consider a one-dimensional lattice random walk in which steps are permitted to the nearest-neighbor sites only. We normalize the length and duration of a step to 1. Apparently, on a one-dimensional lattice the walker can move either to the right (+) or to the left (−) direction. Each site $\boldsymbol{j}$ is characterized by forward and backward transmittances $t_{j,j+1}$ and $t'_{j,j-1}$, respectively: On arriving at site $\boldsymbol{j}$, a walker moving in the right (left) direction takes a step in the same direction with the probability $t_{j,j+1}$ ($t'_{j,j-1}$). Here we assume asymmetric transmittances, i.e., $t_{j,j+1}\neq t'_{j,j-1}$.

We assume that (i) transmittances t and t' at each site are random variables. In general, these random variables are not independent, (ii) transmittances at two different sites are independent, (iii) transmittances at all sites are distributed according to a given normalized probability density $\mathcal{F}(t,t')$. Apparently $\int_0^1\int_0^1\mathcal{F}(t,t')dtdt'=1$. The probability density functions of t and t' are $f^+(t)=\int_0^1\mathcal{F}(t,t')dt'$ and $f^-(t')=\int_0^1\mathcal{F}(t,t')dt$, respectively. The joint probability distribution can be written as $\mathcal{F}(t,t')=f^+(t)f^-(t')$ when random variables t and t' are independent. For any function $h(t,t')$, we define $\langle h(t,t')\rangle=\int_0^1\int_0^1 h(t,t')\mathcal{F}(t,t')dtdt'$.

We denote by $P^+(n,\boldsymbol{j})$ [$P^-(n,\boldsymbol{j})$] the probability that the walker after its nth step arrives at site $\boldsymbol{j}$ with positive (negative) momentum. A set of two master equations can be established to couple the probabilities at step $n+1$ to the probabilities at step n:

$$P^+(n+1,\boldsymbol{j}) = t_{j-1,j}P^+(n,\boldsymbol{j}-1) + r'_{j-1,j}P^-(n,\boldsymbol{j}-1),$$

$$P^-(n+1,\boldsymbol{j}) = r_{j+1,j}P^+(n,\boldsymbol{j}+1) + t'_{j+1,j}P^-(n,\boldsymbol{j}+1), \qquad (1)$$

where $r_{j,j-1}=1-t_{j,j+1}$ and $r'_{j,j+1}=1-t'_{j,j-1}$ denote forward and backward reflectances at site $\boldsymbol{j}$, respectively.

We are mainly interested in the probability that the photon arrives at position $\boldsymbol{j}$ at step n, i.e., $P(n,\boldsymbol{j})=P^+(n,\boldsymbol{j})+P^-(n,\boldsymbol{j})$, from which we extract the first and second moments after n steps as the characteristic features of a random walk

$$\langle\langle \boldsymbol{j}\rangle\rangle_n = \left\langle \sum_j \boldsymbol{j}P(n,\boldsymbol{j})\right\rangle,$$

$$\langle\langle \boldsymbol{j}^2\rangle\rangle_n = \left\langle \sum_j \boldsymbol{j}^2P(n,\boldsymbol{j})\right\rangle. \qquad (2)$$

Here the first bracket represents an ensemble average over all random transmittances, and the second bracket signifies an average with respect to the distribution $P(n,\boldsymbol{j})$.

Assuming a constant forward transmittance t and a backward transmittance t' at each site, translational invariance of the medium can be invoked to deduce the exact solution of $P(n,\boldsymbol{j})$ in the framework of characteristic functions [2]. Furthermore, the mean square-displacement of photons after $n\to\infty$ steps can be obtained as $\langle \boldsymbol{j}^2\rangle_n-\langle \boldsymbol{j}\rangle_n^2=2Dn$, where the the diffusion constant D is

$$D = \frac{2(1-t)(1-t')(t+t')}{(2-t-t')^3}. \qquad (3)$$

In the limit $t=t'$ one obtains $D=t/(2-2t)$, a known result in the realm of the the classical persistent random walk.

The disorder not only may affect the value of diffusion constant as compared to the ordered system, but also may lead to the subdiffusive or superdiffusive behavior. In our model, even a few sites with small transmittances may drastically hinder the photon transport: In the extreme limit where at two different sites $\boldsymbol{j}$ and $\boldsymbol{j}'$, transmittances $t_{j,j+1}=t'_{j,j-1}=t_{j',j'+1}=t'_{j',j'-1}=0$, photons either do not visit the segment between $\boldsymbol{j}$ and $\boldsymbol{j}'$, or are caged in this segment. In the following section, we determine which distributions of transmittances $\mathcal{F}(t,t')$ lead to the normal diffusion of photons.

III. EFFECTIVE MEDIUM APPROXIMATION

Many of the approaches to the transport in disordered media have the disadvantage of being restricted to one-dimensional problems. Here we adopt the effective medium approximation (EMA) which is applicable to two- and three-dimensional media. We generalize a variant of effective medium approximation introduced by Sahimi, Hughes, Scriven, and Davis [19,36].

First we simplify the set of coupled linear difference Eq. (1) using the method of the z transform [2,37] explained in the Appendix:

$$\frac{P^+(z,\boldsymbol{j})}{z} - \frac{P^+(n=0,\boldsymbol{j})}{z} = t_{j-1,j}P^+(z,\boldsymbol{j}-\mathbf{1}) + r'_{j-1,j}P^-(z,\boldsymbol{j}-\mathbf{1}),$$

$$\frac{P^-(z,\boldsymbol{j})}{z}-\frac{P^-(n=0,\boldsymbol{j})}{z}=r_{\boldsymbol{j+1},\boldsymbol{j}}P^+(z,\boldsymbol{j+1})+t'_{\boldsymbol{j+1},\boldsymbol{j}}P^-(z,\boldsymbol{j+1}). \tag{4}$$

We assume the initial conditions $P^+(n=0,\boldsymbol{j})=P^-(n=0,\boldsymbol{j})=\delta_{\boldsymbol{j},0}/2$. To facilitate solution of Eq. (4) we introduce probabilities $P_e^{\pm}(z,\boldsymbol{j})$, and a reference lattice or average medium with all forward transmittances equal to $t_e(z)$ and all backward transmittances equal to $t'_e(z)$, so that

$$\frac{P_e^+(z,\boldsymbol{j})}{z}-\frac{P^+(n=0,\boldsymbol{j})}{z}=t_e(z)P_e^+(z,\boldsymbol{j-1})+r'_e(z)P_e^-(z,\boldsymbol{j-1}),$$

$$\frac{P_e^-(z,\boldsymbol{j})}{z}-\frac{P^-(n=0,\boldsymbol{j})}{z}=r_e(z)P_e^+(z,\boldsymbol{j+1})+t'_e(z)P_e^-(z,\boldsymbol{j+1}). \tag{5}$$

Here $r_e(z)=1-t_e(z)$ and $r'_e(z)=1-t'_e(z)$ denote effective reflectances.

EMA determines $t_e(z)$ and $t'_e(z)$ in a self-consistent manner, in which the role of distribution $\mathcal{F}(t,t')$ is manifest. This is done by taking a cluster of random transmittances from the original distribution, and embedding it into the effective medium. We then require that average of site occupation probabilities of the decorated medium duplicate $P_e^{\pm}(z,\boldsymbol{j})$ of the effective medium. We will sketch the method in the following.

Subtracting Eqs. (4) and (5), we obtain

$$\frac{1}{z}\begin{pmatrix}Q^+(z,\boldsymbol{j})\\Q^-(z,\boldsymbol{j})\end{pmatrix}-\mathbf{T}^-(z)\begin{pmatrix}Q^+(z,\boldsymbol{j-1})\\Q^-(z,\boldsymbol{j-1})\end{pmatrix}-\mathbf{T}^+(z)\begin{pmatrix}Q^+(z,\boldsymbol{j+1})\\Q^-(z,\boldsymbol{j+1})\end{pmatrix}$$
$$=\left[\begin{pmatrix}t_{\boldsymbol{j-1},\boldsymbol{j}} & r'_{\boldsymbol{j-1},\boldsymbol{j}}\\0&0\end{pmatrix}-\mathbf{T}^-(z)\right]\begin{pmatrix}P^+(z,\boldsymbol{j-1})\\P^-(z,\boldsymbol{j-1})\end{pmatrix}$$
$$+\left[\begin{pmatrix}0&0\\r_{\boldsymbol{j+1},\boldsymbol{j}} & t'_{\boldsymbol{j+1},\boldsymbol{j}}\end{pmatrix}-\mathbf{T}^+(z)\right]\begin{pmatrix}P^+(z,\boldsymbol{j+1})\\P^-(z,\boldsymbol{j+1})\end{pmatrix}, \tag{6}$$

where

$$\begin{pmatrix}Q^+(z,\boldsymbol{j})\\Q^-(z,\boldsymbol{j})\end{pmatrix}=\begin{pmatrix}P^+(z,\boldsymbol{j})\\P^-(z,\boldsymbol{j})\end{pmatrix}-\begin{pmatrix}P_e^+(z,\boldsymbol{j})\\P_e^-(z,\boldsymbol{j})\end{pmatrix},$$

$$\mathbf{T}^-(z)=\begin{pmatrix}t_e(z)&r'_e(z)\\0&0\end{pmatrix},$$

$$\mathbf{T}^+(z)=\begin{pmatrix}0&0\\r_e(z)&t'_e(z)\end{pmatrix}. \tag{7}$$

To solve Eq. (6), we introduce the Green function

$$\mathbf{G}(z,\boldsymbol{j})=\begin{pmatrix}G_{11}&G_{12}\\G_{21}&G_{22}\end{pmatrix}$$

which satisfies the equation

$$\frac{1}{z}\mathbf{G}(z,\boldsymbol{j})-\mathbf{T}^-(z)\mathbf{G}(z,\boldsymbol{j-1})-\mathbf{T}^+(z)\mathbf{G}(z,\boldsymbol{j+1})=\delta_{\boldsymbol{j},0}\mathbf{I}. \tag{8}$$

Here $\mathbf{I}$ is the identity matrix. Multiplying both sides of the above equation by $e^{\iota j\theta}$ and then summing over all the sites, the Fourier transform of the Green function, i.e., $\mathbf{G}(z,\boldsymbol{\theta})=\sum_{\boldsymbol{j}=-\infty}^{\infty}e^{\iota \boldsymbol{j}\boldsymbol{\theta}}\mathbf{G}(z,\boldsymbol{j})$, can be obtained as

$$\mathbf{G}(z,\boldsymbol{\theta})=\frac{z^2}{\Delta(z,\boldsymbol{\theta})}\begin{pmatrix}\frac{1}{z}-t'_e(z)e^{-\iota\boldsymbol{\theta}} & r'_e(z)e^{\iota\boldsymbol{\theta}}\\ r_e(z)e^{-\iota\boldsymbol{\theta}} & \frac{1}{z}-t_e(z)e^{\iota\boldsymbol{\theta}}\end{pmatrix}, \tag{9}$$

where $\Delta(z,\theta)=1-z[t_e(z)e^{\iota\theta}+t'_e(z)e^{-\iota\theta}]+z^2[t_e(z)+t'_e(z)-1]$.

For the present, we consider only the simplest approximation, and embed one random transmittance at site $\boldsymbol{l}$ of the effective medium. Then solution of Eq. (6) is

$$\begin{pmatrix}Q^+(z,\boldsymbol{j})\\Q^-(z,\boldsymbol{j})\end{pmatrix}=\int_0^{2\pi}\mathbf{G}(z,\boldsymbol{\theta})\mathbf{S}(z,\boldsymbol{\theta})e^{-\iota\boldsymbol{\theta}(\boldsymbol{j}-\boldsymbol{l})}\times\begin{pmatrix}P^+(z,\boldsymbol{l})\\P^-(z,\boldsymbol{l})\end{pmatrix}\frac{d\boldsymbol{\theta}}{2\pi}, \tag{10}$$

where

$$\mathbf{S}(z,\boldsymbol{\theta})=\begin{pmatrix}[t_{\boldsymbol{l},\boldsymbol{l+1}}-t_e(z)]e^{\iota\boldsymbol{\theta}} & [t'_e(z)-t'_{\boldsymbol{l},\boldsymbol{l-1}}]e^{\iota\boldsymbol{\theta}}\\ [t_e(z)-t_{\boldsymbol{l},\boldsymbol{l+1}}]e^{-\iota\boldsymbol{\theta}} & [t'_{\boldsymbol{l},\boldsymbol{l-1}}-t'_e(z)]e^{-\iota\boldsymbol{\theta}}\end{pmatrix}. \tag{11}$$

The self-consistency equation is $\langle P^{\pm}(z,\boldsymbol{l})\rangle=P_e^{\pm}(z,\boldsymbol{l})$ or

$$\left\langle\left[\mathbf{I}-\int_0^{2\pi}\mathbf{G}(z,\boldsymbol{\theta})\mathbf{S}(z,\boldsymbol{\theta})\frac{d\boldsymbol{\theta}}{2\pi}\right]^{-1}\right\rangle=\mathbf{I}. \tag{12}$$

The above matrix equation leads to these conditions:

$$\int_0^1\int_0^1\frac{\mathcal{F}(t,t')dtdt'}{1-[t-t_e(z)]U(z)-[t'-t'_e(z)]U'(z)}=1,$$

$$\int_0^1\int_0^1\frac{t\mathcal{F}(t,t')dtdt'}{1-[t-t_e(z)]U(z)-[t'-t'_e(z)]U'(z)}=t_e(z),$$

$$\int_0^1\int_0^1\frac{t'\mathcal{F}(t,t')dtdt'}{1-[t-t_e(z)]U(z)-[t'-t'_e(z)]U'(z)}=t'_e(z), \tag{13}$$

where

$$U(z)=\frac{V(z)\{1+z^2[-t_e(z)+t'_e(z)-1]\}-1}{2t_e(z)},$$

$$U'(z)=\frac{V(z)\{1+z^2[t_e(z)-t'_e(z)-1]\}-1}{2t'_e(z)},$$

$$V(z)=\frac{1}{\sqrt{\{1+z^2[t_e(z)+t'_e(z)-1]\}^2-4z^2t_e(z)t'_e(z)}}. \tag{14}$$

It turns out that one of the self-consistency conditions (13) can be trivially satisfied. Consistency equations determine $t_e(z)$ and $t'_e(z)$, in which the role of distribution $\mathcal{F}(t,t')$ is manifest. For symmetric transmittances where $\mathcal{F}(t,t')=f(t)\delta(t-t')$, our consistency conditions (13) indeed duplicate that of Ref. [16]

The translational invariance of the effective medium can be invoked to access the z transform of the the first and second moments of the photon distribution

$$\sum_{n=0}^{\infty} \langle \boldsymbol{j} \rangle_n z^n = \frac{z}{(1-z)^2} \frac{t_e(z) - t'_e(z)}{1 - z[t_e(z) + t'_e(z) - 1]},$$

$$\sum_{n=0}^{\infty} \langle \boldsymbol{j}^2 \rangle_n z^n = \frac{2z^2}{(1-z)^3} \frac{(t_e(z) - t'_e(z))^2}{\{1 - z[t_e(z) + t'_e(z) - 1]\}^2} + \frac{z}{(1-z)^2} \frac{1 + z[t_e(z) + t'_e(z) - 1]}{1 - z[t_e(z) + t'_e(z) - 1]}. \qquad (15)$$

We are interested in the long time behavior, thus Tauberian theorems suggest to analyze Eqs. (13) and (15) in the limit $z \to 1$.

To find which distributions of transmittances $\mathcal{F}(t,t')$ lead to the normal diffusion of photons, we assume that $t_e(z)$ and $t'_e(z)$ have no singularity in the limit $z \to 1$. We find two distinct classes. (i) If $\langle t'/t \rangle < 1$ then

$$t_e(z) = \frac{1}{\langle 1/t \rangle},$$

$$t'_e(z) = \frac{\langle t'/t \rangle}{\langle 1/t \rangle}. \qquad (16)$$

The first class of admissible distribution $\mathcal{F}(t,t')$ is such that $\langle 1/t \rangle$ is finite and $\langle t'/t \rangle$ is less than 1. (ii) If $\langle t/t' \rangle < 1$ then

$$t_e(z) = \frac{\langle t/t' \rangle}{\langle 1/t' \rangle},$$

$$t'_e(z) = \frac{1}{\langle 1/t' \rangle}. \qquad (17)$$

The second class of admissible distribution $\mathcal{F}(t,t')$ is such that $\langle 1/t' \rangle$ is finite and $\langle t/t' \rangle$ is less than 1. The above effective transmittances do not depend on z, thus one can directly use Eq. (3) to access the diffusion constant of photons.

The Cauchy-Schwarz inequality states that for two random variables α and β, $\langle \alpha\beta \rangle^2 \leq \langle \alpha^2 \rangle \langle \beta^2 \rangle$. With $\alpha = \sqrt{t/t'}$ and $\beta = \sqrt{t'/t}$, we find that $1 \leq \langle t/t' \rangle \langle t'/t \rangle$. This clearly shows that $\langle t/t' \rangle$ and $\langle t'/t \rangle$ are not simultaneously less than 1, thus two mentioned classes are quite distinct. EMA does not predict any result when both $\langle t/t' \rangle$ and $\langle t'/t \rangle$ are greater than 1.

EMA is a simple and conceptually attractive scheme for exploring transport in a disordered medium. However, no estimate of the accuracy of this scheme is available. Thus, only an *exact* solution will unquestionably clarify whether the above mentioned constraints on $\mathcal{F}(t,t')$ lead to the diffusive transport.

IV. NUMERICAL SIMULATIONS

The predictions of EMA can be inspected by numerical simulations. The computer program produces 50 media, whose transmittances are distributed according to a given distribution $\mathcal{F}(t,t')$. We deliberately focus on cases where both $\langle t/t' \rangle$ and $\langle t'/t \rangle$ are not simultaneously greater than 1. For each medium, the program takes 10^4 photons at the initial position $\boldsymbol{j}=0$ and generates the trajectory of each photon

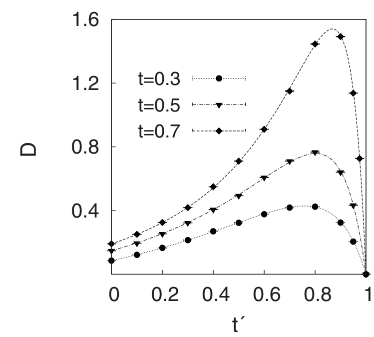

FIG. 1. The diffusion constant D as a function of transmittances t and t' of an ordered medium. Theoretical and Monte Carlo simulation results are denoted, respectively, by lines and points.

following a standard Monte Carlo procedure. The statistics of the photon cloud is evaluated at times $n \in [10\,000, 12\,000, \ldots, 68\,000]$. $\langle\langle \boldsymbol{j}^2 \rangle\rangle_n - \langle\langle \boldsymbol{j} \rangle\rangle_n^2$ is computed for each snapshot at time n, and then fitted to $2Dn + O$ by the method of linear regression. An offset O takes into account the initial ballistic regime. We compare our numerical diffusion constant with the analytical one based on Eqs. (3), (16), and (17). Note that to increase the accuracy of numerical simulation, it is better to use the exact enumeration technique [38].

First we assume that transmittances of all sites are equal. For $t \in [0.3, 0.5, 0.7]$ and $t' \in [0.0, 0.1, 0.2, \ldots, 0.9, 1.0]$, our numerical and analytical predictions for the diffusion constant are compared in Fig. 1. In the absence of disorder, translational invariance of the medium has been invoked to deduce the exact diffusion constant (3). Thus, Fig. 1 is a partial test for the quality of our numerical simulations.

Next we consider $\mathcal{F}(t,t') = f^+(t) f^-(t')$ such that $f^+(t)$ is a uniform distribution for $t_1 < t < t_2$, and $f^-(t') = \delta(t' - t'_3)$. We choose $t_1 = 0.2$, $0.2 < t_2 < 0.6$, and $t'_3 \in [0.1, 0.7, 0.9]$. Our numerical and analytical predictions are compared in Fig. 2(a). We also considered the case where $f^+(t)$ is a uniform distribution for $t_1 < t < t_2$, and $f^-(t')$ is a uniform distribution for $t'_3 < t' < t'_4$. We choose $t_1 = 0.1$, $0.2 < t_2 < 0.6$, $t'_3 = 0.6$ and $t'_4 \in [0.7, 0.9]$. Our results are shown in Fig. 2(b).

We also present two other examples. We consider $\mathcal{F}(t,t') = f^+(t) f^-(t')$ such that $f^+(t) = (1-\alpha) t^{-\alpha}$ for $0 < t < 1$, and $f^-(t')$ is a uniform distribution for $0.7 < t' < 0.9$. Figure 3(a) depicts D as a function of α. Our results for the case $f^+(t) = (1-\alpha) t^{-\alpha}$ and $f^-(t') = \delta(t' - t'_1)$ are illustrated in Fig. 3(b). Figures 1–3 vividly show that the effective medium approach to the diffusion constant D is quite successful.

V. DISCUSSION

In the present paper, we address the persistent random walk of photons on a one-dimensional lattice of random asymmetric transmittances. Clearly, a photon steps back by each reflection. Intuitively, one expects the abundance of large reflectances (e.g., two different sites with $t_{j,j+1} = t'_{j,j-1} = t_{j',j'+1} = t'_{j',j'-1} = 0$) to drastically decrease excursion of the photons. As percolation properties [1,6], this feature is induced by the dimensionality of the lattice. We focus on de-

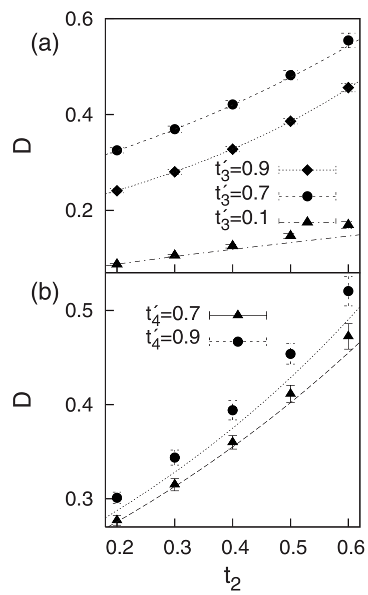

FIG. 2. (a) The diffusion constant D as a function of t_2 for different values of t_3'. $\mathcal{F}(t,t')=f^+(t)f^-(t')$, such that $f^+(t)$ is a uniform distribution for $0.2<t<t_2$ and $f^-(t')=\delta(t'-t_3')$. (b) D as a function of t_2 for different values of t_4'. $f^+(t)$ is a uniform distribution for $0.1<t<t_2$, and $f^-(t')$ is a uniform distribution for $0.6<t'<t_4'$. Theoretical and simulation results are denoted, respectively, by lines and points.

termining distributions of transmittances $\mathcal{F}(t,t')$ which lead to the normal diffusion of photons. The probability distribution $P^{\pm}(n,\boldsymbol{j})$ as an exact solution of the master equation (1) is quite hard to obtain. However the relatively simple but approximate effective medium approach reveals intriguing aspects of the system. In two cases, the transport of photons is diffusive: (i) $\langle 1/t\rangle$ is finite and $\langle t'/t\rangle$ is less than 1. (ii) $\langle 1/t'\rangle$ is finite and $\langle t/t'\rangle$ is less than 1. Monte Carlo simulations confirm our predictions. EMA does not predict any result when both $\langle t/t'\rangle$ and $\langle t'/t\rangle$ are greater than 1.

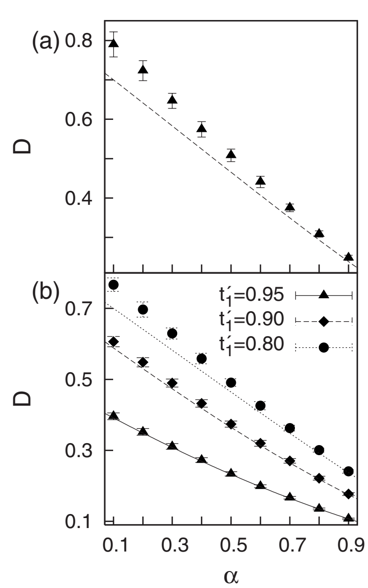

FIG. 3. The diffusion constant D as a function of α. $\mathcal{F}(t,t')=f^+(t)f^-(t')$ such that (a) $f^+(t)=(1-\alpha)t^{-\alpha}$ for $0<t<1$, and $f^-(t')$ is a uniform distribution for $0.7<t'<0.9$. (b) $f^+(t)=(1-\alpha)t^{-\alpha}$ and $f^-(t')=\delta(t'-t_1')$. Theoretical and simulation results are denoted, respectively, by lines and points.

It would be instructive to compare our problem with the transport on a one-dimensional lattice with random asymmetric hopping rates. Independent steps are the base of the hopping transport, while correlated steps are the essence of the persistent random walk. Hopping conduction is described by the master equation

$$\frac{\partial P(\tau,\boldsymbol{j})}{\partial\tau}=w_{\boldsymbol{j},\boldsymbol{j}+1}P(\tau,\boldsymbol{j}+1)+w_{\boldsymbol{j},\boldsymbol{j}-1}P(\tau,\boldsymbol{j}-1)-(w_{\boldsymbol{j}+1,\boldsymbol{j}}+w_{\boldsymbol{j}-1,\boldsymbol{j}})P(\tau,\boldsymbol{j}), \qquad (18)$$

where $P(\tau,\boldsymbol{j})$ is the probability for the particle to be on site $\boldsymbol{j}$ at continuous time τ, and $w_{\boldsymbol{j},\boldsymbol{j}'}$ denotes the probability of jumping from site $\boldsymbol{j}'$ to site $\boldsymbol{j}$ per unit time. In the asymmetric hopping model $w_{\boldsymbol{j},\boldsymbol{j}+1}\neq w_{\boldsymbol{j}+1,\boldsymbol{j}}$. First we note that EMA does not predict any result when both $\langle w_{\boldsymbol{j},\boldsymbol{j}+1}/w_{\boldsymbol{j}+1,\boldsymbol{j}}\rangle=\langle w_{\leftarrow}/w_{\rightarrow}\rangle$ and $\langle w_{\boldsymbol{j}+1,\boldsymbol{j}}/w_{\boldsymbol{j},\boldsymbol{j}+1}\rangle=\langle w_{\rightarrow}/w_{\leftarrow}\rangle$ are greater than 1 [23,26]. Making use of a periodization of the medium, Derrida obtained exact expressions for the velocity and diffusion constant [21]. In the case $\langle\ln(w_{\leftarrow}/w_{\rightarrow})\rangle<0$, he found the following. (i) The velocity V vanishes if $\langle w_{\leftarrow}/w_{\rightarrow}\rangle\geq 1$. (ii) For $\langle w_{\leftarrow}/w_{\rightarrow}\rangle<1<\langle(w_{\leftarrow}/w_{\rightarrow})^2\rangle$ the velocity is finite but the diffusion coefficient is infinite. (iii) For $\langle(w_{\leftarrow}/w_{\rightarrow})^2\rangle<1$ both V and D are finite. All these results are easy to transpose when $\langle\ln(w_{\leftarrow}/w_{\rightarrow})\rangle>0$.

As already mentioned in Sec. I, the metamaterial introduced by Fedetov *et al.* [31] can be used to realize a lattice of random asymmetric transmittances. Suppose that the twist vector of this metamaterial is along the right direction of the lattice. For a normally incident circularly polarized light of wavelength 630 nm propagating to the right (left) direction, the intensity transmittance of this metamaterial is $t=0.43$ ($t'=0.57$). We suggest a simple arrangement where a fraction ε of the lattice sites are randomly occupied by the metamaterial, and the rest of lattice is occupied by half transparent dielectric slabs with $t=t'=0.5$. Then the distribution of transmittances can be written as $\mathcal{F}(t,t')=\varepsilon\,\delta(t-0.43)\delta(t'-0.57)+(1-\varepsilon)\delta(t-0.5)\delta(t'-0.5)$. We find

$$D=\frac{2(1-0.2456\varepsilon)}{(2-0.2456\varepsilon)^2}.$$

One can measure the diffusion constant of photons as a function of ε to test our predictions.

EMA does not predict any result when both $\langle t/t'\rangle$ and $\langle t'/t\rangle$ are greater than 1. It would be interesting to investigate the anomalous diffusion of photons and self-averaging quantities of the system following Refs. [21–23]. Our studies can also be extended to higher-dimensional lattices.

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APPENDIX: z-TRANSFORM

The z transform $F(z)$ of a function $F(n)$ of a discrete variable $n=0,1,2,\ldots$, is defined by

$$F(z) = \sum_{n=0}^{\infty} F(n) z^n. \tag{A1}$$

One then derives the z transform of $F(n+1)$ simply as $F(z)/z - F(n=0)/z$. Note the similarities of this rule with the Laplace transform of the time derivative of a continuous function [2,37].

Under specified conditions the singular behavior of $F(z)$ can be used to determine the asymptotic behavior of $F(n)$ for large n (Tauberian theorems) [2]. For example, the identity $\Gamma(1-\alpha)(1-z)^{\alpha-1} = \Sigma_{n=0}^{\infty}\Gamma(n-\alpha+1)z^n/n!$ shows that

$$F(z) \sim \frac{\Gamma(1-\alpha)}{(1-z)^{1-\alpha}} \to F(n) = \frac{\Gamma(n-\alpha+1)}{n!}, \tag{A2}$$

where $\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt$. Particularly,

$$F(z) \sim \frac{1}{(1-z)^2} \to F(n) = n+1,$$

$$F(z) \sim \frac{1}{(1-z)^3} \to F(n) = \frac{1}{2}(n^2+3n+2). \tag{A3}$$

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