# Thermally activated switching in the presence of non-Gaussian noise

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We study the effect of a non-Gaussian noise on interstate switching activated primarily by Gaussian noise. Even weak non-Gaussian noise can strongly change the switching rate. The effect is determined by all moments of the noise distribution. It is expressed in a closed form in terms of the noise characteristic functional. The analytical results are compared with the results of simulations for an overdamped system driven by white Gaussian noise and a Poisson noise. Switching induced by a purely Poisson noise is also discussed.

Much progress has been made recently in the studies of switching between coexisting stable states, primarily because switching can be now investigated for a large variety of well-controlled microscopic and mesoscopic systems ranging from trapped electrons and atoms to Josephson junctions and to nanomechanical and micromechanical oscillators [1–10]. Fluctuations in these systems are usually due to thermal or externally applied Gaussian noise. However, as the systems become smaller, an increasingly important role may be played also by non-Gaussian noise. It may come, for example, from one or a few two-state fluctuators hopping at random between the states; the fluctuator noise may be often described as a telegraph noise [11].

The switching probability is sensitive to a non-Gaussian noise. This sensitivity attracted much attention after it was proposed [12] to use switching in Josephson junctions to measure the full counting statistics in electronic circuits [13,14]. Several theoretical [15–18] and experimental [19,20] papers on measuring the third moment of the current distribution from the switching rates were presented recently, and different theoretical approaches were compared in Refs. [21,22].

In this paper we study switching induced by Gaussian noise in the presence of an additional non-Gaussian noise. Even where the latter has a smaller intensity than the Gaussian noise, its effect on the switching rate may be exponentially strong. We show that it can be described in a simple form in terms of the noise characteristic functional, thus accounting for all moments of the noise distribution. The analytical results are compared with simulations for an overdamped system driven by white Gaussian noise and a Poisson noise. We also consider switching induced by a Poisson noise alone; here, the result for the rate may be qualitatively different from that in the weakly non-Gaussian noise approximation.

The potentially strong effect of an extra modulation, whether random or regular, on the rate of Gaussian noise induced switching can be understood from the well-established picture of the switching dynamics. Switching events result from large rare noise outbursts. For Gaussian noise, the switching rate is $W \propto \exp(-R/D)$, where $R$ is the activation energy and $D$ is the noise intensity [23]. Even though switching happens at random, the system trajectories followed in switching form a narrow tube in the space of dynamical variables $\mathbf{q} = (q_1, q_2, \dots)$ centered at the most probable (optimal) switching path $\mathbf{q}_{\mathrm{opt}}(t)$, as observed in a recent experiment [24].

One can think of the effect of an additional modulation in terms of a generalized work done by the modulation on the system moving along $\mathbf{q}_{\mathrm{opt}}(t)$ [25,26]. This work changes the activation barrier. The change $\delta R$ is proportional to the modulation amplitude. Therefore, the overall change of the switching rate $\propto \exp(-\delta R/D)$ depends on the modulation amplitude exponentially. The switching rate gives the probability current from the occupied state [27]. It is an observable quantity. As such, it must be averaged over realizations of the modulation, if the modulation is random, i.e., for random modulation one must average $\exp(-\delta R/D)$.

Since $\delta R$ is linear in the characteristic amplitude of random modulation, the ratio $\delta R/D$ does not have to be small even where the modulation intensity, which is quadratic in the amplitude, is smaller than $D$. However, the distribution of non-Gaussian modulation may decay slower than Gaussian on the tail. To determine whether the effect of a non-Gaussian noise on switching may be regarded as a perturbation one must compare the probabilities of appropriate large fluctuations induced by the Gaussian and non-Gaussian noises, taking into account all moments of the distribution.

We study switching for a system described by the Langevin equation

$$\dot{\mathbf{q}} = \mathbf{K}(\mathbf{q}) + \mathbf{f}(t) + \boldsymbol{\xi}(t). \tag{1}$$

We assume that, in the absence of noise, the system has a stable stationary state $\mathbf{q}_A$ and a saddle point $\mathbf{q}_S$ on the boundary of the basin of attraction to $\mathbf{q}_A$, with $\mathbf{K}(\mathbf{q}_A) = \mathbf{K}(\mathbf{q}_S) = 0$. Switching from the stable state is due to the forces $\mathbf{f}(t)$ and $\boldsymbol{\xi}(t)$, which are the Gaussian and non-Gaussian noises, respectively. We separate them, since physically they often come from different sources. It is convenient to characterize $\mathbf{f}(t)$ by its probability density functional $\mathcal{P}_{\mathbf{f}}[\mathbf{f}(t)] = \exp(-\mathcal{R}_{\mathbf{f}}/D)$,

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$$\mathcal{R}_{\mathbf{f}}[\mathbf{f}(t)] = \frac{1}{4} \int dt\,dt'\, \mathbf{f}(t)\hat{\mathcal{F}}(t-t')\mathbf{f}(t'), \qquad (2)$$

where $\hat{\mathcal{F}}(t-t')/2D$ is the inverse of the pair correlator of $\mathbf{f}(t)$. The characteristic noise intensity $D$ is small, so that the switching rate $W \ll t_r^{-1}, t_c^{-1}$, where $t_r$ is the relaxation time of the system and $t_c$ is the noise correlation time. The non-Gaussian noise is more conveniently described for our purpose by the characteristic functional

$$\widetilde{\mathcal{P}}_{\boldsymbol{\xi}}[\mathbf{k}] = \left\langle \exp\!\left( i \int dt\, \mathbf{k}(t)\boldsymbol{\xi}(t) \right) \right\rangle_{\boldsymbol{\xi}}, \qquad (3)$$

where $\langle \cdots \rangle_{\boldsymbol{\xi}}$ means averaging over $\boldsymbol{\xi}(t)$.

We first consider the case where the intensity of the non-Gaussian noise $\boldsymbol{\xi}$ is smaller than $D$. We will disregard corrections proportional to this intensity, but the ratio of the characteristic amplitude $g_0$ of $\boldsymbol{\xi}$ to $D$ will not be assumed small. The switching rate can be written as

$$W = C \langle \exp(-\mathcal{R}[\boldsymbol{\xi}]/D) \rangle_{\boldsymbol{\xi}},$$

$$\mathcal{R}[\boldsymbol{\xi}] = \min\!\left( \mathcal{R}_{\mathbf{f}} + i \int dt\, \mathbf{k}(t)[\dot{\mathbf{q}} - \mathbf{K} - \mathbf{f}(t) - \boldsymbol{\xi}(t)] \right), \quad (4)$$

where prefactor $C$ weakly depends on $g_0, D$. The minimum is taken over trajectories $\mathbf{f}(t), \mathbf{q}(t), \mathbf{k}(t)$ that satisfy boundary conditions $\mathbf{f}(t), \mathbf{k}(t) \to 0$ for $t \to \pm\infty$, $\mathbf{q}_{t\to-\infty} \to \mathbf{q}_A, \mathbf{q}_{t\to\infty} \to \mathbf{q}_S$. This formulation was proposed in the weak-noise limit [28,29] for a time-periodic $\boldsymbol{\xi}(t)$, in which case $\mathbf{q}_{A,S}$ are also periodic and there is no averaging over $\boldsymbol{\xi}$. The variational problem (4) describes coupled optimal trajectories $\mathbf{f}_{\mathrm{opt}}(t), \mathbf{q}_{\mathrm{opt}}(t), \mathbf{k}_{\mathrm{opt}}(t)$, with $\mathbf{f}_{\mathrm{opt}}(t)$ being the most probable noise realization that brings the system to the saddle on the basin boundary of the initially occupied state.

It is known from variational calculus that, to first order in $\boldsymbol{\xi}$, the effect of $\boldsymbol{\xi}(t)$ on $\mathcal{R}$ can be calculated along the optimal trajectory unperturbed by $\boldsymbol{\xi}(t)$. Such a trajectory is an instanton. Its typical duration is $\sim \max(t_r, t_c)$. It is translation invariant with respect to time and can be centered at any time $t_0$. The time-translation symmetry (degeneracy) requires special care when using perturbation theory. In particular, a periodic in time $\boldsymbol{\xi}(t)$ lifts the degeneracy by fixing $t_0$ (modulo the period) so as to minimize $\mathcal{R}[\boldsymbol{\xi}]$ and maximize the rate $W$ [25].

If $\boldsymbol{\xi}(t)$ is a stationary noise, the switching rate $W$ is independent of time. In this case one can think not of the adjustment of the instanton center $t_0$ to $\boldsymbol{\xi}(t)$, but, equivalently, of the adjustment of $\boldsymbol{\xi}(t)$ to $t_0$ so as to maximize the overall probability of switching. This adjustment provides the major contribution to the value of $W$ when the averaging over realizations of $\boldsymbol{\xi}(t)$ is performed in Eq. (4) using a solution with a given $t_0$.

From Eqs. (3) and (4) one obtains a simple expression for the switching rate,

$$W = W^{(0)} A_{\mathrm{sw}}, \quad A_{\mathrm{sw}} = \widetilde{\mathcal{P}}_{\boldsymbol{\xi}}[i\boldsymbol{\chi}/D], \qquad (5)$$

where $W^{(0)}$ is the switching rate in the absence of non-Gaussian noise. The factor $A_{\mathrm{sw}}$ describes the effect of non-Gaussian noise. It is expressed in a closed form in terms of the noise characteristic functional calculated for function $\boldsymbol{\chi}(t) = -i\mathbf{k}_{\mathrm{opt}}^{(0)}(t)$, where $\mathbf{k}_{\mathrm{opt}}^{(0)}(t)$ is the solution of the variational problem (4) for $\boldsymbol{\xi} = \mathbf{0}$. The real function $\boldsymbol{\chi}(t)$ is the logarithmic susceptibility which describes the linear response of $\ln W$ to a perturbation [25,26,30]. The structure of Eq. (5) resembles that of the expression for a large fluctuation probability in a birth-death system with non-Gaussian modulation of reaction rates [31].

From Eq. (5), the effect of a non-Gaussian noise on the switching rate is determined by the ratio of the noise amplitude to the Gaussian noise intensity $D$. Equation (5) applies to both underdamped and overdamped systems. Examples of calculating $\mathbf{k}_{\mathrm{opt}}^{(0)}$ can be found in Refs. [25,26,30], and papers cited therein.

As an illustration we will consider the case of a one-component $\delta$-correlated Poisson noise $\xi(t) = g\sum_n \delta(t-t_n)$ with mean pulse frequency $\nu$. Using the explicit form of the noise characteristic functional [32], we obtain

$$A_{\mathrm{sw}} = \exp\!\left( -\nu \int dt\{1 - \exp[-\chi(t)g/D]\} \right), \qquad (6)$$

where $\chi(t)$ is the corresponding component of the logarithmic susceptibility. If $|g|/D \ll 1$, $\ln A_{\mathrm{sw}}$ is a series in $g/D$. The coefficients in this series describe the effects of the moments of the Poisson noise on the switching rate. In the opposite case, $|g|/D \gg 1$ (but the Poisson noise intensity $\nu g^2 \ll D$), if $g\chi(t)$ becomes negative, then $\ln A_{\mathrm{sw}} \approx \nu[2\pi D/g\ddot{\chi}(t_m)]^{1/2} \exp[-\chi(t_m)g/D]$ where $t_m$ is the instant where $-g\chi(t)$ is maximal. If $g\chi(t) \geq 0$ for all $t$ and $|g|/D \gg 1$, the major contribution to $A_{\mathrm{sw}}$ comes from the region of small $|\chi(t)|$. If $g\chi(t)$ is small only for $|t| \to \infty$, where it decays exponentially with $|t|$, then $\ln A_{\mathrm{sw}} \propto \nu t_r \ln(|g|/D)$, to leading order in $g/D$.

The Poisson noise distribution does not fall off as steeply as the Gaussian. This imposes a limitation on the range of $g/D$ where Poisson noise may be treated as a perturbation and the above theory applies. To see the far-tail effect we consider switching due to a purely Poisson noise, where $\mathbf{f} = \mathbf{0}$ in the equation of motion (1). We will use the method of optimal fluctuation, as for some other types of non-Gaussian noise [33].

The switching rate is determined, to logarithmic accuracy, by the integral over trajectories $\mathbf{k}(t), \mathbf{q}(t)$ of the functional $\langle \exp\{-i\int dt\, \mathbf{k}(t)[\dot{\mathbf{q}} - \mathbf{K} - \boldsymbol{\xi}(t)]\} \rangle_{\boldsymbol{\xi}}$ [34]. We consider a zero-mean noise, $\boldsymbol{\xi}(t) \to \boldsymbol{\xi}(t) - \langle\boldsymbol{\xi}\rangle$, $\mathbf{K} \to \mathbf{K} + \langle\boldsymbol{\xi}\rangle$, and assume that $\mathbf{q}_A, \mathbf{q}_S$ are also appropriately shifted. The formulation is more compact if different components of the Poisson noise are independent short pulses with areas $\mathbf{g} = (g_1, \ldots, g_M)$ and average frequencies $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_M)$. Of interest for switching are trajectories that approach the saddle point [28]. In the spirit of the method of optimal fluctuation, for small $|\mathbf{g}|$ and for $|\boldsymbol{\nu}| \lesssim t_r^{-1}$ the integral over trajectories $\mathbf{q}(t), \mathbf{k}(t)$ can be calculated by steepest descent. This gives

$$W = C' \exp(-R_P), \quad R_P = \min \int dt\,(i\mathbf{k}\dot{\mathbf{q}} - H),$$

$$H(\mathbf{q}, i\mathbf{k}) = -\sum_j \nu_j(1 + ig_jk_j - e^{ig_jk_j}) + i\mathbf{k}\mathbf{K}(\mathbf{q}). \qquad (7)$$

The variational problem (7) determines the optimal switching trajectory $\mathbf{q}_{\text{opt}}(t), \mathbf{k}_{\text{opt}}(t)$. It starts at $t \to -\infty$ at $\mathbf{q} \to \mathbf{q}_A, \mathbf{k} \to 0$ and goes to $\mathbf{q} \to \mathbf{q}_S, \mathbf{k} \to 0$ for $t \to \infty$. On this trajectory $H = 0$. As in systems driven by white Gaussian noise [23], the optimal trajectory is a Hamiltonian trajectory of an auxiliary system with coordinate $\mathbf{q}$, momentum $i\mathbf{k}$, and Hamiltonian $H$. A similar formulation, but in different terms and with extra assumptions, was developed for fluctuations in electric circuits [17,22,35]. In our approach the noise is characterized by average pulse frequencies $\nu_i$ and, independently, by pulse areas $g_i$ ($g_i \gtrless 0$); the approach can be immediately generalized to a noise with a finite correlation time.

The switching exponent in Eq. (7) is $R_P \gg 1$ for small $|\mathbf{g}|$. However, in contrast to the case of Gaussian noise, $R_P$ is not proportional to the reciprocal noise intensity $\Sigma_i \nu_i g_i^2$ nor does it scale like reciprocal noise amplitude $|\mathbf{g}|^{-1}$, although $R_P|\mathbf{g}|$ often slowly varies with $|\mathbf{g}|$.

An explicit dependence of the switching rate on the Poisson noise parameters can be found for a one-variable overdamped system with equation of motion

$$\dot{q} = -U'(q) + f(t) + \xi(t). \qquad (8)$$

Here, $U(q)$ is the effective potential. The stationary states $q_A$ and $q_S$ correspond to the minimum and the barrier top of $U(q)$.

If $f(t)$ is white Gaussian noise, $\langle f(t)f(t') \rangle = 2D\delta(t - t')$, and the Poisson noise is weak, the Poisson-noise induced factor in the switching rate $A_{\text{sw}}$ is described by Eq. (6) with $\chi(t) = -f_{\text{opt}}^{(0)}(t)/2 = -\dot{q}_{\text{opt}}^{(0)}(t)$ and with $\dot{q}_{\text{opt}}^{(0)} = U'(q_{\text{opt}}^{(0)})$.

In the opposite case where switching is due to purely Poisson noise, i.e., $f = 0$ in Eq. (8), $H$ can be reduced to the same form as in Ref. [35]. From Eq. (7),

$$R_P = \frac{1}{g}\int_{\tilde{q}_A}^{\tilde{q}_S} dq\,\kappa(q), \quad \kappa = \ln\{1 + [\kappa U'(q)/g\nu]\}. \qquad (9)$$

Here, $\tilde{q}_A$ and $\tilde{q}_S$ are the shifted extrema of the potential given by equation $U'(q) = g\nu$. From Eq. (9), $R_P \sim r_P \ln(r_P/\nu t_r)$, with $r_P = (\tilde{q}_S - \tilde{q}_A)/g$.

A qualitative feature of unipolar (pulses of one sign) Poisson noise is that, for an overdamped system, it causes switching only provided the noise pulses push the system from the stable state towards the saddle. In this case $r_P > 0$. There is no switching for pulses of the opposite sign. The "one-sidedness" of fluctuations in overdamped systems has other manifestations, which includes the work fluctuation distribution [36]. On the other hand, we expect that an underdamped system should be able to switch for Poisson pulses of any sign, in which case there should be a critical value of damping for which switching from the state is possible for a given sign of $g$. Equations (7) and (9) apply if $r_P, r_P/\nu t_r \gg 1$.

As a cause of switching, a Poisson noise is effectively weaker than a Gaussian noise only if the switching exponent $R_P$ is larger than the switching exponent for the Gaussian noise, which for white noise is $\Delta U/D$ with $\Delta U = U(q_S)$
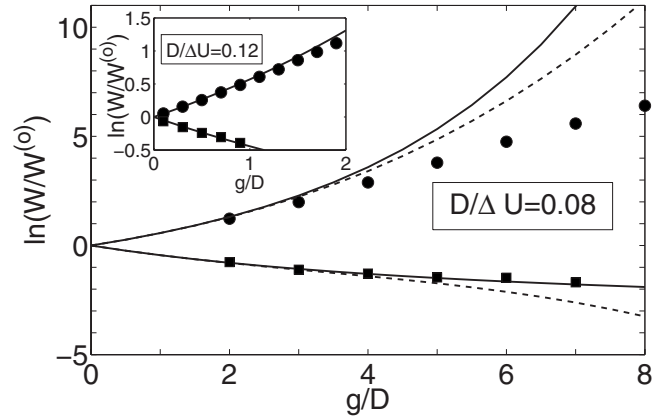


FIG. 1. Poisson noise induced change of the switching exponent for an overdamped Brownian particle in potential (10). The dimensionless Poisson noise frequency is $\nu = 0.5$. The data of numerical simulations are shown with circles and squares for the cases where in escape the particle moves along and opposite to Poisson pulses, respectively. The solid curves show Eq. (6) for these cases, and the dashed curves show the approximation adopted in Refs. [16–18] where only three moments of Poisson noise are taken into account.

$-U(q_A)$ [27]. The condition $R_P > \Delta U/D$ also effectively limits the range of applicability of Eq. (6). We see that $\ln A_{\text{sw}}$ becomes large provided $-\chi(t_m)g/D \gg 1$. An order of magnitude estimate shows that $-\chi(t_m)g/D \sim (\Delta U/D)R_P^{-1}\ln(r_P/\nu t_r)$, and therefore from Eq. (9) the large $\ln A_{\text{sw}}$ asymptotics applies provided $\ln(r_P/\nu t_r) \gg 1$. This condition is compatible with $-\chi(t_m)g/D \gg 1$ only for very small $D$.

We now apply the above results to an overdamped system (8) with a double-well potential

$$U(q) = -q^2/2 + q^4/4, \qquad (10)$$

which has been extensively studied in the context of white-noise driven systems. In the absence of Poisson noise, the escape rate in this case is $W^{(0)} = (\sqrt{2}/\pi)\exp(-1/4D)$, and the logarithmic susceptibility for escape from the negative-$q$ well ($q_A = -1$) is $\chi(t) = -\exp(t/2)(2\cosh t)^{-3/2}$ [$\chi(t)$ has opposite sign for switching from $q_A = 1$].

In Fig. 1 we present results of Monte Carlo simulations of switching of an overdamped Brownian particle described by Eqs. (8) and (10). They are compared with prediction (6) and the approximation where only terms up to $g^3$ are kept in Eq. (6). The Poisson noise intensity $\nu g^2 < D$ in the whole range of studied $g/D$. For $g/D \lesssim 1$ the effect of Poisson noise is small and comes, primarily, to the change of the activation barrier, $\Delta U \to \Delta U \pm \nu g$, and the effective noise intensity, $D \to D + \nu g^2$. For larger $g/D \gtrsim 3$ the switching exponent changes significantly, as expected.

For switching in the direction opposite to Poisson noise pulses, where $\ln W/W^{(0)} < 0$, the numerics agree well with Eq. (6). This is to be expected since the far tail of Poisson noise distribution is immaterial here; for $g/D \gtrsim 5$ the results differ noticeably from the three-moments approximation. For switching along the pulses, because of the far-tail effect, with increasing $g$ Poisson noise quickly becomes as important as
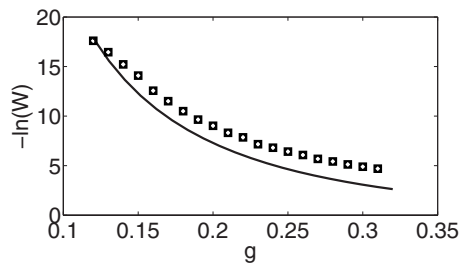
FIG. 2. Switching exponent $R_P = -\ln W$ for an overdamped particle in a potential (10) $[t_r = 1/2]$ driven by a Poisson noise with dimensionless mean frequency $\nu = 0.5$ and pulse area $g$. Squares show the results of Monte Carlo simulations and the solid line is the asymptotic theory (9).

white noise for chosen $D$. Therefore, the perturbation theory fails and the dependence of the switching exponent on $g$ is much weaker than the exponential dependence expected from Eq. (6).

Numerical simulations of the switching rate for purely Poisson noise are shown in Fig. 2. There is good agreement between the data and the asymptotic theory (9) for small $g$. In this range $R_P g$ slowly varies with $g$.

In conclusion, we have considered switching in systems simultaneously driven by a Gaussian and a non-Gaussian noise. Even where the non-Gaussian noise has intensity smaller than that of the Gaussian, it may strongly change the switching rate. The effect is determined by the ratio of the non-Gaussian noise amplitude to the Gaussian noise intensity. It is described by the characteristic functional of the non-Gaussian noise calculated for a function determined by the system dynamics without this noise. A non-Gaussian tail of the noise distribution may strongly modify the switching rate even for a small noise intensity. We demonstrate this effect using Poisson noise as an example. Analytical results and Monte Carlo simulations show agreement in the appropriate parameter regions.

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