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Missing levels in acoustic resonators

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It is shown that the deviations of the experimental statistics of six chaotic acoustic resonators from Wigner-Dyson random matrix theory predictions are explained by a recent model of random missing levels. In these resonatorsa made of aluminum plates a the larger deviations occur in the spectral rigidity (SRs) while the nearest-neighbor distributions (NNDs) are still close to the Wigner surmise. Good fits to the experimental NNDs and SRs are obtained by adjusting only one parameter, which is the fraction of remaining levels of the complete spectra. For two Sinai stadiums, one Sinai stadium without planar symmetry, two triangles, and a sixth of the three-leaf clover shapes, was found that 7%, 4%, 7%, and 2%, respectively, of eigenfrequencies were not detected.

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Wigner's statistical theory of spectra based on random matrix ensembles has become, since its introduction in the 1950s, an important theoretical and experimental tool to understand physical phenomena in many different areas, such as atomic nuclei, atoms, molecules, microwave cavities, acoustic resonators, quantum dots, quantum gravity, etc. [1]. After developments made by Dyson and Mehta [2] the theory was prepared to be fully confronted with experimental data. The breakthrough in this direction was achieved in the beginning of the 1980s by Haq, Pandey, and Bohigas, who have verified the theoretical predictions using what they called the nuclear data ensemble [3]. About the same time the connection between spectral statistics properties and classical chaos was put on firm ground [4]. This connection states that quantum systems whose classical analogs are chaotic have the same statistics properties as the eigenvalues of matrices of the random matrix theory (RMT) ensembles. In particular, if the system has time-reversal symmetry, the statistics to be used are those of real symmetrical matrices of the Gaussian orthogonal ensemble (GOE).

A remarkable feature of the theory is its sensitivity to the amount of system symmetries. The statistical measures react to even small breaking of a given symmetry [5–7]. Of course, this sensitivity works in favor of the theory by enlarging its applications. We mention, for instance, that it has been conjectured that spectral analysis can become a useful nondestructive test of material imperfections. However, at the same time, this sensitivity brings difficulties in obtaining experimentally the statistical measures. Dyson realized this difficulty and put it in the language of information theory, considering spectra as messages that may be corrupted. Following this idea, he used spectral correlations to work out an error-correction code to detect isolated spurious or missing levels. Unfortunately, his test has proved to be impractical [8].

Recently the problem of missing levels has been reexamined, focusing on the case when the fraction of levels missing is randomly distributed along the spectrum [9]. From a practical point of view it is reasonable to assume the randomness condition, since there are many reasons why levels cannot be detected.

The main effect on the statistical properties of randomly incomplete spectra is a reduction of the correlations among levels. Roughly speaking, the levels behave more independently, letting the spectrum be more Poissonian [9]. This is reflected in the decrease of the repulsion between adjacent levels and in the increase of long-range statistics.

In a recent paper the application in the analysis of spectra of frequencies of elastomechanical vibrations has revealed that in that kind of system the missing levels can be an important issue [10]. In fact, Bertelsen *et al.* [12] verified that the experimental density of states (DOS) was slightly below the Weyl theoretical formula and attributed that discrepancy to the loss of levels.

The purpose of this article is to report the effect of missing levels in the data set of frequencies of acoustic resonators by using the model of Ref. [9]. We used the data of two resonators extracted from Refs. [11,12] and also our own data of four resonators.

Our aluminum plate resonators have the profiles shown in Fig. 1. To couple the *flexural* and *in-plane* modes, the planar symmetries were removed by digging two straight channels in one face of each sample as represented by the darker lines in Fig. 1, and we call them asymmetrical samples.To remove the samples internal stress an annealing process was applied keeping them in an oven at 300 °C for 24 h. We obtained data of four resonator plates cut in the shape of chaotic billiards: namely, a Sinai stadium, an asymmetrical Sinai, and a rectangular and a scalene asymmetrical triangle with irrational angles $(\alpha,\beta,\gamma)=(\frac{1}{2},\frac{\sqrt{5}-1}{4},\frac{1}{2}-\frac{\sqrt{5}-1}{4})\pi$ and $(1-\frac{\sqrt{2}-1}{2}-\frac{\sqrt{5}-1}{4},\frac{\sqrt{5}-1}{4},\frac{\sqrt{2}-1}{2})\pi$, respectively.

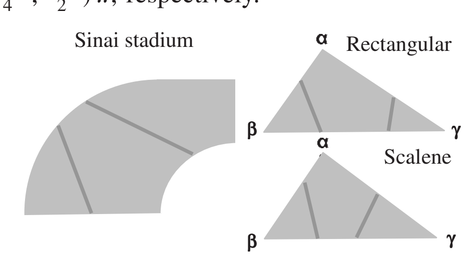

FIG. 1. Asymmetrical samples profiles. The darker lines represent the channels dug to break the samples' planar symmetry. The triangular angles are irrationals, and the scalene triangle is almost rectangular with angle $\alpha\approx 87.1°$.

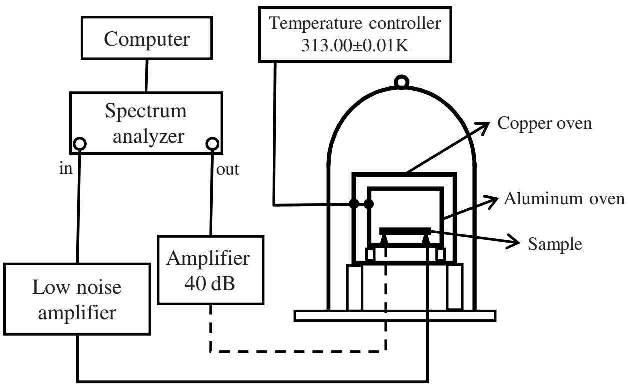

FIG. 2. The experimental apparatus diagram.

The experimental apparatus diagram is shown in Fig. 2. In a vacuum chamber, the samples were placed over three ruby spheres of 1 mm diameter, which are the end of the transducers' systems. All these parts were mounted inside an aluminum oven which was also inside a copper oven, and both had the temperature controlled at $T=313.00\pm0.01$ K. The traces were obtained by exciting the samples through one of the transducers (the transmitter), increasing the frequency driven by a spectrum analyzer and measuring the sample response with one of the receivers. The sample temperature was monitored by following the peak position of a particular resonance which after two days stabilized, showing that the sample temperature was stable. To avoid the loss of levels that could happen due to the transducer being located at the resonator node position, we collected five traces with the transducers in different positions and the statistics measures were derived from all five traces. Examples of traces are shown in Fig. 3.

In order to analyze the data, cubic polynomials were fitted to the cumulative staircases of the experimental spectra. The polynomial values calculated at the experimental frequencies correspond to the so-called unfolded spectra which have an average DOS equal to 1. We obtained two statistical measures, the so-called nearest-neighbor distribution (NND) and the Δ_3 statistics [spectral rigidity (SR)].

The NND gives the probability $P(s)$ that there is no level between a pair of levels separated by a distance s. If a fraction $1-f$ of levels is randomly removed, we have an incomplete spectrum. It is shown in Refs. [9,13] that in this case the NND is given by

$$p(s,f) = \sum_{k=0}^{\infty} (1-f)^k P\left(k, \frac{s}{f}\right), \tag{1}$$

where the functions $P(k,x)$ are the probabilities that initially there were k levels between the pair of levels.

The SR statistics gives the deviations from the straight line fitted to the ladder cumulative spectrum in a window of length L. For an incomplete spectra the SR is given by [9]

$$\delta_3(L,f) = (1-f)\frac{L}{15} + f^2\Delta_3\left(\frac{L}{f}\right). \tag{2}$$

In Eqs. (1) and (2) we followed the convention that quantities of the incomplete spectrum are represented by lowercase letters while uppercase letters are used for those of the complete ones.

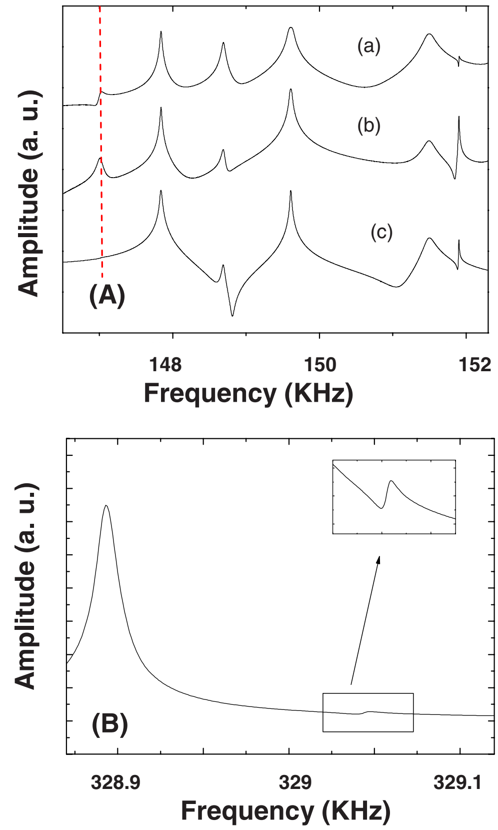

FIG. 3. (Color online) In (A) are shown three traces in the range 148–152 kHz, illustrating a missing level due to a node transducer position, pointed out by the vertical dashed line. In (B) a trace is shown in the range 138.9–139.1 kHz, showing the existence of a very weak and narrow resonance.

Equations (1) and (2) are general relations that apply to systems of any nature. By assumption, the acoustic resonators analyzed here behave as quantum systems whose classical analogs are chaotic with time-reversal symmetry. Accordingly, their eigenfrequencies are expected to have the same statistical properties as the real symmetric matrices of the GOE of the random matrix theory.

Following this idea we proceed as in Refs. [9,13]. The NND $p(s,f)$ given by Eq. (1) was calculated using the Wigner surmise $P(s)=P(0,s)=\frac{\pi}{2}s\exp(-\frac{\pi}{4}s^2)$, the next-nearest-neighbor distribution $P(1,s)=\frac{8}{3\pi^3}(\frac{4}{3})^5 s^4\exp(-\frac{16s^2}{9\pi})$, and for higher spacing distributions $P(k,s)$ with Gaussians centered at $k+1$ having variances $V(k)$ given by [14]

$$V^2(k) \simeq k - 2\int_0^k (k-x)Y_2(x)dx - \frac{1}{6}, \tag{3}$$

where

$$Y_2(x) = \left(\frac{\sin(\pi x)}{\pi x}\right)^2 - [\mathrm{Si}(\pi x) - \pi\epsilon(x)]\left(\frac{\cos(\pi x)}{\pi x} - \frac{\sin(\pi x)}{(\pi x)^2}\right), \tag{4}$$

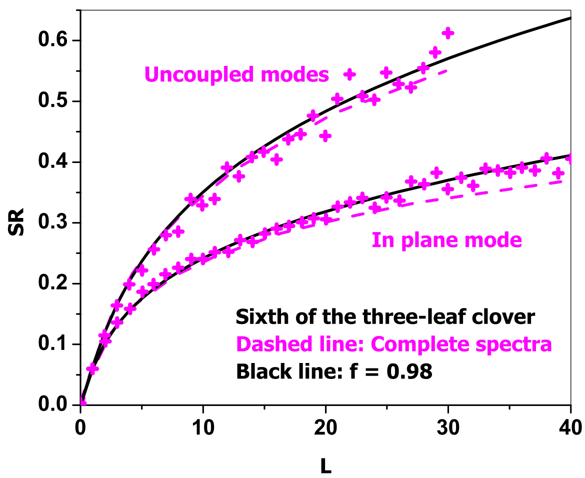

FIG. 4. (Color online) In magenta (or gray) are symbols of the experimental spectral rigidities of the in-plane mode and of the uncoupled modes from Ref. [11] of the sixth of the three-leaf clover shape resonator. The dashed magenta (gray) lines are the simulations obtained with the complete spectra and the black lines by supposing 2% of missing levels.

and $\mathrm{Si}(\pi x)$ is the sine integral and $\epsilon(x)=0$ if $x=0$ [2].

The spectral rigidity $\delta_3(L,f)$ given by Eq. (2) was calculated with the GOE expression for $\Delta_3(L)$ given by

$$\Delta_3(L) = \frac{L}{15} - \frac{1}{15L^4}\int_0^L [L-x]^3[2L^2-9xL-3x^2]Y_2(x)dx. \tag{5}$$

In an experimental study of a sixth of the three-leaf clover shape resonator Andersen *et al.* [11] obtained the statistics of the *in-plane* mode separated from the *flexural* one. They observed that the flexural mode follows the GOE statistics described well by the Wigner distribution and by the Δ_3 statistics. However, for the in-plane mode the spectral rigidity lies above the complete spectra Δ_3 statistics, although $P(s)$ still lies close to the Wigner distribution. Similar results were found in the experimental statistics when the two modes are uncoupled. If we suppose that frequencies are missing, this unexpected result of Andersen *et al.* [11] can be described by the present SR model with 2% of missed levels (f=0.98) as shown by the black line labeled as "in-plane mode" in Fig. 4.

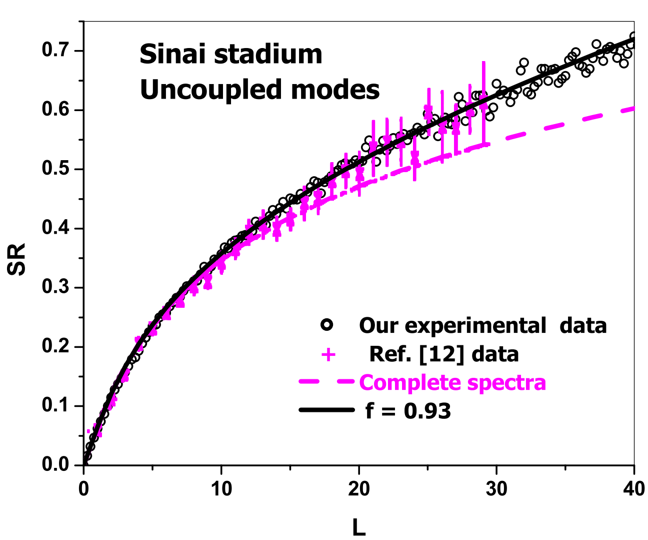

FIG. 5. (Color online) The experimental spectra rigidities for the Sinai stadium. The circles in black are our data (910 eigenfrequencies), and the other symbols correspond to the data extracted from Ref. [12]. The magenta (or gray) dashed line is the SR of the complete spectra, and the black solid line we obtained by supposing 7% (f=0.93) of missing levels.

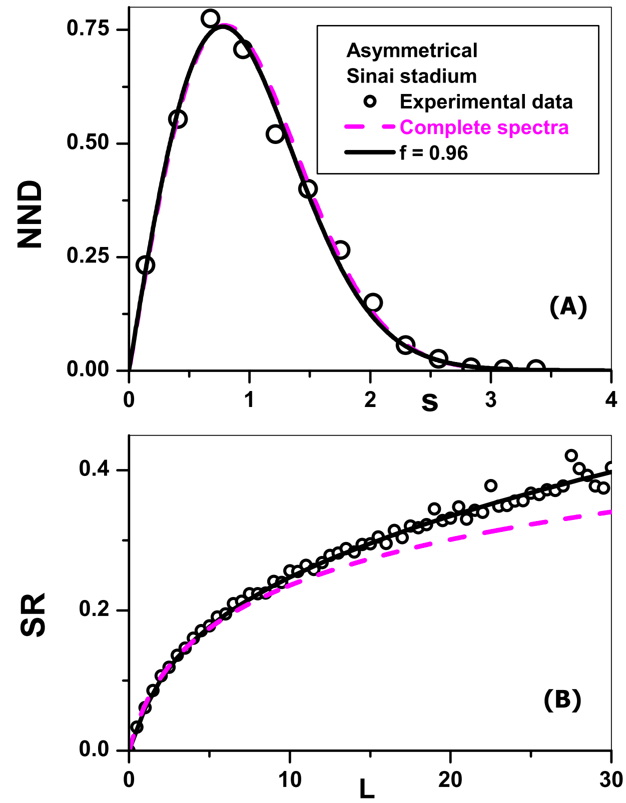

FIG. 6. (Color online) In (A) the NND statistics and in (B) the SR ones for the asymmetrical Sinai stadium sample. In both cases the open circles are the experimental data (990 eigenfrequencies). The dashed lines correspond to the complete spectra statistics of the coupled modes. The solid lines show the good agreement when we consider 4% of missing levels.

When the two modes are uncoupled the statistical quantities must be obtained by the superposition of two independent GOE spectra; this amounts to replacing $\Delta_3(\frac{L}{f})$ in Eq. (2) by $2\Delta_3(\frac{L}{2f})$ [15]. Therefore, also the 2% of missing levels can explain the experimental data as shown by the upper black line in Fig. 4, which means we do not need to separate the two modes to measure the amount of missing levels.

Another example of uncoupled modes is shown by the Sinai stadium in Fig. 5, where the circles correspond to our experimental data while the other symbols to the 1997 data of Bertelsen *et al.* [12]. In Fig. 5 the dashed line corresponds to the complete spectrum simulation and the solid line is the SR missing-level model with f=0.93, showing a good fit to

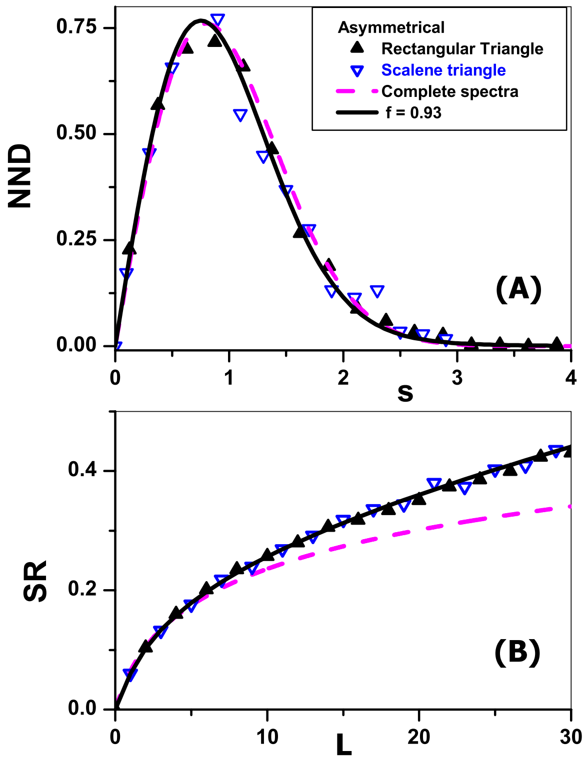

FIG. 7. (Color online) In (A) the NND statistics and in (B) the SR ones for the asymmetrical triangular samples. In both cases the triangular symbols are our experimental data, 1265 eigenfrequencies for the rectangular triangle and 1237 for the scalene one. The gray dashed lines are the complete spectrum statistical measures. The black solid lines show the good agreement when we consider 7% of missing levels for the two triangular samples. The scalene triangle is close to a rectangular one with $\alpha \approx 87.1°$.

all data, which confirms the assumption of Bertelsen *et al.* [12] about the lack of levels in the experimental spectra.

For the asymmetrical Sinai stadium the results are shown in Fig. 6, where the dashed lines are the complete spectrum predictions for coupled modes and the solid lines are the fit with 4% of missing levels. The NND does not differ too much from the Wigner surmise.

Similar results were obtained with the asymmetrical rectangular and scalene triangles as shown in Fig. 7. As before, the deviations from the Δ_3 statistics are more significant than NND ones, and they are also well described by the missing-level model with $f=0.93$.

In conclusion, the predictions given by the GOE statistics for the complete spectra ($f=1$) describes reasonably just the experimental NND $P(S)$, but the spectral rigidities Δ_3 clearly show deviations from the experimental spectral rigidity. However, when one takes into account the effect produced by removing a fraction $(1-f)$ of levels at random, the situation changes completely. By doing this, agreement between experimental and theoretical results is achieved by adjusting only one parameter: the fraction f of the detected frequencies. Regarding the use of the Weyl formula to detect missing events, the present results show a more stringent method to do this job as we are looking to the effect of the losses using the correlations among the frequencies. As a consequence, several statistics can be investigated and, in particular, in long-range ones the effect of spectral imperfections are magnified. We also remark that the use only of the DOS can be strongly affected by the lack of certitude in the theoretical determination of these quantities for each mode.

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