Effect of a time-dependent field on subdiffusing particles

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We analyze the effect of a time-dependent external field on non-Markovian migration described by the continuous time random walk (CTRW) approach. The rigorous method of treating the problem is proposed which is based on the Markovian representations of the CTRW approach and field modulation. With the use of this method we derive the non-Markovian stochastic Liouville equation (SLE), that describes the effect of this field, and thoroughly analyze the relation of the derived SLE with equations proposed earlier. This SLE is applied to the case of subdiffusive migration in which the exact formulas for the first and second moments of spatial distribution are obtained. In the case of oscillating external field they predict unusual dependence of the first moment on oscillation phase and anomalous time behavior of field dependent contribution to the dispersion which agree with results of earlier works. Anomalous time dependence is also found in the case of a fluctuating field. The specific features of this time dependence are analyzed in detail.

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I. INTRODUCTION

Brownian motion in an external time-dependent field is the important stage of many physical and chemical processes which often strongly affect their kinetics [1,2]. Close attention has been given to the anomalous (subddifusive) jump-like motion typical for disordered systems [3,4] and, in particular, to the effect of a time-dependent field on this type of migration [5,6]. Usually the motion anomaly is assumed to be a manifestation of the long memory in the kinetics of jumps. In such a case the serious difficulty in the theoretical treatment of time-dependent field effects occurs because of the subtle interplay of field and anomalous memory effects which should be properly described.

Subdiffusive processes in time-independent potential $V(x)$ are traditionally described by the fractional Smoluchowski equation (FSE) for the probability distribution function (PDF) $\rho(x,t)$ [4],

$$\dot{\rho} = -{}_0D_t^{1-\alpha}\hat{\mathcal{L}}_\alpha \rho, \tag{1}$$

where ${}_0D_t^{1-\alpha}$ is the Riemann-Liouville fractional derivative defined by

$${}_0D_t^{1-\alpha}\psi = \frac{1}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^t dt_1 \frac{\psi(t_1)}{(t-t_1)^{1-\alpha}} \tag{2}$$

and

$$\hat{\mathcal{L}}_\alpha = -D_\alpha \nabla_x[\nabla_x - F(x)] \tag{3}$$

is the Smoluchowski operator, in which D_α is a subdiffusion constant, $\nabla_x \equiv \partial/\partial x$, and $F(x) = -\nabla_x V(x)/(k_B T)$ is a force. The FSE (1) can be derived within the continuous time random walk (CTRW) approach [4] assuming the long time tailed behavior of the waiting time distribution $W(t)$ for CTRW jumps: $W(t) \sim 1/t^{1+\alpha} (\alpha < 1)$.

In the case of a time-dependent field $F(x,t)$ [i.e., time-dependent $\mathcal{L}_\alpha(t)$], however, no analogs of the FSE are rigorously derived as yet. The main difficulty is in the correct treatment of the effect of $\mathcal{L}_\alpha(t)$ evolution during the time of waiting for jumps. The variant of the FSE which has recently been proposed in Refs. [5,6] looks quite reasonable but it is rigorously justified only for some particular time dependences $F(x,t)$.

In this work within the CTRW approach we rigorously derive the FSE describing the influence of a time-dependent field. The derivation is based on the recently proposed Markovian representation of the CTRW and the non-Markovian stochastic Liouville equation (SLE) [7]. In the case of deterministic (nonstochastic) time dependence of the field the rigorous FSE is shown to reduce to that proposed earlier [5,6]. In the case of a fluctuating field, however, the field effect can be described only with the derived rigorous FSE. The solutions of this FSE for different time dependences of the force $F(t)$, for simplicity, assumed to be independent of x, are proposed and discussed in detail.

II. MARKOVIAN SLE

Here we will present the method of treating the effect of a time-dependent field $F(x,t)$ on CTRW-like processes by reduction of the problem to solving the SLE with time-independent operators.

To clarify the method we first consider the Markovian (normal diffusion) case: $\alpha=1$, in which the evolution of the system is described by the Smoluchowski equation

$$\dot{\rho} = -\hat{\mathcal{L}}_1(t)\rho = D_1 \nabla_x[\nabla_x \rho - F(x,t)\rho]. \tag{4}$$

For our further analysis it is useful to introduce the evolution operator determined by this equation,

$$\hat{G}_m(t) = Te^{-\int_0^t d\tau \hat{\mathcal{L}}_1(\tau)}. \tag{5}$$

The method is based on the representation of the time dependence of $F(x,t)$ in terms of the dependence on some Markovian (in general, stochastic) variable $z(t)$:

$$F(x,t) \equiv F(x,z(t)), \quad \hat{\mathcal{L}}_1(t) \equiv \hat{\mathcal{L}}_1(z(t)), \tag{6}$$

whose evolution is described by the PDF $\sigma(z,t)$ satisfying the Markovian equation

$$\dot{\sigma} = -\hat{L}\sigma, \quad \text{with } \sigma(z,0) = \sigma_i(z), \tag{7}$$

in which $\hat{L}$ is the linear operator in $\{z\}$ space and $\int dz \sigma_i(z) = 1$. For brevity, formulas are written assuming that $\{\mathbf{z}\}$ space is one dimensional, though they are, evidently, valid for any dimensionality of $\{\mathbf{z}\}$ space. The corresponding examples will be discussed below.

In this representation the kinetics of the process described by Eq. (4) is determined by the average evolution operator which in the space $\{x \otimes z\}$ is given by the formula

$$U_m(x,z;x_i,z_i|t) = \langle xz|\langle Te^{-\int_0^t d\tau \hat{\mathcal{L}}_1(z(\tau))}\rangle|x_i z_i\rangle, \tag{8}$$

where the average (denoted as $\langle\cdots\rangle$) is taken over trajectories of the stochastic Markovian process in $\{z\}$ space with fixed initial (z_i) and final (z) coordinates. In particular, the PDF of interest, $\bar{\rho}_F(x,x_i|t)$ averaged over $F(t)$ fluctuations, can be calculated as

$$\bar{\rho}_F(x,x_i|t) = \int dz \int dz_i U_m(x,z;x_i,z_i|t)\sigma_i(z_i). \tag{9}$$

The important point of the proposed representation consists in the fact that for Markovian processes in $\{x \otimes \mathbf{z}\}$ space the operator $\hat{U}_m$ satisfies the Markovian SLE with time-independent operators [8],

$$\dot{\hat{U}}_m = -[\hat{\mathcal{L}}_1(z) + \hat{L}]\hat{U}_m. \tag{10}$$

This equation should be solved with the initial condition $U(x,z;x_i,z_i|0) = \delta(x-x_i)\delta(z-z_i)$.

Thus we have reduced the problem to solving the SLE (10) with time-independent operators, though at the cost of the extension of the space of the process, which describes the evolution of the system.

Noteworthy is that the representation (8)–(10) is valid not only for stochastic functions $z(t)$ but also for dynamical ones, which are known to be Markovian as well. For example, in the model of harmonically oscillating force,

$$z(t) = z_c(t) = z_0 \sin(\omega t + \varphi), \tag{11}$$

the dependence $z_c(t)$ can be considered as a coordinate part of the trajectory of dynamical motion (in the harmonic potential), described by the operator

$$\hat{L} = v\nabla_z - \omega^2 z \nabla_{v_z}, \tag{12}$$

in the phase space $\{\mathbf{z}\} = \{z, v_z\}$ ($v_z = \dot{z}$ is the velocity) with

$$\sigma_i(\mathbf{z}) = \delta[\mathbf{z} - \mathbf{z}_c(\mathbf{z}_i|t)], \tag{13}$$

where $\mathbf{z}_c(\mathbf{z}_i|t)$ is the classical trajectory satisfying the initial condition $\mathbf{z}_c(t=0) = \mathbf{z}_i$ [in the model (11) $\mathbf{z}_i = (z_0 \sin\phi, z_0 \cos\phi)$].

Evidently, the case of $z(t)$ represented as a linear combination of, say, N oscillating functions $z_j(t) = z_{0_j}\sin(\omega_j t + \phi_j)$: $z(t) = \Sigma_0^N z_j(t)$, can be modeled by coupling of the system under study to N harmonic coordinates $\mathbf{z}_N = (z_1, z_2, \ldots, z_N)$.

In this representation one can model any deterministic (dynamical) dependence $F(x,t)$ on time. It is important to note, however, that the proposed SLE treatment is valid not only for deterministic but for any Markovian stochastic time dependences $F(x,t)$ as well.

It is also worth noting that in the case of deterministic (dynamical) process $\mathbf{z}(t) \equiv \mathbf{z}_c(\mathbf{z}_i|t)$ (in general, in the multidimensional space $\{\mathbf{z}\}$), corresponding to the dynamical time modulation of the force (11), the evolution operator $\hat{U}_m(t)$ predicted by Eq. (10) can be directly related to that $\hat{G}_m(t)$ of Eq. (4):

$$\hat{U}_m(t) \equiv \hat{U}_m(\mathbf{z},\mathbf{z}_i|t) = \hat{G}_m(t)\delta[\mathbf{z} - \mathbf{z}_c(\mathbf{z}_i|t)]. \tag{14}$$

The validity of this formula can easily be verified by substituting it into Eq. (10) and taking into account that $\hat{\mathcal{L}}_1(\mathbf{z})\delta[\mathbf{z} - \mathbf{z}_c(t)] = \hat{\mathcal{L}}_1(\mathbf{z}_c(t))\delta[\mathbf{z} - \mathbf{z}_c(t)]$ and $\hat{\mathcal{L}}_1(\mathbf{z}_c(t)) \equiv \hat{\mathcal{L}}_1(t)$.

III. NON-MARKOVIAN CTRW AND MARKOVIAN REPRESENTATION

The main goal of this work is the analysis of the effect of a time-dependent field on CTRW-type (subdiffusive) migration.

In the CTRW approach the stochastic motion in $\{x\}$ space is treated as a set of jumps with jump statistics described by the waiting time distribution $W(t)$ [3,4]. For time-independent driving force the non-Markovian equation for the PDF $\rho(x,t)$ is conventionally derived by summing up the contributions of all sets of jumps. In terms of the Laplace transform $R(\epsilon) = \int_0^\infty dt \rho(t) e^{-\epsilon t}$, this equation is written as [3,4]

$$\epsilon R(\epsilon) = \rho_i - M(\epsilon)\hat{\mathcal{L}}_\alpha R(\epsilon). \tag{15}$$

In this equation $\rho_i(x)$ is the initial PDF and

$$M(\epsilon) = [1 - \widetilde{W}(\epsilon)]/[\epsilon \widetilde{W}(\epsilon)], \tag{16}$$

where $\widetilde{W}(\epsilon) = \int_0^\infty dt W(t) e^{-\epsilon t}$. Note that in the case of subdiffusion, when $M(\epsilon) = \epsilon^{1-\alpha}$, Eq. (15) reduces to the Laplace transform variant of the FSE (1).

CTRW-type processes can conveniently be analyzed within the Markovian representation [7] in which these processes are assumed to result from jumplike $\hat{\mathcal{L}}_1(t)$ fluctuations determined by the dependence of $\hat{\mathcal{L}}_1(y(t))$ on some Markovian stochastic variable $y(t)$ whose PDF $\eta(y,t)$ satisfies the equation

$$\dot{\eta} = -\hat{\Lambda}\eta, \quad \text{with } \eta(y|0) = \eta_i(y). \tag{17}$$

In this equation $\hat{\Lambda}$ is a linear operator in $\{y\}$ space and $\eta_i(y)$ is the initial condition $[\int dy\, \eta_i(y) = 1]$.

The dependence $\hat{\mathcal{L}}_1(y)$ is taken in the form $\hat{\mathcal{L}}_1(y(t)) \equiv \delta[y_0 - y(t)]\hat{\mathcal{L}}_1$, where y_0 is the coordinate at which the system undergoes the jump described by $\hat{\mathcal{L}}_1$. Similar to the above-considered model of $z(t)$ modulation of $\hat{\mathcal{L}}_1$, in the case of $y(t)$ modulation the evolution of the system is described by the PDF $p(x,y|t)$ in the combined space $\{x \otimes y\}$. This PDF satisfies the SLE type of Eq. (10), which as applied to the Laplace transform $P(\epsilon) = \int_0^\infty dt p(t) e^{-\epsilon t}$ is given by

$$\epsilon P = \delta(x - x_i)\delta(y - y_i) - [\hat{\Lambda} + \delta(y - y_0)\hat{\mathcal{L}}_1]P. \quad (18)$$

Of special interest is the PDF averaged over the $y(t)$ process,

$$\bar{\rho}_y(x,x_i|t) = \int dy \int dy_i P(x,y;x_i,y_i|t)\,\eta_i(y_i). \quad (19)$$

The $y(t)$-controlled (or modulated) process in $\{x\}$ space proves to be of CTRW type. Thus the obtained CTRW depends on the initial condition $p_i(y)$ and the form of the operator $\hat{\Lambda}$. In what follows we will consider the nonstationary variant realized for $p_i(y)=\delta(y-y_0)$ [7]. In this variant the average PDF $\bar{\rho}_y(x,x_i|t)$ is known to satisfy the CTRW-like equation usually written as applied to $R(\epsilon)=\bar{R}_y(\epsilon)=\int_0^\infty dt\bar{\rho}_y(t)e^{-\epsilon t}$ [7]. This equation coincides with Eq. (15) but with

$$M(\epsilon) = \bar{M}(\epsilon) = (D_1/D_\alpha)\epsilon\langle y_0|(\epsilon + \hat{\Lambda})^{-1}|y_0\rangle. \quad (20)$$

The behavior of $\bar{M}(\epsilon)$ is completely determined by the specific features of the controlling process (17). Various models of this process are discussed in Ref. [7].

IV. CTRW-BASED SLE

The Markovian representation is very suitable for treating the effect of a time-dependent field on CTRW-like processes.

The corresponding equation is straightforwardly derived by taking into account that, in accordance with the Markovian representation, this equation describes the Markovian process in $\{x\otimes y\}$ space affected by the the driving force which can be modeled by interaction with the Markovian $z(t)$ variable. This means that the equation sought is actually the Markovian SLE [8] for the PDF $q(\mathbf{r};\mathbf{r}_i|t)$ in the extended space $\{\mathbf{r}\}\equiv\{x\otimes y\otimes \mathbf{z}\}$ space. For the Laplace transform $Q(\epsilon)=\int_0^\infty dtq(t)e^{-\epsilon t}$ this equation is written as

$$\epsilon Q = \delta(\mathbf{r} - \mathbf{r}_i) - [\hat{\Lambda} + \delta(y - y_0)\hat{\mathcal{L}}_1 + \hat{L}]Q. \quad (21)$$

This equation is seen to differ from Eq. (18) for $P(x,y|\epsilon)$ only in the replacement of ϵ by $\hat{\Omega}=\epsilon+\hat{L}$ and the evident change of δ-function describing the initial condition. Naturally, for the PDF averaged over $y(t)$ process,

$$R(\epsilon) = \bar{R}_y(\epsilon) = \int_0^\infty dt\bar{\rho}_y(t)e^{-\epsilon t}, \quad (22)$$

one gets a CTRW-like equation (sometimes called the non-Markovian SLE [7]) similar to Eq. (15). In particular, for the Laplace transform of the average evolution operator $R(\epsilon)=\hat{\tilde{U}}_n(t)$ [generalizing $\hat{U}_m(t)$ defined in Eq. (8)] this equation reads

$$[\hat{\Omega} + \hat{\mathcal{L}}_\alpha(z)M(\hat{\Omega})]\hat{\tilde{U}}_n = \rho_i\sigma_i \quad \text{with } \hat{\Omega} = \epsilon + \hat{L} \quad (23)$$

and the initial condition $\rho_i\sigma_i=\delta(x-x_i)\delta(\mathbf{z}-\mathbf{z}_i)$. Notice that the order of operators $\hat{\mathcal{L}}_\alpha$ and $M(\hat{\Omega})$ is important since they do not commute with each other.

In the time representation [i.e., for $U_n(t)$] Eq. (23) is written as

$$\dot{\hat{U}}_n = -\hat{L}\hat{U}_n - \hat{\mathcal{L}}_\alpha(z)\int_0^t d\tau M(\tau)e^{-\hat{L}\tau}\hat{U}_n(t-\tau). \quad (24)$$

This equation should be solved with the initial condition $U_n(0)=\delta(x-x_i)\delta(\mathbf{z}-\mathbf{z}_i)$.

To qualitatively interpret Eqs. (23) and (24) within the CTRW approach it is worth noting that, according to the SLE representation (10), the time-dependent-field affected CTRW can be considered as a sequence of jumps governed by the z-dependent operator $\mathcal{L}_1(z)$. The CTRW process is accompanied by the simultaneous evolution of the parameter $z(t)$, determined by the operator $e^{-\hat{L}t}$. This operator will enter in formulas describing CTRW evolution in the form of the product $W(t)e^{-\hat{L}t}$ which for the Laplace transform $\tilde{W}(\epsilon)$ just corresponds to the replacement ϵ by $\hat{\Omega}=\epsilon+\hat{L}$ in $\tilde{W}(\epsilon)$ thus resulting in Eqs. (23) and (24).

Rigorous equations (23) and (24) allow one to describe CTRW evolution affected by a time-dependent field in very general assumptions on the form of the time dependence which can be either deterministic (dynamical) or stochastic.

Noteworthy is that in the case of the deterministic process governed by $\hat{L}$, i.e., a dynamical type of time dependence of $\hat{\mathcal{L}}_\alpha(t)\equiv\hat{\mathcal{L}}_\alpha(\mathbf{z}_c(\mathbf{z}_i|t))$, the solution $\hat{U}_n(t)$ of Eq. (24) can be represented in the form similar to that obtained above for Markovian processes [see Eq. (14)]:

$$\hat{U}_n(t) \equiv \hat{U}_n(\mathbf{z},\mathbf{z}_i|t) = \hat{G}_n(t)\,\delta[\mathbf{z} - \mathbf{z}_c(\mathbf{z}_i|t)], \quad (25)$$

where $\hat{G}_n(t)$ is the operator in $\{x\}$ space satisfying a non-Markovian CTRW type equation, that can be considered as a generalization of Eq. (4):

$$\dot{G}_n = -\hat{\mathcal{L}}_\alpha(z_c(t))\int_0^t d\tau M(\tau)G_n(t-\tau). \quad (26)$$

In deriving Eq. (26) one should take into account the relations $e^{-\hat{L}\tau}\delta[\mathbf{z}-\mathbf{z}_c(t-\tau)]=\delta[\mathbf{z}-\mathbf{z}_c(t)]$ and $\hat{\mathcal{L}}_\alpha(\mathbf{z})\delta[\mathbf{z}-\mathbf{z}_c(t)]=\hat{\mathcal{L}}_\alpha(\mathbf{z}_c(t))\,\delta[\mathbf{z}-\mathbf{z}_c(t)]$.

This equation coincides with that proposed in Ref. [5]. It is easily seen that the average PDF $\bar{\rho}_F(x,x_i|t)$ calculated with the use of Eq. (9), which is applicable in the non-Markovian case as well, also satisfies Eq. (26).

The above analysis shows that in the case of the deterministic time dependence of force, the force effect on the subdiffusing particles is correctly described by Eq. (26) for the PDF $\rho(x,t)$ in $\{x\}$ space, which was derived in Ref. [5] with the use of not quite rigorous (somewhat intuitive) arguments.

V. APPLICATIONS

As we have already pointed out, the SLE approach presented above is valid for any dependence $F(x,t)\equiv F(x,z(t))$. In this work, however, we will restrict ourselves to the analysis of the simple model of x-independent force:

$$F(x,t) \equiv F(x,z(t)) = F_0 z(t). \quad (27)$$

This analysis will illustrate the correctness of predictions of works [5,6] in the case of dynamical (deterministic) $F(x,t)$

time dependence. It will also demonstrate the specific features of this effect for the stochastic $F(x,t)$ time dependence.

A. General results

Our study is based on the analysis of the time evolution of the moments of the PDF $\bar{\rho}_{yz}(x|t)$ [averaged over $y(t)$ and $z(t)$ processes], $m_n(t)=\int dx x^n \bar{\rho}_{yz}(t)$. For this analysis we need to specify the initial PDF $\rho_i(x)$. For simplicity we will assume the initial condition $\rho_i(x)=\delta(x)$, for which, taking into account the relation (9), the problem reduces to manipulations with $\hat{U}_n(t)$ satisfying Eqs. (23) or (24).

Instead of moments $m_n(t)$, it is more convenient to analyze their Laplace transforms $\tilde{m}_n(\epsilon)$ which can be expressed in terms of moment operators in the $\{\mathbf{z}\}$ space

$$\hat{M}_n = \int dx x^n \bar{R}_y(x,\hat{\Omega}): \tag{28}$$

$$\tilde{m}_n(\epsilon) = \langle \hat{M}_n \rangle_z = \int dz \int d\bar{z} \langle z|\hat{M}_n|\bar{z}\rangle \sigma_i(\bar{z}). \tag{29}$$

As is seen from Eq. (23) the operators $\hat{M}_n(\epsilon)$ satisfy simple equations,

$$\hat{\Omega}\hat{M}_n = nfzM(\hat{\Omega})\hat{M}_{n-1} + n(n-1)D_\alpha M(\hat{\Omega})\hat{M}_{n-2}, \tag{30}$$

for $n \geqslant 2$ with $\hat{M}_{-1}=0$ and $\hat{M}_0=\hat{\Omega}^{-1}$, in which $f=D_\alpha F_0 z_0$. Solution of these equations and substitution into Eq. (29) yields for Laplace transforms of time derivatives of the moments

$$\tilde{\dot{m}}_1(\epsilon) = f\langle z\Phi(\hat{\Omega})\rangle_z, \quad \tilde{\dot{m}}_2(\epsilon) = \tilde{\dot{\mu}}_0(\epsilon) + \tilde{\dot{\mu}}_2(\epsilon), \tag{31}$$

with

$$\tilde{\dot{\mu}}_0(\epsilon) = 2D_\alpha\langle\Phi(\hat{\Omega})\rangle_z, \quad \tilde{\dot{\mu}}_2(\epsilon) = f^2\langle(z\Phi(\hat{\Omega}))^2\rangle_z. \tag{32}$$

Here $\Phi(\hat{\Omega})=\hat{\Omega}^{-1}M(\hat{\Omega})$.

In what follows we will concentrate on the analysis of force-dependent terms. The force-independent contribution $\tilde{\dot{\mu}}_0(\epsilon)$ was discussed in detail earlier [4].

After the inverse Laplace transformation of expressions (31) and (32) one gets $\dot{\mu}_0(t)=2D_\alpha\phi(t)$,

$$\dot{m}_1(t) = f\iint d\mathbf{z}d\mathbf{z}_i z g(\mathbf{z},\mathbf{z}_i|t)\sigma_i(\mathbf{z}_i), \tag{33}$$

$$\dot{\mu}_2(t) = f^2 \iiint d\mathbf{z}d\bar{\mathbf{z}}d\mathbf{z}_i z\bar{z}\int_0^t d\bar{t}\, g(\mathbf{z},\bar{\mathbf{z}}|t-\bar{t})g(\bar{\mathbf{z}},\mathbf{z}_i|\bar{t})\sigma_i(\mathbf{z}_i). \tag{34}$$

In these formulas

$$g(\bar{\mathbf{z}},\bar{\mathbf{z}}|t) = \phi(t)\langle\mathbf{z}|e^{-\hat{L}t}|\bar{\mathbf{z}}\rangle, \tag{35}$$

where $\langle\mathbf{z}|e^{-\hat{L}t}|\bar{\mathbf{z}}\rangle$ is controlled by the model of $z(t)$ modulation, while $\phi(t)=(2\pi i)^{-1}\int_{-i\infty}^{i\infty}d\epsilon\epsilon^{-1}M(\epsilon)e^{\epsilon t}$ is determined by the CTRW model considered and for the subdiffusion model $[M(\epsilon)=\epsilon^{1-\alpha},(\alpha<1)]$

$$\phi(t) = \Gamma^{-1}(\alpha)t^{\alpha-1}. \tag{36}$$

B. Harmonically oscillating force

In the model of harmonically oscillating force (11) in which $\hat{L}$ [Eq. (12)] describes dynamical motion in the harmonic potential, one gets $\langle\mathbf{z}|e^{-\hat{L}t}|\bar{\mathbf{z}}\rangle=\delta[\mathbf{z}-\mathbf{z}_c(\bar{\mathbf{z}}|t)]$, where $\mathbf{z}_c(\bar{\mathbf{z}}|t)=(z_c(\bar{\mathbf{z}}|t),v_{z_c}(\bar{\mathbf{z}}|t))$ is the trajectory of dynamical motion with $\mathbf{z}_c(t=0)=\bar{\mathbf{z}}$ in the phase space $\{\mathbf{z}\}$.

Substituting this formula into Eqs. (33)–(35) one obtains $\mu_0(t)=2D_\alpha\Gamma^{-1}(1+\alpha)t^\alpha$,

$$m_1(t) = f\int_0^t d\tau z_c(\tau)\phi(\tau), \tag{37}$$

$$\mu_2(t) = f^2\int_0^t d\bar{t}z_c(\bar{t})\int_0^{\bar{t}} d\tau z_c(\tau)\phi(\bar{t}-\tau)\phi(\tau), \tag{38}$$

where $z_c(t)$ is given by Eq. (11).

For brevity, we will restrict ourselves to the discussion of the long time behavior of the moments only:

$$m_1(t) \overset{t\to\infty}{\simeq} (f/\omega^\alpha)\sin(\varphi+\pi\alpha/2), \tag{39}$$

$$\mu_2(t) \overset{t\to\infty}{\simeq} \gamma_2(\alpha)f^2(t/\omega)^\alpha, \tag{40}$$

where $f=D_\alpha F_0 z_0$ and $\gamma_2(\alpha)=\cos(\pi\alpha/2)/[2\Gamma(1+\alpha)]$.

These formulas predict some peculiarities of the subdiffusion response to oscillating force. First, $m_1(t)$ appears to be nonzero with asymptotic value (at $t\to\infty$) independent of time and harmonically oscillating as a function of the initial phase ϕ of force oscillations. Second, the force-dependent part $\mu_2(t)$ is anomalously large, increasing in time, so that $\mu_2(t)/\mu_0(t)=\cos(\pi\alpha/2)[f^2/(4D_\alpha\omega^\alpha)]$ is independent of time. Third, in the case of conventional diffusion, i.e., at $\alpha\to 1$, $\mu_2(t)/t^\alpha\to 0$, as should be.

The analysis of exact formulas (39) and (40) shows that for $\phi=0$ they are similar to those derived in Ref. [5] (as expected) except for a slight difference in the analytical representation of results.

C. Stepwise oscillating force

Here we will briefly discuss the model of stepwise oscillating force. In this model the equation proposed in Ref. [5] is found to be correct [6]. The exact method allows one to check the results obtained in Ref. [6].

The model is defined as $z(t)=z_0(-1)^{[2t/\tau_0]}$, where τ_0 is the oscillation period and $[x]$ denotes the integer part of x. It can also be represented as $z(t)=z_c(t)=\Sigma_{n=-\infty}^{\infty}z_n e^{in\omega t}$, where $\omega=2\pi/\tau_0$ and z_n are given by $z_{2n}=0$, $z_{2n+1}=-2i/[\pi(2n+1)]$ with $z_{-n}=z_n^*$.

As mentioned above, the case of $z_c(t)$ represented as a superposition of harmonically oscillating functions can be described by assuming $z_c(t)$ modulation to result from dynamical motion in a harmonic potential in the multidimensional space $\mathbf{z}=(z_1,v_{z_1};\ldots;z_n,v_{z_n};\ldots)$. In this case formulas

(33) and (34) predict the same expressions (37) and (38) for the moments, which yield

$$m_1(t) \overset{t\to\infty}{\simeq} \bar{\gamma}_1(\alpha)(2f/\omega^{\alpha}), \tag{41}$$

$$\mu_2(t) \overset{t\to\infty}{\simeq} \bar{\gamma}_2(\alpha)f^2(t/\omega)^{\alpha}. \tag{42}$$

Here the functions $\bar{\gamma}_1(\alpha)$ and $\bar{\gamma}_2(\alpha)$ are defined as $\bar{\gamma}_1(\alpha)=2[1-2^{-(1+\alpha)}]\zeta(1+\alpha)\sin(\pi\alpha/2)$, $\bar{\gamma}_2(\alpha)=[1-2^{-(2+\alpha)}]\zeta(2+\alpha)\Gamma^{-1}(1+\alpha)$, where $\zeta(x)$ is the Riemann's zeta function.

Obtained results agree with those of Ref. [6] thus confirming the correctness of equations applied in this work.

D. Fluctuating force

Of great interest is also the case of fluctuating force $F(t)$, i.e., Markovian fluctuating $z(t)$, which can easily be analyzed with the use of the proposed approach. In this case the moments $m_1(t)$ and $\mu_2(t)$ can be obtained in analytical form under fairly general assumptions on the nature of $z(t)$ fluctuations. The operator $\hat{L}$, which determines the properties of Markovian fluctuations [see Eq. (7)], is assumed to have the equilibrium state $\sigma_e(z)$ and the initial distribution in $\{z\}$ space is taken to be equilibrium: $\sigma_i(z)=\sigma_e(z)$. We also assume that $z(t)$ is a stochastic process with vanishing mean value, $\langle z(t)\rangle=\langle\sigma_e|z(t)|\sigma_e\rangle=0$, which actually means that $m_1(t)=0$ (in this bra-ket notation $\langle\sigma_e|\equiv\int d\mathbf{z}$ [7] and $|\sigma_i\rangle=|\sigma_e\rangle$).

The behavior of the second moment $\mu_2(t)$ essentially depends on that of the pair correlation function $K(t)=\langle z(0)z(t)\rangle=\langle\sigma_e|ze^{-\hat{L}t}z|\sigma_e\rangle$ or more exactly on its asymptotic behavior at $t\to\infty$. To understand the main specific features of the function $\mu_2(t)$ we will consider the simple model

$$K(t)=\langle z^2\rangle/[1+(t/\tau_c)^{\beta}], \tag{43}$$

where τ_c is a characteristic correlation time. The time dependence $\mu_2(t)$ appears to be substantially different for two cases:

(a) rapidly decreasing $K(t)$, $\beta>\alpha$,

(b) slowly decreasing $K(t)$, $\beta\leqslant\alpha$.

Both cases can conveniently be analyzed using Eqs. (31) and (32) for the Laplace transforms.

For brevity, we consider only the long time limit (i.e., the limit $\epsilon\to 0$) in which

$$\tilde{\mu}_2(\epsilon)\approx f^2\epsilon^{-1}\Phi(\epsilon)\langle\sigma_e|\, z\Phi(\hat{\Omega})z)|\sigma_e\rangle, \tag{44}$$

where $|\sigma_e\rangle$ and $\langle\sigma_e|$ are the equilibrium states of the operator $\hat{L}$.

With the use of this expression one can easily get the analytical expressions for $\mu_2(t)$ for both types of $K(t)$ dependence.

1. Rapidly decreasing K(t), β>α

In this case

$$m_1(t)=0 \text{ and } \mu_2(t) \overset{t\to\infty}{\simeq} \Gamma^{-1}(1+\alpha)\bar{f}^2(\bar{\tau}t)^{\alpha}, \tag{45}$$

where $\bar{f}=D_{\alpha}F_0\sqrt{\langle z^2\rangle}$ and the characteristic time $\bar{\tau}$ is expressed in terms of the correlation function $K(t)=\langle z(0)z(t)\rangle=\langle\sigma_e|ze^{-\hat{L}t}z|\sigma_e\rangle$:

$$\bar{\tau}^{\alpha}=\int_0^{\infty}dtK(t)\phi(t)/\langle z^2\rangle. \tag{46}$$

Notice that the inequality $\beta>\alpha$ ensures finiteness of the integral in Eq. (46) and thus finiteness of $\bar{\tau}$.

It is seen that for $\beta>\alpha$ the characteristic features of $\mu_2(t)$ are similar to those obtained in the case of oscillating force with f and ω replaced by $\bar{f}$ and $\bar{\tau}^{-1}$, respectively. Noteworthy is, however, that unlike this case, for fluctuating force $\mu_2(t)/t^{\alpha}$ is finite at $\alpha\to 1$, as expected. Note also that for $\beta>\alpha$ the specific properties of $z(t)$ fluctuations manifest themselves only in the value of the characteristic time $\bar{\tau}$, i.e., actually, in the amplitude of the asymptotic dependence $\mu_2(t)$.

2. Slowly decreasing K(t), β⩽α

In the case of slow decay of $K(t)$ [when the integral (46) diverges] the properties of force fluctuations determine not only the amplitude of the function $\mu_2(t)$ but also the form of time dependence itself.

In accordance with formula (44) the behavior of $\tilde{\mu}_2(\epsilon\to 0)$ is essentially depends on

$$\langle\sigma_e|\, z\Phi(\hat{\Omega})z)|\sigma_e\rangle=\int_0^{\infty}dte^{-\epsilon t}K(t)\phi(t)\sim\epsilon^{\beta-\alpha}. \tag{47}$$

Substitution of this relation into Eq. (44) and subsequent inverse Laplace transformation yields

$$m_1(t)=0 \text{ and } \mu_2(t) \overset{t\to\infty}{\sim} t^{2\alpha-\beta}. \tag{48}$$

For $\beta=\alpha$ this expression predicts the time dependence $\mu_2(t)\sim t^{\alpha}$ coinciding with that obtained at $\beta>\alpha$ [Eq. (45)], as expected.

Of certain interest is the limit $\beta\to 0$, corresponding to the case of static fluctuations, in which $\mu_2(t)\sim t^{2\alpha}$. This behavior can easily be understood taking into account that in the static limit the second moment $\mu_2(t)$ is directly related to the square of the first moment $m_1(z,t)$ for fixed, i.e., time-independent, parameter z (for time-independent force F), averaged over distribution of z (over distribution of forces): $\mu_2(t)\sim\langle m_1^2(z,t)\rangle_z$. In this relation the time dependence $m_1(z,t)$ can be obtained, for example, with use of Eq. (37) with $z_c(t)=\text{const}$, i.e., with $z_c(t)$ independent of time: $\langle m_1(z,t)\rangle\sim t^{\alpha}$. Therefore we arrive at the estimation $\mu_2(t)\sim t^{2\alpha}$ which agrees with Eq. (48) (for $\beta=0$).

Thus, according to formula (48), in the limit of anomalously slow decrease of the correlation function $K(t)$ of $z(t)$ fluctuations the increase of the parameter β, i.e. more fast decay of $K(t)$, results in pronounced slowing down of the growth of the second moment $\mu_2(t)$ with time.

VI. CONCLUDING REMARKS

We analyzed the effect of a time-dependent field on CTRWs using the rigorous method based on the Markovian representations of the CTRW and a time-dependent field. With the use of this method the rigorous non-Markovian SLE is derived, which describes the effect of a time-dependent field on non-Markovian CTRWs. In the case of deterministic (dynamical) time dependence the obtained equation turns out to coincide with the those proposed in Refs. [5,6]. For stochastic time dependence, however, the equation, proposed earlier, is inapplicable and the effect of the fluctuating force can be treated only by means of the non-Markovian SLE.

The rigorous SLE is applied to describing the field effect on subdiffusive motion. Obtained formulas (39)–(48) demonstrate some interesting specific features of the response of anomalous subdiffusive systems:

(i) In the case of deterministic time dependence of the force, i.e., deterministic dependence $z(t)$, the first moment (average displacement) is, in general, nonzero (even in the long time limit) depending on the oscillation phase. The time-dependent force results in the anomalously strong force-dependent contribution to the second moment $\mu_2(t)$, which increases with time. At long times the dependence on time $\mu_2(t)$ is universal and is determined only by the diffusion anomaly parameter $\alpha(\alpha<1)$: $\mu_2(t)\sim t^{\alpha}$.

(ii) For stochastic time dependence of the force $F(z(t))$ the contribution to the second moment $\mu_2(t)$ also grows with time but the function form is not universal, depending on the parameter β which characterizes the decay of the pair correlation of $z(t)$ fluctuations [see Eq. (43)]: $\mu_2(t)\sim t^{\alpha}$ for $\beta>\alpha$ and $\mu_2(t)\sim t^{2\alpha-\beta}$ for $\beta\leqslant\alpha$.

The results obtained in our work are not only of principle value but also of certain interest for applications. A number of possible applications are discussed in reviews [3,4]. In addition, also worth mentioning are recent investigations of the kinetics of charge carrier recombination and transient conductivity in some disordered semiconductor, in which the mobility of charge carriers was shown to be dispersive (subdiffusive) [9,10]. The proposed theoretical approach can be very helpful for the interpretation of the results of such investigations.

In conclusion, it is worth noting that the proposed Markovian SLE approach (10) for describing the influence of time-dependent fields is applied only to one particular problem of the theory of force induced effects in stochastic systems. This approach is, however, fairly general and can be very suitable in studying many time-dependent-field affected stochastic processes [1,2,11] since it reduces the study to the analysis of characteristic features of time-independent operators (their spectra, eigenfunctions, etc.). Some applications of the SLE approach (10) are currently under consideration.

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