Diffusion and relaxation controlled by tempered α-stable processes

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We derive general properties of anomalous diffusion and nonexponential relaxation from the theory of tempered α-stable processes. The tempering results in the existence of all moments of operational time. The subordination by the inverse tempered α-stable process provides diffusion (relaxation) that occupies an intermediate place between subdiffusion (Cole-Cole law) and normal diffusion (exponential law). Here we obtain explicitly the Fokker-Planck equation and the Cole-Davidson relaxation function. This model includes subdiffusion as a particular case.

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I. INTRODUCTION

Many studies have been reported on the phenomenon of subdiffusion which is typically observed when, due to dominating influence of traps (see [1–3] and references therein), the waiting times of random walks become α stable, $\alpha < 1$, with an infinite mean. However, this picture is only an idealization of the physical world. In reality the time of trap life can be restricted. It should be taken into account that the traps can be located in some spatial regions from which a walker may easily escape. Consequently, in a more general representation, the random walks start as subdiffusion, but their characteristics become very similar to those of normal diffusion at large time scales. One of such clear examples is a random motion of bright points (BPs) associated with magnetic fields at the solar photosphere. The BPs transport in the intergranular lanes with times less than 20 min has a subdiffusive character, but the analysis of the BPs motion supports the normal diffusion behavior for larger times. The experimental result is reported in [4]. The anomalous diffusion at short times and normal at long times was also noticed in [5,6] for molecules diffusing in living cells. The experiments cover extremely large time scales (more than 5 decades) and different experimental conditions. Then the subdiffusive regimes can cover up to 3 decades.

The present paper is just devoted to such a problem. For this purpose we are going to apply the tempered α-stable processes for the description of diffusion and relaxation. In comparison with the purely α-stable process such a process has finite moments, but it saves some important rudiments of the stable process too [7–9]. Therefore if its inverse process is taken as a subordinator, it provides then a diffusive picture occupying an intermediate place between subdiffusion and normal diffusion. We begin with a description of the tempered α-stable process and its inverse. This allows one to build a subordinated process responsible for anomalous diffusion and nonexponential relaxation. Then we study their properties following from the subordinator features.

It should be noted that our methods and results differ from the earlier interesting investigations presented in [10,11]. These studies do not include the tempered α-stable inverse subordinator and their results give opposite asymptotic properties for subdiffusion. To develop a particle tracking solution to the movement of passive tracers in heterogeneous media, a tempered model has been considered recently in [12] to capture the slow convergence of subdiffusion to a diffusion limit. However, the model is only asymptotically equivalent to ours as $t \to 0$ or $t \to \infty$.

II. CONTINUOUS TIME RANDOM WALKS AND SUBORDINATION

The model of subdiffusion is based on a flexible Montroll-Weiss idea on continuous time random walks (CTRWs) [13]. Briefly, the representation of anomalous diffusion by means of the CTRWs methodology is the following (see [14–16]). Consider a sequence T_i, $i=1,2,\ldots$, of non-negative, independent, identically distributed (IID) random variables which represent waiting-time intervals between subsequent jumps of a walker. The random time interval of n jumps in space equals $T(n)=\sum_{i=1}^{n} T_i$ with $T(0)=0$. The random number N_t of jumps, performed by the walker up to time $t>0$, is determined by the largest index n for which the sum of n interjump time intervals does not exceed the observation time t, namely $N_t = \max\{n : T(n) \leq t\}$. The position of the walker after the random number N_t of jumps, performed by the walker up to time $t>0$, becomes then

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$$R(N_t) = \sum_{i=1}^{N_t} R_i,$$

where R_i are IID variables giving both the length and the direction of the ith jump. The process $R(N_t)$ is just known as the CTRW. In a more general case the variables R_i can be multidimensional vectors.

If the time intervals T_i belong to the domain of attraction of a completely asymmetric α-stable distribution with the index $0<\alpha<1$, the generalization of the central limit theorem yields the continuous limit $a^{-1/\alpha}T([a\tau])\overset{d}{\rightarrow}U_\tau$ as $a\rightarrow\infty$, where U_τ is a strictly increasing α-stable Lévy process, $a>0$ parameter, $[x]$ denotes the integer part of x, and "$\overset{d}{\rightarrow}$" means "tends in distribution." Similarly, let the jumps R_i belong to the domain of attraction of a γ-stable distribution $S_{\gamma,\beta}(x), 0<\gamma\leqslant 2, |\beta|\leqslant 1$ so that $a^{-1/\gamma}R([a\tau])\overset{d}{\rightarrow}X_\tau$ as $a\rightarrow\infty$, where X_τ is a γ-stable Lévy process known as the parent process. If $\gamma=2$, the parent process is the classical Brownian motion. Both the process U_τ and the process X_τ are indexed by random operational (internal) time τ. In order to find a particle position at the observable time t, we have to introduce the notion of the inverse-time α-stable subordinator S_t relating the internal and the observable times,

$$a^{-\alpha}N_{at}\overset{d}{\rightarrow}S_t = \inf\{\tau: U_\tau > t\} \tag{1}$$

as $a\rightarrow\infty$. Then, as $a\rightarrow\infty$, the continuous limit of the CTRW process $R(N_t)$ obtains the following form:

$$a^{-\alpha/\gamma}R(N_{at}) \approx (a^\alpha)^{-1/\gamma}R([a^\alpha S_t])\overset{d}{\rightarrow}X(S_t), \tag{2}$$

known as the anomalous diffusion process [14], directed by the inverse α-stable subordinator S_t. It should be pointed out that the process U_τ does not have any finite p moments for $p\leqslant\alpha$. Therefore the subdiffusion is characterized by a power mean-square displacement in time [15–17].

III. TEMPERED α-STABLE PROCESS AND ITS INVERSE

However, there are physical phenomena, for example, the random motion of BPs in intergranular lanes on the Sun, where it would be desirable to get also a model that overcomes the infinite-moment difficulty while preserving the subdiffusive behavior for short times [18]. The remedy was first proposed in the physical literature by Mantegna and Stanley [19]. Their idea of truncated Lévy flights served as a model for random phenomena which exhibit at small scales properties similar to those of Lévy flights, but have distributions which at large scales have cutoffs and thus have finite moments of any order. Koponen [20], building on Mantegna and Stanley's ideas, defined the smoothly truncated Lévy flights which stressed the advantage of a nice analytic form. Independently, the same family of distributions was described earlier by Hougaard [21] in the context of a biological application. However, different methods for the truncation were suggested also in the economic and statistical sciences [22–24], but until Rosiński's paper [7] there was a lack of invariance under linear transformations for the distributions introduced, a significant property that the α-stable laws possess. He succeeded in finding the appropriate class of tempered stable distributions and processes [9,24].

Motivated by [8] we discuss properties of a diffusion process which is related to an inverse tempered α-stable subordinator. The Laplace image for the probability density function (PDF) of a tempered non-negative α-stable variable is

$$\tilde{f}(u) = \exp[\delta^\alpha - (u+\delta)^\alpha], \tag{3}$$

where δ is a positive constant and $0<\alpha<1$ [8]. If δ equals zero, the tempered α-stable PDF becomes simply α-stable. Equation (3) describes probabilistic properties of the tempered α-stable Lévy process $U(\tau)$ which generalizes the above-mentioned process U_τ.

Next, we will find its inverse process $S(t)$ as in Eq. (1), where $U(\tau)$ substitutes U_τ. If $f(t,\tau)$ is the PDF of $U(\tau)$, then the PDF $g(\tau,t)$ of its inverse $S(t)$ can be represented as

$$g(\tau,t) = -\frac{\partial}{\partial\tau}\int_{-\infty}^{t} f(t',\tau)dt'.$$

Taking the Laplace transform of $g(\tau,t)$ with respect to t, we get

$$\tilde{g}(\tau,u) = \frac{(u+\delta)^\alpha - \delta^\alpha}{u}e^{-\tau[(u+\delta)^\alpha-\delta^\alpha]}. \tag{4}$$

When $u\gg 1$ $(t\ll 1)$ or $\delta\rightarrow 0$, Eq. (4) tends to

$$\tilde{g}(\tau,u) = u^{\alpha-1}e^{-\tau u^\alpha}, \tag{5}$$

which is the Laplace image of an inverse α-stable PDF typical for subdiffusion. If $u\ll 1$ $(t\gg 1)$ or $\alpha\rightarrow 1$, then Eq. (4) becomes the Laplace image of the Dirac delta function. It follows from Eq. (5) that the PDF of the inverse α-stable process is

$$g(\tau,t) = \frac{1}{2\pi i}\int_{\mathrm{Br}} e^{ut-\tau u^\alpha}u^{\alpha-1}du = t^{-\alpha}F_\alpha(\tau/t^\alpha),$$

where Br denotes the Bromwich path, and the function $F_\alpha(z)$ has a Taylor series expansion

$$F_\alpha(z) = \sum_{k=0}^{\infty}\frac{(-z)^k}{k!\Gamma(1-\alpha-k\alpha)}.$$

For more details see [15,25]. It should be recalled that this function is a specific case of the Wright function

$$W(a,b;z) = \frac{1}{2\pi i}\int_H e^{s+zs^{-a}}s^{-b}ds,$$

where H denotes the Hankel path with a cut along the negative real semiaxis around zero [26]. The latter plays a key role in derivation of $g(\tau,t)$ for $\delta\neq 0$.

Using general properties of the Laplace transform

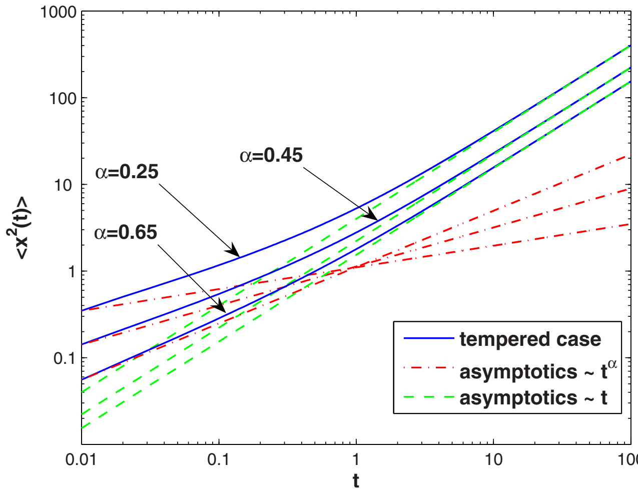

FIG. 1. (Color online) Mean-square displacement of anomalous diffusion subordinated by an inverse tempered α-stable process. Observed already in [8] for $\alpha=0.5$.

$$\int_0^t q(t')dt' \xrightarrow{\mathcal{L}} \frac{1}{s}Q(s), \quad e^{-at}h(t) \xrightarrow{\mathcal{L}} H(s+a),$$

Eq. (4) can be expressed in terms of the Wright functions, namely

$$g(\tau,t) = e^{\tau\delta^\alpha}\int_0^t [W(-\alpha,-\alpha;\tau/x^\alpha)x^{-\alpha-1} - \delta^2 W(-\alpha,0;\tau/x^\alpha)/x]e^{-\delta x}dx.$$

This description generalizes continuous time random walks of anomalous diffusion in the case when the subordinator is an inverse tempered α-stable process. To derive properties of the process $X[S(t)]$ we should assume more detailed features of the process $X(\tau)$.

IV. SUBORDINATION BY AN INVERSE TEMPERED α-STABLE PROCESS

If now the independent identically distributed random space jumps $X_1, X_2, \ldots$ have either Gaussian or Lévy distribution, then formally we can pass from the discrete representation to the continuous limit in the form of the stochastic process $X(\tau)$ depending on the internal time τ. Let the parent process $X(\tau)$ have the PDF $h(x,\tau)$. Then the PDF of the subordinated process $X[S(t)]$ obeys the integral relationship between the probability densities of the parent and directing processes, $X(\tau)$ and $S(t)$, respectively,

$$p(x,t) = \int_0^\infty h(x,\tau)g(\tau,t)d\tau. \tag{6}$$

In fact, this relation connects the probability to find a particle at x on the operational time τ with the probability for the operational time τ to be equal to real time t. In the Laplace space the probability density $p(x,t)$ has the most simple form. Based on the form, it is easy to calculate all the moments of $p(x,t)$. Taking into account Eq. (4), the Laplace transform of Eq. (6) with respect to t gives

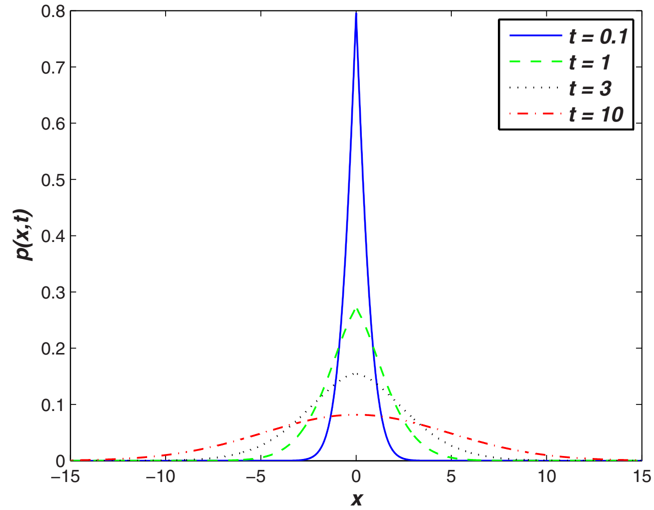

FIG. 2. (Color online) Propagator $p(x,t)$ for the tempered Gaussian diffusion with a constant potential, $\alpha=2/3$ and $\delta=0.5$, drawn for consecutive dimensionless instances of time $t=0.1$, 1, 3, and 10. The cusp shape of the PDF disappears.

$$\tilde{p}(x,u) = \frac{(u+\delta)^\alpha - \delta^\alpha}{u}\tilde{h}[x,(u+\delta)^\alpha - \delta^\alpha]. \tag{7}$$

For $\delta=0$ the latter expression becomes $u^{\alpha-1}\tilde{h}(x,u^\alpha)$.

If the moments of the process $X(\tau)$ are known, it is not difficult to find the moments of the process $X[S(t)]$. For example, for the Gaussian process ($\gamma=2$) the second moment is $\langle X^2(\tau)\rangle = D\tau$, where D is a diffusive constant. Then the mean-square displacement of $X[S(t)]$ can be written as

$$\langle X^2[S(t)]\rangle = \int_0^\infty \langle X^2(\tau)\rangle g(\tau,t)d\tau.$$

The Laplace image $\langle \tilde{X}_u^2\rangle$ of $\langle X^2[S(t)]\rangle$ has the form

$$\langle \tilde{X}_u^2\rangle = \frac{D}{u[(u+\delta)^\alpha - \delta^\alpha]}. \tag{8}$$

Consequently, using properties of the Laplace transform and after simple algebraic transformations, the inverse Laplace transform of Eq. (8) reads

$$\frac{1}{u[(u+\delta)^\alpha - \delta^\alpha]} \xrightarrow{\mathcal{L}^{-1}} \int_0^t e^{-\delta y}y^{\alpha-1}E_{\alpha,\alpha}(\delta^\alpha y^\alpha)dy, \tag{9}$$

where

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty z^k/\Gamma(\alpha k+\beta), \quad \alpha>0, \quad \beta>0$$

is the Mittag-Leffler function [25]. The function (9) gives rise to interesting asymptotic properties of the mean-square displacement $\langle X^2[S(t)]\rangle$. For $t \ll 1$ this displacement behaves as $Dt^\alpha/\Gamma(\alpha+1)$, but for $t \gg 1$ it increases linearly in time Dt/α (see Fig. 1). Thus the anomalous diffusion, governed by the inverse tempered α-stable subordinator, occupies an intermediate place between subdiffusion and the normal diffusion. For short times it behaves as subdiffusion whereas for

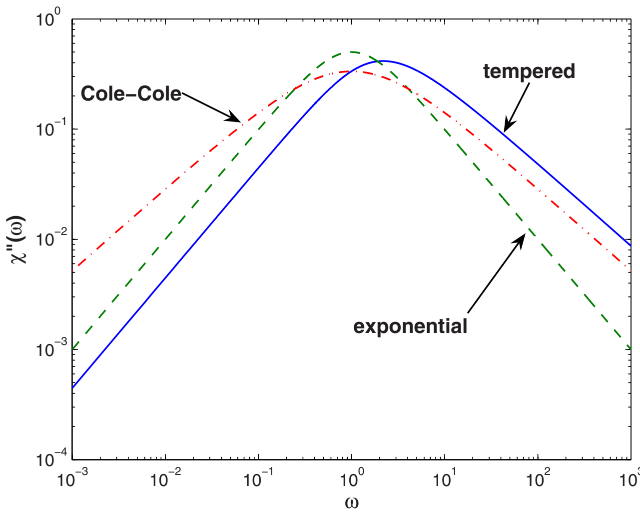

FIG. 3. (Color online) Absorption (imaginary) term of the frequency-domain relaxation function $\chi(\omega)=\chi'(\omega)-i\chi''(\omega)$ for $\alpha=0.75$.

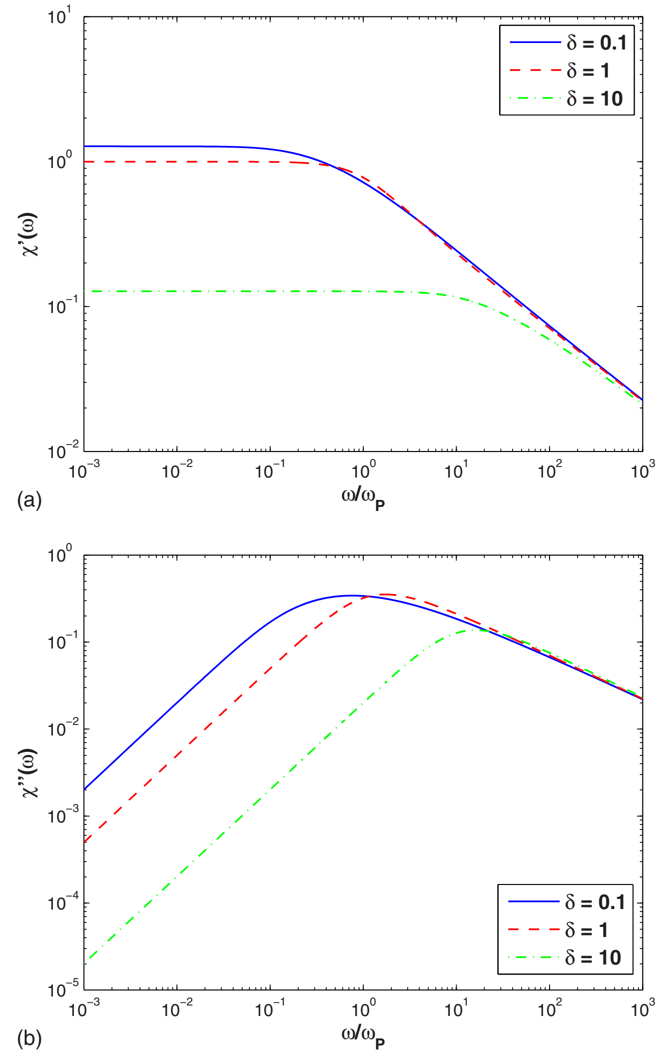

FIG. 4. (Color online) Frequency-domain relaxation function $\chi(\omega)=\chi'(\omega)-i\chi''(\omega)$ with different values δ for $\alpha=0.5$.

the long times it resembles the properties of the normal diffusion. Let us call the diffusion subordinated by the inverse tempered α-stable process a "tempered subdiffusion." As is well known [11], the inverse α-stable process accounts for the amount of time when a walker does not participate in a motion. If the walker only randomly moves all time ($\alpha=1$), the internal time τ coincides with the physical time t. By analogy, we may conclude that the process $S(t)$ for the tempered subdiffusion represents a case when a walker does not participate in a motion only for restricted intervals of time. At large time scales the walker begins to move randomly without any stopping as if $\alpha=1$.

V. EQUATION OF TEMPERED SUBDIFFUSION

Let $\hat{L}(x)$ be a time-independent Fokker-Planck operator, whose exact form is not important here. Let the ordinary Fokker-Planck equation (FPE),

$$\partial h(x,\tau)/\partial\tau=\hat{L}(x)h(x,\tau),$$

describe evolution of a particle subject to the operation time τ. Acting by the operator $\hat{L}(x)$ on the image $\tilde{p}(x,u)$ from Eq. (7), we find

$$\hat{L}(x)\tilde{p}(u,x)=[(u+\delta)^{\alpha}-\delta^{\alpha}]\tilde{p}(x,u)-q(x)\frac{[(u+\delta)^{\alpha}-\delta^{\alpha}]}{u}, \tag{10}$$

where $q(x)$ is an initial condition. When $\delta=0$, the inverse Laplace transform of the latter expression gives a fractional representation of the FPE [1,15],

$$p(x,t)=q(x)+\frac{1}{\Gamma(\alpha)}\int_0^t d\tau(t-\tau)^{\alpha-1}\hat{L}(x)p(x,\tau). \tag{11}$$

In the case of the tempered subdiffusion the kernel in the integral representation of the FPE will be more complex, containing as a special case the kernel of Eq. (11) for $\delta\to 0$. Using the formal integral representation [27] of the FPE,

$$p(x,t)=q(x)+\int_0^t d\tau M(t-\tau)\hat{L}(x)p(x,\tau), \tag{12}$$

and taking the inverse Laplace transform of Eq. (10), we obtain the explicit form of the kernel $M(t)$, namely

$$M(t)=e^{-\delta t}t^{\alpha-1}E_{\alpha,\alpha}(\delta^{\alpha}t^{\alpha}). \tag{13}$$

For $t\ll 1$ (or $\delta\to 0$) this function takes the power form $t^{\alpha}/\Gamma(\alpha)$ as the kernel in Eq. (11). However, for $t\gg 1$ (or $\alpha\to 1$) $M(t)$ becomes constant and, as a result, Eq. (12) transforms into the integral form of the ordinary FPE.

In Fig. 2, as an example, the propagator $p(x,t)$ for the tempered diffusion with $\alpha=2/3$ and $\delta=0.5$ is drawn. The cusp shape of the PDF disappears when time increases.

VI. TEMPERED RELAXATION

The commonly accepted theoretical approaches to model relaxation phenomena assume [1] decay of an excitation undergoing diffusion in the system under consideration. In this framework, the relaxation function $\phi(t)$ describes the temporal decay of a given mode k and can be expressed [16] through the Fourier transform of the diffusion process $X[S(t)]$,

$$\phi(t) = \langle e^{-kX[S(t)]} \rangle. \tag{14}$$

Here $k>0$ has the physical meaning of a wave number (the Fourier image of spatial coordinates). Starting with Eq. (4), we can write the Laplace image of Eq. (14) as

$$\tilde{\phi}(u) = \frac{[(u+\delta)^{\alpha} - \delta^{\alpha}]}{u[\Phi(k) + (u+\delta)^{\alpha} - \delta^{\alpha}]}, \tag{15}$$

where $\Phi(k)$ is the logarithm of the characteristic function of the process $X(\tau)$.

To expose the characteristic properties of the "tempered relaxation" we use the frequency-domain description [28,29] of the relaxation phenomenon

$$\chi(\omega) = \int_0^{\infty} e^{-i\omega t}\left(-\frac{d\phi(t)}{dt}\right)dt. \tag{16}$$

Then, for the relaxation under the inverse tempered α-stable process the function (16) takes the form

$$\chi(\omega) = \frac{1}{1 - \sigma^{\alpha} + (i\omega/\omega_p + \sigma)^{\alpha}}, \tag{17}$$

where $0 \leqslant \sigma < \infty$ is a constant, and ω_p is the characteristic frequency of the relaxing system. The value σ is proportional to δ.

The above frequency-domain relaxation function for $\sigma=0$ recovers the Cole-Cole law. If $\alpha=1$, the dependence (17) becomes Debye's. In the case of $\sigma=1$ it has the Cole-Davidson form. The relaxation directed by the inverse tempered α-stable process just takes an intermediate place between the superslow relaxation and the exponential one (see Fig. 3). Such a type of evolution is observed in relaxation experiments (see, for example, [28]).

Let us mention that for any positive $\sigma \neq 0$ the absorption (imaginary) term of the frequency-domain relaxation function $\chi(\omega)=\chi'(\omega)-i\chi''(\omega)$ behaves as $\chi''(\omega) \sim \alpha\sigma^{\alpha-1}\omega/\omega_p$ for $\omega/\omega_p \ll \sigma$ and $\chi''(\omega) \sim \omega_p^{\alpha} \sin(\pi\alpha/2)/\omega^{\alpha}$ for $\omega/\omega_p \gg \sigma$. If $\sigma=0$, the value $\chi''(\omega)$ tends to ω^{α} for $\omega/\omega_p \ll 1$ and $\omega^{-\alpha}$ for $\omega/\omega_p \gg 1$. The evolution of the frequency-domain relaxation function $\chi(\omega)$ under different δ is presented in Fig. 4. Note that the shift of $\chi''(\omega)$ in the low frequency range depends on $\delta^{\alpha-1}$.

For $t \ll 1$ the relaxation function $\varphi(t)$, corresponding to the Cole-Cole law, decreases in its initial stage as a power $1-t^{\alpha}/\Gamma(\alpha+1)$. In the case of $t \gg 1$ and any positive $\delta \neq 0$ the function $\varphi(t)$ tends to an exponential law $\alpha\exp(-\delta^{1-\alpha}t)$. The relaxation function demonstrates clearly something intermediate in its evolution between the power-law decay and the exponential one.

VII. CONCLUSIONS

In summary we have developed an approach to anomalous diffusion and nonexponential relaxation from tempered α-stable processes. The model is broader than the purely subdiffusive case. It is very important that they both can be considered on the unique base following the theory of subordinated random processes. We have derived a tempered form [Eqs. (12) and (13)] of the FPE and we have calculated the mean-square displacement [Eqs. (8) and (9)] of anomalous diffusion subordinated by an inverse tempered α-stable process. Using the formal integral representation of the tempered FPE and taking the inverse Laplace transform of Eq. (10), we have obtained the explicit form of the kernel $M(t)$. Moreover, we have found the explicit form of the tempered relaxation function (17).

The cusp shape (characteristic for the subdiffusion) of the PDF of the tempered diffusion process disappears with time; see Fig. 2. Thus our model occupies an intermediate place between subdiffusion and normal diffusion. We expect that our results will yield insights into the coexistence of subdiffusion and normal diffusion in nature.

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