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# Maximum distributions of bridges of noncolliding Brownian paths

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One-dimensional Brownian motion starting from the origin at time $t=0$, conditioned to return to the origin at time $t=1$ and to stay positive during time interval $0<t<1$, is called the Bessel bridge with duration 1. We consider an $N$-particle system of such Bessel bridges conditioned never to collide with each other in $0<t<1$, which is the continuum limit of the vicious walk model in watermelon configuration with a wall. Distributions of maximum values of paths attained in the time interval $t\in(0,1)$ are studied to characterize the statistics of random patterns of the repulsive paths on the spatiotemporal plane. For the outermost path, the distribution function of maximum value is exactly determined for general $N$. We show that the present $N$-path system of noncolliding Bessel bridges is realized as the positive-eigenvalue process of the $2N\times 2N$ matrix-valued Brownian bridge in the symmetry class C. Using this fact, computer simulations are performed and numerical results on the $N$ dependence of the maximum-value distributions of the inner paths are reported. The present work demonstrates that the extreme-value problems of noncolliding paths are related to random matrix theory, the representation theory of symmetry, and number theory.

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## I. INTRODUCTION

Random walk (RW) models are important in physics, chemistry, and computer sciences. They can be used effectively when we explain basic concepts of statistical physics [1], stochastic processes in physics and chemistry [2], and stochastic algorithms [3]. In particular, RW models have been used to provide simple and useful models to discuss various physical phenomena in far from equilibrium, such as interface dynamics [4–6], polymer networks [7–9], wetting and melting transitions [10], and so on. If we take the proper spatiotemporal continuum limit, called the *diffusion scaling limit*, of the RW models, Brownian motion (BM) models are obtained. By virtue of the limit procedure, the BM models are enriched with mathematics. The following example of a conditional BM will demonstrate this statement [11–13].

Consider the one-dimensional standard BM $B(t)$, $t\geq 0$, where $\langle B(t)\rangle=0$ and $\langle B(s)B(t)\rangle=\min\{s,t\}$. The BM conditioned to stay positive $B(t)>0$, $0<t<\infty$, is called the *three-dimensional Bessel process*, abbreviated as BES(3), since it is equivalent in distribution with the radial part of the three-dimensional BM and its transition probability density is given by a special case of the modified Bessel function (see Sec. II A below and 3.3 C in [14], VI.3 in [15], and IV.34 in [16]). When we consider the case that it starts from the origin at time $t=0$ and returns to the origin at time $t=1$, this conditional BES(3) is called the *three-dimensional Bessel bridge with duration 1* starting from 0, since as illustrated by Fig. 1, the path drawn on the $(1+1)$-dimensional spatiotemporal plane seems to be a bridge over the time axis. (Note that by the scaling property between space and time of BM, no generality is lost by setting the time duration be 1.) Let $r(t)$, $0\leq t\leq 1$, denote the three-dimensional Bessel bridge. By symmetry, we can expect that the height of the bridge attains its maximum value with the highest probability at time $t=1/2$. The probability density for $r(t)\in dx$ at time $t=1/2$ is readily calculated as

$$p_{\mathrm{BESb}}(1/2,x)=8\sqrt{\frac{2}{\pi}}x^2e^{-2x^2},\quad 0\leq x<\infty \tag{1}$$

[see below Eq. (15)], which gives the moments $\langle[r(1/2)]^s\rangle=2^{-s}(s+1)!!$ if $s$ is even and $\langle[r(1/2)]^s\rangle=\{2/\sqrt{\pi}\}2^{-s/2}[(s+1)/2]!$ if $s$ is odd. The shape of the present bridge is, however, randomly deformed, and as we can see in Fig. 1 the time $0<\tau<1$, at which $r(t)$ attains its maximum, will fluctuate around the mean time $\langle\tau\rangle=1/2$. We define

$$H_1^{(1)}=\max_{0<t<1} r(t)=r(\tau).$$

We can show that the probability density for $H_1^{(1)}\in dh$ is given as

$$p_{H_1^{(1)}}(h)=8\sum_{n=1}^{\infty}e^{-2n^2h^2}(4n^4h^3-3n^2h),\quad 0\leq h<\infty, \tag{2}$$

which gives the moments

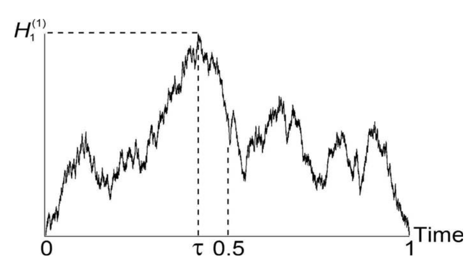


FIG. 1. A sample of path of the three-dimensional Bessel bridge with duration 1. In this example the time $\tau$, at which the height of the bridge attains its maximum, is less than 1/2.

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$$\langle (H_1^{(1)})^s \rangle = \frac{s(s-1)}{2^{s/2}} \Gamma(s/2)\zeta(s), \quad s = 0,1,2,\ldots,$$

with the gamma function $\Gamma(z) = \int_0^\infty e^{-u} u^{z-1} du$. Here $\zeta(s)$ is Riemann's zeta function,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

which is an important special function in number theory [12,17]. In the present paper, as multivariate generalization of the Bessel bridge [18], we study the $N$-path systems of the three-dimensional Bessel bridges with duration 1, conditioned never to collide with each other in $0<t<1$; $\mathbf{r}^{(N)}(t) = (r_1^{(N)}(t), r_2^{(N)}(t), \ldots, r_N^{(N)}(t))$, $0 \le t \le 1$, with the conditions $\mathbf{r}^{(N)}(0) = \mathbf{r}^{(N)}(1) = \mathbf{0}$ and $0 < r_1^{(N)}(t) < r_2^{(N)}(t) < \cdots < r_N^{(N)}(t)$, $0 < t < 1$.

The systems of RWs with nonintersecting condition were introduced by Fisher as mathematical models to describe the wetting and melting transitions and named *vicious walk models* [10]. They have been used not only to discuss the dynamics of domain walls and melting of commensurate surfaces [19], but also to study polymer networks [8,9], the related enumerative combinatorial problems [20–22], and nonequilibrium critical phenomena [23]. In general the noncolliding diffusion particle systems are obtained as the diffusion scaling limits of the vicious RW models [24–26]. In particular, the version of the vicious RW model, whose continuum limit is the *noncolliding Bessel bridges*, $\mathbf{r}^{(N)}(t)$, $0 \le t \le 1$, is called the *N-watermelons with a wall* [9,21,22,26,27].

In the discrete mathematics the one-dimensional RWs conditioned to visit only nonnegative sites $\mathbf{N}_0 = \{0,1,2,\ldots\}$ is called the Dyck paths and the asymptotics of the average height of the Dyck paths in the long-step limit was studied by de Bruijn, Knuth, and Rice [28]. Recently Fulmek generalized this classical result by evaluating the asymptotics of the average height of the 2-watermelons with a wall [29]. In this calculation, he showed the fact that the number-theoretical special functions, such as Jacobi's theta function and the double Dirichlet series, are useful to describe the asymptotics. Motivated by this important observation, the present authors [13] studied the $N=2$ case of the noncolliding Bessel bridges, $\mathbf{r}^{(2)}(t) = (r_1^{(2)}(t), r_2^{(2)}(t))$, which is the continuum limit of the 2-watermelons with a wall, and clarified that this phenomenon is indeed the generalization of the relationship between the maximum-value distribution of the three-dimensional Bessel bridge and Riemann's zeta function mentioned above.

We will report in this paper both the exact results and the numerical results on the maximum-value distributions of $N$ paths in the noncolliding Bessel bridges. The main exact result is the following determinantal expression for the maximum-value distribution of the outermost path (i.e., the height of the continuum limit of watermelons),

$$H_N^{(N)} = \max_{0<t<1} r_N^{(N)}(t), \tag{3}$$

for the $N$-path system $\mathbf{r}^{(N)}(t) = (r_1^{(N)}(t), r_2^{(N)}(t), \ldots, r_N^{(N)}(t))$;

$$\mathbf{P}(H_N^{(N)} < h) = \frac{(-1)^N}{2^{N^2}\prod_{j=1}^N (2j-1)!} \times \det_{1\le j,k\le N}\left[\sum_{n=-\infty}^{\infty} H_{2(j+k-1)}(\sqrt{2}nh) e^{-2n^2h^2}\right], \tag{4}$$

where $H_n(x)$ denotes the $n$th Hermite polynomial defined by

$$H_n(x) = n! \sum_{k=0}^{[n/2]} \frac{(-1)^k (2x)^{n-2k}}{k!(n-2k)!}, \tag{5}$$

with $[a]$=the largest integer that is not greater than $a$. [We can see that, since $H_2(x) = 4x^2 - 2$, Eq. (4) with $N=1$ and the relation $p_{H_1^{(1)}}(h) = d\mathbf{P}(H_1^{(1)} < h)/dh$ give the result (2).] The long-step asymptotics of all moments of the height of the $N$-watermelons with a wall have been fully studied for arbitrary $N \ge 1$ by Feierl [30,31]. We will show that our result (4) for the distribution functions of the continuous model is consistent with the results by Feierl for the moments of his discrete model. Quite recently Schehr *et al.* [32] showed their study on the same problem and others by the path-integral technique, a different method both from ours and Feierl's. They also reported an expression for the maximum-value distribution [Eq. (5) in their paper [32]], which is different from (4). We will show that the equivalence of these two expressions is guaranteed by the functional equation satisfied by Jacobi's theta function $\vartheta_3(x,y)$.

Dyson introduced a matrix-valued BM, $M(t) = [M_{jk}(t)]$, $t \ge 0$, in the space of Hermitian matrices. If the matrix size is $N$, the $N$ diagonal elements $M_{jj}(t)$, $1 \le j \le N$, the $N(N-1)/2$ real parts of the upper-triangle elements, $\mathrm{Re}[M_{jk}(t)]$, $1 \le j < k \le N$, and the $N(N-1)/2$ imaginary parts of them, $\mathrm{Im}[M_{jk}(t)]$, $1 \le j < k \le N$, are given by independent one-dimensional standard BMs, the total number of which is $N^2$. Dyson showed that the $N$ eigenvalues of $M(t)$ behaves as an interacting diffusion particle system on the real axis $\mathbf{R}$, in which the long-ranged repulsive forces work between any pair of particles with the strength proportional to the inverse of the distance of two particles [33]. This eigenvalue process is called Dyson's BM model, and it has been proved to be equivalent in distribution with the system of $N$-BMs conditioned never to collide with each other [34,35]. The correspondence between eigenvalue processes of matrix-valued diffusion processes and noncolliding particle systems has been studied [35–39]. In the present paper we will use the fact that the $N$-noncolliding Bessel bridges can be realized as the positive-eigenvalue process of the $2N \times 2N$ *matrix-valued Brownian bridge*, whose distribution at each time $0 < t < 1$ is related to the random matrix theory [40] with a special symmetry called the class C in [41,42]. Figure 2 shows a sample of paths of the $N=10$ noncolliding Bessel bridges realized by this eigenvalue process. One can imagine that it is very hard to simulate such paths all starting from the origin and returning to the origin with noncolliding condition by direct computer simulation. The present paper will demonstrate that the relationship between the random matrix ensembles and the noncolliding particle systems [39] provides a practical method to study such conditional processes effectively by computer simulations. We will report the numerical

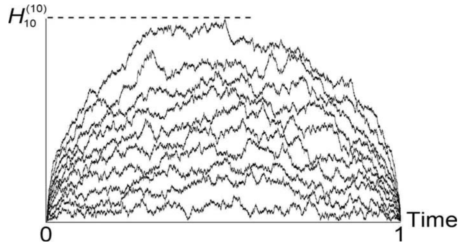


FIG. 2. A sample of paths of $N=10$ noncolliding Bessel bridges with duration 1, all starting from 0 and returning to 0, realized by the eigenvalue process.

evaluation of the maximum-value distributions of not only the outermost path $r_N^{(N)}(t)$, but also inner paths $r_k^{(N)}(t)$, $k=1,2,\dots,N-1$.

Both from the viewpoints of statistical physics and of random matrix theory [18,40], the study of the $N\to\infty$ limit is interesting and important for noncolliding paths [43–45]. For the average value of the maximum values of the outermost path, we can read the following behavior from the numerical work by Bonichon and Mosbah for the watermelons with a wall [46],

$$\langle H_N^{(N)}\rangle \simeq \sqrt{1.67N-0.06}, \quad N\to\infty. \tag{6}$$

Recently, Schehr *et al.* [32] gave an argument that the numerical data of Bonichon and Mosbah, relation (6), shows only a preasymptotic behavior at large $N$ and the true asymptotics should be

$$\langle H_N^{(N)}\rangle \simeq \sqrt{2N}, \quad N\to\infty. \tag{7}$$

In the present paper, we report the numerical study of the $N$ dependence of the maximum-value distributions systematically not only for the outermost path, but also for all inner paths, and discuss the $N\to\infty$ asymptotics based on our numerical data.

The paper is organized as follows. In Sec. II A, after giving brief explanations of the three-dimensional Bessel bridge and the Karlin-McGregor formula [47], we define the $N$-noncolliding Bessel bridges by giving the transition probability densities. A matrix representation of the symmetry class C is shown in Sec. II B and the matrix-valued Brownian bridge in the symmetry class C is introduced. The equivalence in distribution between its positive-eigenvalue process and the noncolliding Bessel bridges is then stated. The problems studied in this paper is announced in Sec. II C. In Sec. III A the exact expression of the distribution function of the maximum value for the outermost path, which is the same as Eq. (5) in [32], is derived by our method (Lemma 1). This exact expression is then transformed into two kinds of determinantal expressions (Proposition 2 and Theorem 4) in Sec. III B, one of which is Eq. (7) given above. The key lemma in the transformation is a set of equalities between infinite series involving the Hermite polynomials (Lemma 3) derived from the functional equation of Jacobi's theta function $\vartheta_3(x,y)$. The numerical study is reported in Sec. IV. Concluding remarks are given in Sec. V. Appendixes are used to give proofs of some formulas.

## II. MODELS AND PROBLEMS

### A. Transition probability density of noncolliding Bessel bridges

Let $B(t)$, $t\geqslant 0$, be the one-dimensional standard BM starting from the origin; $B(0)=0$. For any series of times $t_0\equiv 0 < t_1 < t_2 < \cdots < t_M$, $M=1,2,\dots$, the probability that the BM stays in interval $[a_m,b_m]$ at each time $t_m$, $m=1,2,\dots,M$, is given by

$$\mathbf{P}(B(t_m)\in[a_m,b_m], m=1,2,\dots,M) = \int_{a_1}^{b_1}dx_1\int_{a_2}^{b_2}dx_2\cdots\int_{a_M}^{b_M}dx_M \times\prod_{m=0}^{M-1}p(t_{m+1}-t_m,x_{m+1}|x_m),$$

where the transition probability density $p(t,y|x)$ from the position $x$ to $y$ during time period $t$ is given by the probability density of the normal distribution with mean 0 and variance $t$,

$$p(t,y|x)=\frac{1}{\sqrt{2\pi t}}\exp\left\{-\frac{(y-x)^2}{2t}\right\}, \quad t>0, \quad x,y\in\mathbf{R}.$$

We consider the situation that an absorbing wall is set at the origin and BM is absorbed if it arrives at the wall. By the reflection principle of BM [14], the transition probability density of such an absorbing BM is given by

$$p_{\text{abs}}(t,y|x)=p(t,y|x)-p(t,-y|x) = \frac{1}{\sqrt{2\pi t}}\{e^{-(y-x)^2/(2t)}-e^{-(y+x)^2/(2t)}\} \tag{8}$$

for $x,y>0$, $t>0$. The survival probability of the absorbing BM starting from $x>0$ for the time period $T$ is then given by

$$\mathcal{N}(T,x)=\int_0^\infty p_{\text{abs}}(T,y|x)dy,$$

whose asymptotics in $x/\sqrt{T}\to 0$ is easily evaluated as

$$\mathcal{N}(T,x)\simeq\sqrt{\frac{2}{\pi}}\frac{x}{\sqrt{T}} \quad \text{in} \quad \frac{x}{\sqrt{T}}\to 0.$$

The transition probability density of the BM under the condition that it stay forever in the positive region $\mathbf{R}_+=\{x\in\mathbf{R}:x>0\}$ is then given by

$$p_{\text{BES}(3)}(t,y|x)\equiv\lim_{T\to\infty}\frac{\mathcal{N}(T-t,y)p_{\text{abs}}(t,y|x)}{\mathcal{N}(T,x)}=\frac{y}{x}p_{\text{abs}}(t,y|x) \tag{9}$$

for $x>0$, $y\geqslant 0$, $0<t<\infty$. The diffusion process whose transition probability density is given by (9) is called the three-dimensional Bessel process, abbreviated as BES(3), for the following reasons. Consider the $d$-dimensional BM, $\mathbf{B}^{(d)}(t)=(B_1(t),B_2(t),\dots,B_d(t))$, whose coordinates are given by independent one-dimensional standard BMs $\{B_j(t)\}_{j=1}^d$. The distance from the origin of the $d$-dimensional BM,

$$R^{(d)}(t) = |\mathbf{B}^{(d)}(t)| = \sqrt{[B_1(t)]^2 + [B_2(t)]^2 + \cdots + [B_d(t)]^2},$$

can be regarded as a diffusion process in $\mathbf{R}_+ \cup \{0\}$ and its transition probability density is given by

$$p_{\mathrm{BES}(d)}(t,y|x) = \frac{y^{\nu+1}}{x^{\nu}} \frac{1}{t} e^{-(x^2+y^2)/(2t)} I_{\nu}\left(\frac{xy}{t}\right) \tag{10}$$

for $x>0$, $y \geq 0$, $t>0$ with

$$\nu = \frac{d-2}{2},$$

where $I_\nu(z) \equiv \Sigma_{n=0}^{\infty} (z/2)^{2n+\nu} / \{\Gamma(n+1)\Gamma(\nu+n+1)\}$ is the modified Bessel function with the gamma function $\Gamma(z) = \int_0^\infty e^{-u} u^{z-1} du$. The process $R^{(d)}(t)$ is called the $d$-dimensional Bessel process BES($d$) [14–16]. The transition probability density (9) of the BM conditioned to stay positive is equal to (10) with $d=3$, i.e., $\nu=1/2$, since $I_{1/2}(z) = (e^z - e^{-z})/\sqrt{2\pi z}$.

Consider the space of all configurations of $N$ particles in $\mathbf{R}_+$ with a fixed order of positions,

$$\mathbf{W}_N^{\mathrm{C}} = \{\mathbf{x} = (x_1, x_2, \ldots, x_N) \in \mathbf{R}_+^N : x_1 < x_2 < \cdots < x_N\},$$

which is called the Weyl chamber of type $\mathrm{C}_N$ in representation theory [48]. For $\mathbf{x} = (x_1, x_2, \ldots, x_N)$, $\mathbf{y} = (y_1, y_2, \ldots, y_N) \in \mathbf{W}_N^{\mathrm{C}}, t>0$, consider the determinant

$$\det_{1 \leq j,k \leq N} [p_{\mathrm{BES}(3)}(t, y_j | x_k)] = \prod_{j=1}^N \frac{y_j}{x_j} \det_{1 \leq j,k \leq N} [p_{\mathrm{abs}}(t, y_j | x_k)],$$

where the equality is given by the relation (9) and multilinearity of determinant. By the theory of Karlin and McGregor [47] (see also [49] with [50,51]), the probability that an $N$-particle system of BES(3)'s starting from the configuration $\mathbf{x} \in \mathbf{W}_N^{\mathrm{C}}$ can keep the order of $N$-particle positions by avoiding any collision of particles for time period $T>0$ is given by

$$\mathcal{N}_N^{\mathrm{C}}(T, \mathbf{x}) = \int_0^\infty dy_1 \cdots \int_0^\infty dy_N \det_{1 \leq j,k \leq N} [p_{\mathrm{BES}(3)}(T, y_j | x_k)].$$

By the Markov property of diffusion processes, if we assume that the configuration at time $t=1$ is fixed to be $\mathbf{y} \in \mathbf{W}_N^{\mathrm{C}}$, for $0<t_1<t_2<1$, the transition probability density from $\mathbf{x}^{(1)} \in \mathbf{W}_N^{\mathrm{C}}$ at time $t_1$ to $\mathbf{x}^{(2)} \in \mathbf{W}_N^{\mathrm{C}}$ at time $t_2$ is given as

$$\begin{aligned} p_{\mathbf{y}}^{(N)}(t_1, \mathbf{x}^{(1)}; t_2, \mathbf{x}^{(2)}) &= \frac{\det_{1 \leq j,k \leq N} [p_{\mathrm{BES}(3)}(1-t_2, y_j | x_k^{(2)})] \det_{1 \leq j,k \leq N} [p_{\mathrm{BES}(3)}(t_2 - t_1, x_j^{(2)} | x_k^{(1)})]}{\det_{1 \leq j,k \leq N} [p_{\mathrm{BES}(3)}(1-t_1, y_j | x_k^{(1)})]} \\ &= \frac{\det_{1 \leq j,k \leq N} [p_{\mathrm{abs}}(1-t_2, y_j | x_k^{(2)})] \det_{1 \leq j,k \leq N} [p_{\mathrm{abs}}(t_2 - t_1, x_j^{(2)} | x_k^{(1)})]}{\det_{1 \leq j,k \leq N} [p_{\mathrm{abs}}(1-t_1, y_j | x_k^{(1)})]}. \end{aligned} \tag{11}$$

Let $|\mathbf{x}|^2 = \Sigma_{j=1}^N x_j^2$ and define

$$h_N^{\mathrm{C}}(\mathbf{x}) = \prod_{1 \leq j<k \leq N} (x_k^2 - x_j^2) \prod_{\ell=1}^N x_\ell \tag{12}$$

for $\mathbf{x} = (x_1, x_2, \ldots, x_N) \in \mathbf{W}_N^{\mathrm{C}}$. Since we have known the asymptotics

$$\det_{1 \leq j,k \leq N} [p_{\mathrm{abs}}(t, y_j | x_k)] \simeq \frac{t^{-N(2N+1)/2} e^{-|\mathbf{x}|^2/(2t)}}{2^{N(2N-1)/2} \prod_{j=1}^N \{\Gamma(j)\Gamma(j+1/2)\}} h_N^{\mathrm{C}}(\mathbf{x}) h_N^{\mathrm{C}}(\mathbf{y})$$

in $|\mathbf{y}| \to 0$ [the $\nu=1/2$ case of Eq. (33) in [39]], Eq. (11) gives the following:

$$\begin{aligned} p^{(N)}(t_1, \mathbf{x}^{(1)}; t_2, \mathbf{x}^{(2)}) &\equiv \lim_{|\mathbf{y}| \to 0} p_{\mathbf{y}}^{(N)}(t_1, \mathbf{x}^{(1)}; t_2, \mathbf{x}^{(2)}) = \left(\frac{1-t_2}{1-t_1}\right)^{-N(2N+1)/2} \frac{h_N^{\mathrm{C}}(\mathbf{x}^{(2)})}{h_N^{\mathrm{C}}(\mathbf{x}^{(1)})} \\ &\times \det_{1 \leq j,k \leq N} [p_{\mathrm{abs}}(t_2 - t_1, x_j^{(2)} | x_k^{(1)})] \exp\left\{-\frac{|\mathbf{x}^{(2)}|^2}{2(1-t_2)} + \frac{|\mathbf{x}^{(1)}|^2}{2(1-t_1)}\right\}, \end{aligned} \tag{13}$$

for $\mathbf{x}^{(1)}, \mathbf{x}^{(2)} \in \mathbf{W}_N^{\mathrm{C}}$ and $0<t_1<t_2<1$, and

$$p^{(N)}(0, \mathbf{0}; t, \mathbf{x}) \equiv \lim_{|\mathbf{x}^{(1)}| \to 0} p^{(N)}(0, \mathbf{x}^{(1)}; t, \mathbf{x}) = \frac{\{t(1-t)\}^{-N(2N+1)/2}}{(\pi/2)^{N/2} \prod_{j=1}^N (2j-1)!} \{h_N^{\mathrm{C}}(\mathbf{x})\}^2 \exp\left\{-\frac{|\mathbf{x}|^2}{2t(1-t)}\right\}, \tag{14}$$

for $\mathbf{x} \in \mathbf{W}_N^{\mathrm{C}}$ and $0<t<1$. The $N$-particle system of noncolliding three-dimensional Bessel bridges with duration 1 all starting from the origin is defined as the diffusion process such that its transition probability density is given by (13) and (14), and denoted by $\mathbf{r}^{(N)}(t)=(r_1^{(N)}(t), r_2^{(N)}(t), \ldots, r_N^{(N)}(t))$, $0 \leq t \leq 1$. That is, for any sequence of times $t_0 \equiv 0 < t_1 < t_2 < \cdots < t_M < 1$, $M=1,2,\ldots$, and for any sequence of regions $\Delta_m \in \mathbf{W}_N^{\mathrm{C}}$, $m=1,2,\ldots,M$,

$$\mathbf{P}(\mathbf{r}^{(N)}(t_m) \in \Delta_m, m=1,2,\ldots,M) = \int_{\Delta_1} d\mathbf{x}^{(1)} \cdots \int_{\Delta_M} d\mathbf{x}^{(M)} \times \prod_{m=0}^{M-1} p^{(N)}(t_m, \mathbf{x}^{(m)}; t_{m+1}, \mathbf{x}^{(m+1)}). \tag{15}$$

Note that no generality is lost by setting the time duration to be 1 by the scaling property between space and time of the present $N$-particle system inherited from BM via BES(3). From now on we call $\mathbf{r}^{(N)}(t)$, $0 \leq t \leq 1$, simply the *$N$-noncolliding Bessel bridges* for short. We remark that, if we set $N=1$ and $t=1/2$ in (14), $h_1^{\mathrm{C}}(x)=x$ and Eq. (1) is obtained.

### B. Matrix-valued Brownian bridge in symmetry class C

For $N \geq 1$, let $I_N$ be the $N \times N$ unit matrix and define the $2N \times 2N$ matrix

$$J = \begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix},$$

where $0_N$ denotes the $N \times N$ zero matrix. Let $\mathcal{H}(N)$ and $\mathcal{S}(N;\mathbf{C})$ be collections of all $N \times N$ Hermitian matrices and all $N \times N$ complex symmetric matrices, respectively. Then consider the following set of $2N \times 2N$ Hermitian matrices:

$$\mathcal{H}^{\mathrm{C}}(2N) = \left\{ C = \begin{pmatrix} H & S \\ S^{\dagger} & -{}^tH \end{pmatrix}; H \in \mathcal{H}(N), S \in \mathcal{S}(N;\mathbf{C}) \right\}, \tag{16}$$

where ${}^tH$ denotes the transpose of $H$ and $S^{\dagger} \equiv {}^t\bar{S}$ denotes the Hermitian conjugate of $S$. We can see that any element $C \in \mathcal{H}^{\mathrm{C}}(2N)$ satisfies the relation

$${}^tCJ + JC = 0, \tag{17}$$

which means that $C \in \mathcal{H}^{\mathrm{C}}(2N) \subset \mathcal{H}(2N)$ satisfies the symplectic Lie algebra, symbolically written as $\mathcal{H}^{\mathrm{C}}(2N) = \mathrm{sp}(2N,\mathbf{C}) \cap \mathcal{H}(2N)$ [48]. Due to the additional symmetry (17), the $2N$ real eigenvalues of $C \in \mathcal{H}^{\mathrm{C}}(2N)$ are given in the form $\lambda=(\lambda_1, \lambda_2, \ldots, \lambda_N, -\lambda_1, -\lambda_2, \ldots, -\lambda_N)$, where $\lambda_j \geq 0$, $1 \leq j \leq N$. Altland and Zirnbauer studied $\mathcal{H}^{\mathrm{C}}(2N)$ as the set of the Hamiltonians in the Bogoliubov–de Gennes formalism for the BCS mean-field theory of superconductivity, with regarding the pairing of positive and negative eigenvalues $(\lambda_j, -\lambda_j)$, $1 \leq j \leq N$, as the particle-hole symmetry in the Bogoliubov–de Gennes theory [41,42]. They called $\mathcal{H}^{\mathrm{C}}(2N)$ (a representation of) the symmetry class C, since $\mathrm{sp}(2N,\mathbf{C})$ is denoted by $\mathrm{C}_N$ in Cartan's notation [52].

The *Brownian bridge with duration 1* starting from the origin is defined as the one-dimensional standard BM starting from 0 conditioned to return to 0 at time $t=1$ and denoted by $b(t)$, $0 \leq t \leq 1$. The transition probability density of $b(t)$ is given by

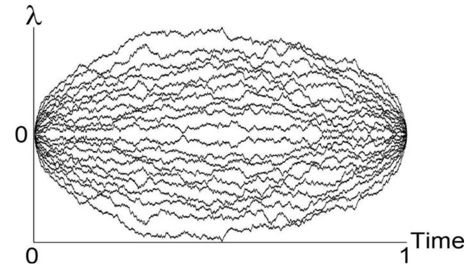


FIG. 3. A sample of paths of the eigenvalue process $\boldsymbol{\lambda}^{(N)}(t)$, $0 \leq t \leq 1$, of the matrix-valued Brownian bridge $C^{(N)}(t)$, $0 \leq t, \leq 1$, in the symmetry class C for $N=10$.

$$p_{\mathrm{Bb}}(s,x;t,y) = \frac{p(1-t,0|y)p(t-s,y|x)}{p(1-s,0|x)} \tag{18}$$

for $0 \leq s < t \leq 1$, $x,y \in \mathbf{R}$. Let $b_{jk}^{\rho}(t)$, $0 \leq \rho \leq 2$, $1 \leq j \leq k \leq N$, and $\tilde{b}_{jk}^{0}(t)$, $1 \leq j < k \leq N$, be independent Brownian bridges with duration 1 starting from the origin. Put

$$s_{jk}^{\rho}(t) = \begin{cases} b_{jk}^{\rho}(t)/\sqrt{2} & \text{if } j<k, \\ b_{jj}^{\rho}(t) & \text{if } j=k, \\ b_{kj}^{\rho}(t)/\sqrt{2} & \text{if } j>k, \end{cases}$$

for $0 \leq \rho \leq 2$, and

$$a_{jk}^{0}(t) = \begin{cases} \tilde{b}_{jk}^{0}(t)/\sqrt{2} & \text{if } j<k, \\ 0 & \text{if } j=k, \\ -\tilde{b}_{kj}^{0}(t)/\sqrt{2} & \text{if } j>k, \end{cases}$$

and consider the $N \times N$ matrices $S^{\rho}(t)=(S_{jk}^{\rho}(t))_{1 \leq j,k \leq N}$, $0 \leq \rho \leq 2$ and $A^0(t)=(a_{jk}^{0}(t))_{1 \leq j,k \leq N}$. Then the $2N \times 2N$ matrix-valued BM is defined by

$$C^{(N)}(t) = \begin{pmatrix} S^0(t)+iA^0(t) & S^1(t)+iS^2(t) \\ S^1(t)-iS^2(t) & -(S^0(t)-iA^0(t)) \end{pmatrix}, \quad 0 \leq t \leq 1. \tag{19}$$

In order to define $C^{(N)}(t)$, we have used $N(N+1)/2 \times 3 + N(N-1)/2 = N(2N+1)$ independent Brownian bridges. By definition (19), $C^{(N)}(t) \in \mathcal{H}^{\mathrm{C}}(2N)$, $0 \leq t \leq 1$, and $C^{(N)}(0) = C^{(N)}(1) = 0_{2N}$. That is, $C^{(N)}(t)$ can be regarded as a Brownian bridge in the $[N(2N+1)]$-dimensional space $\mathcal{H}^{\mathrm{C}}(2N)$.

At each time $0<t<1$, $C^{(N)}(t)$ can be diagonalized by a unitary-symplectic matrix and we can obtain the eigenvalue process $\boldsymbol{\lambda}^{(N)}(t)=(\lambda_1^{(N)}(t), \ldots, \lambda_N^{(N)}(t), -\lambda_1^{(N)}(t), \ldots, -\lambda_N^{(N)}(t))$ with $0 \leq \lambda_1^{(N)}(t) \leq \cdots \leq \lambda_N^{(N)}(t)$. Using the generalized Bru's theorem given in [39], we can determine the transition probability density for the positive part of eigenvalue process $\boldsymbol{\lambda}_+^{(N)}(t)=(\lambda_1^{(N)}(t), \lambda_2^{(N)}(t), \ldots, \lambda_N^{(N)}(t))$. The result is exactly the same as Eqs. (13) and (14). It implies that $\boldsymbol{\lambda}_+^{(N)}(t) \in \mathbf{W}_N^{\mathrm{C}}$, $0<t<1$, with probability 1, and the present positive-eigenvalue process $\boldsymbol{\lambda}_+^{(N)}(t)$ is equivalent in distribution with the noncolliding Bessel bridges $\mathbf{r}^{(N)}(t)$. Figure 3 shows a sample of paths of the eigenvalue process $\boldsymbol{\lambda}^{(N)}(t)$, $0 \leq t \leq 1$, for $N=10$ generated by computer (see Sec. IV for detail). There we can see ten positive paths $\boldsymbol{\lambda}_+^{(N)}(t)$ and their mirror images with respect to the line $\lambda=0$. The sample of paths of the noncolliding Bessel bridges, $\mathbf{r}^{(N)}(t)$, $0\le t\le 1$, shown in Fig. 2 for $N=10$, is just obtained by the upper half of this figure.

### C. Problems

For the $N$-path system of noncolliding Bessel bridges, $\mathbf{r}(t)$, $0\le t\le 1$, we study the maximum values for each path attained in the time interval $t\in(0,1)$,

$$H_k^{(N)} \equiv \max_{0<t<1} r_k^{(N)}(t), \quad 1\le k\le N. \tag{20}$$

The problem considered in the present paper is to clarify the statistical property of the random variables $H_k^{(N)}$, $1\le k\le N$. We will report the exact expressions of the distribution function for the outermost path $H_N^{(N)}$ for general $N\ge 1$ and the numerical results for inner paths. The dependence of $N$ is studied, and the asymptotics in $N\to\infty$ will be discussed.

## III. EXACT RESULTS FOR THE OUTERMOST PATHS

### A. Distribution function of $H_N^{(N)}$

In this section we derive an exact expression for the distribution function of the maximum value of the outermost path, $\mathbf{P}(H_N^{(N)}<h)$. In order to that first we consider the absorbing BM in an interval $(0,h)$ for $h>0$, in which two absorbing walls are put at the origin and at the position $x=h$. The transition probability density $p_{\rm abs}^h(t,y|x)$ for $t>0$, $0\le x,\ y\le h$ is the solution of the diffusion equation $\partial u(t,y)/\partial t=(1/2)\partial^2 u(t,y)/\partial y^2$ with the initial condition $\lim_{t\to 0}u(t,y)=\delta(y-x)$ and with the Dirichlet boundary conditions $u(t,0)=u(t,h)=0$, $t\ge 0$. By the method of separation of variables and the Fourier analysis, the unique solution is determined as (see, for example, [53])

$$p_{\rm abs}^h(t,y|x) = \frac{2}{h}\sum_{n=1}^{\infty} \exp\left(-\frac{n^2\pi^2}{2h^2}t\right)\sin\left(\frac{n\pi}{h}y\right)\sin\left(\frac{n\pi}{h}x\right), \tag{21}$$

for $t>0$, $0\le x$, $y\le h$. Note that it is the different expression of the function

$$\begin{aligned} p_{\rm abs}^h(t,y|x) &= \sum_{n=-\infty}^{\infty}\{p(t,y|x+2hn)-p(t,y|-x+2hn)\} \\ &= \frac{1}{\sqrt{2\pi t}}\sum_{n=-\infty}^{\infty}\left[\exp\left\{-\frac{1}{2t}[y-(x+2hn)]^2\right\} - \exp\left\{-\frac{1}{2t}[y-(-x+2hn)]^2\right\}\right], \end{aligned}$$

which was used in our previous paper [13].

Consider the two determinantal functions

$$q^{(N)}(t,\mathbf{y}|\mathbf{x}) = \det_{1\le j,k\le N}[p_{\rm abs}(t,y_j|x_k)], \quad t>0, \quad \mathbf{x},\mathbf{y}\in \mathbf{W}_N^{\rm C} \tag{22}$$

and

$$q_h^{(N)}(t,\mathbf{y}|\mathbf{x}) = \det_{1\le j,k\le N}[p_{\rm abs}^h(t,y_j|x_k)], \quad t>0, \quad \mathbf{x},\mathbf{y}\in \mathbf{W}_N^h, \tag{23}$$

where $\mathbf{W}_N^h=\{\mathbf{x}\in(0,h)^N : x_1<x_2<\cdots<x_N\}$. By the theory of Karlin and McGregor [47], $q^{(N)}(t,\mathbf{y}|\mathbf{x})$ $[q_h^{(N)}(t,\mathbf{y}|\mathbf{x})]$ is the probability for the $N$-dimensional absorbing BM starting from $\mathbf{x}\in\mathbf{W}_N^{\rm C}$ $[\mathbf{x}\in\mathbf{W}_N^h]$ at time $t=0$ to survive during time period $t$ by avoiding any hitting with the absorbing boundaries of $\mathbf{W}_N^{\rm C}$ $[\mathbf{W}_N^h]$ and to arrive at $\mathbf{y}\in\mathbf{W}_N^{\rm C}$ $[\mathbf{y}\in\mathbf{W}_N^h]$ at time $t$. Note that if we think that the $N$-dimensional vector $\mathbf{x}=(x_1,x_2,\ldots,x_N)$ represents the positions of $N$ particles on $\mathbf{R}_+$, a hitting with the boundary of $\mathbf{W}_N^{\rm C}$ means a hitting of the innermost particle $x_1$ with the origin or any collision between neighboring particles $x_j=x_{j+1}$, $1\le j\le N-1$. Similarly a hitting with the boundary of $\mathbf{W}_N^h$ means $x_1=0$, or any collision of particles, or a hitting of the outer most particle $x_N$ with the wall at $x=h$.

Consider the $N$-particle system of noncolliding BES(3), $\widetilde{\mathbf{R}}^{(N)}(t)=(\widetilde{R}_1^{(N)}(t),\cdots,\widetilde{R}_N^{(N)}(t))$ starting from the configuration $\mathbf{x}\in\mathbf{W}_N^h$ at time $t=0$; $\widetilde{\mathbf{R}}^{(N)}(0)=\mathbf{x}$, and arriving at the configuration $\mathbf{y}\in\mathbf{W}_N^h$ at time $t=1$; $\widetilde{\mathbf{R}}^{(N)}(1)=\mathbf{y}$. Let $\widetilde{H}_N^{(N)}=\max_{0<t<1}\widetilde{R}_N^{(N)}(t)$. Then we can say that $\mathbf{P}(\widetilde{H}_N^{(N)}<h)=q_h^{(N)}(1,\mathbf{y}|\mathbf{x})/q^{(N)}(1,\mathbf{y}|\mathbf{x})$. By definition of the $N$-noncolliding Bessel bridges, $\mathbf{r}^{(N)}(t)$, $0\le t\le 1$, given in Sec. II A, we can conclude that

$$\mathbf{P}(H_N^{(N)}<h) = \lim_{|\mathbf{x}|\to 0,|\mathbf{y}|\to 0}\frac{q_h^{(N)}(1,\mathbf{y}|\mathbf{x})}{q^{(N)}(1,\mathbf{y}|\mathbf{x})}. \tag{24}$$

As shown in Appendix A, the method of the Schur function expansion [39] gives the following asymptotics for $q^{(N)}(1,\mathbf{y}|\mathbf{x})$ and $q_h^{(N)}(1,\mathbf{y}|\mathbf{x})$ in $|\mathbf{x}|\to 0$, $|\mathbf{y}|\to 0$:

$$q^{(N)}(1,\mathbf{y}|\mathbf{x}) = \left(\frac{2}{\pi}\right)^{N/2}\prod_{j=1}^{N}\frac{1}{(2j-1)!}h_N^{\rm C}(\mathbf{x})h_N^{\rm C}(\mathbf{y}) \times\{1+\mathcal{O}(|\mathbf{x}|,|\mathbf{y}|)\}, \tag{25}$$

$$\begin{aligned} q_h^{(N)}(1,\mathbf{y}|\mathbf{x}) &= \left(\frac{2}{h}\right)^{N}\left(\frac{\pi}{h}\right)^{2N^2}\left\{\prod_{j=1}^{N}\frac{1}{(2j-1)!}\right\}^2 h_N^{\rm C}(\mathbf{x})h_N^{\rm C}(\mathbf{y}) \\ &\times\sum_{1\le n_1<n_2<\cdots<n_N}\exp\left\{-\frac{\pi^2}{2h^2}|\mathbf{n}|^2\right\}\{h_N^{\rm C}(\mathbf{n})\}^2 \\ &\times\{1+\mathcal{O}(|\mathbf{x}|,|\mathbf{y}|)\}, \end{aligned} \tag{26}$$

where $h_N^{\rm C}$ was defined by (12). Then Eq. (24) gives the following result.

*Lemma 1*. For $N \geqslant 1$, $h > 0$,

$$\mathbf{P}(H_N^{(N)} < h) = c_N h^{-N(2N+1)} \sum_{1 \leqslant n_1 < n_2 < \cdots < n_N} \{h_N^{\mathrm{C}}(\mathbf{n})\}^2 \times \exp\left\{ -\frac{\pi^2}{2h^2}|\mathbf{n}|^2 \right\}, \tag{27}$$

where $c_N = 2^{N/2}\pi^{N(4N+1)/2}/\prod_{j=1}^N (2j-1)!$.

*Remark 1*. This expression is exactly the same as Eq. (5) in [32], which was derived by the path-integral method using a Selberg's integral. Here we would like to put emphasis on the resemblance between the summand of (27) and (14). As mentioned in Sec. II B, Eq. (14) is the same as the probability density of the eigenvalue distribution of random matrices in the class C [with variance $t(1-t)$]. The exponent of the factor $h^{-N(2N+1)}$ in (27) is the dimension of the space $\mathcal{H}^{\mathrm{C}}(2N)$. Another piece of evidence to show the hidden symmetry of the present maximum-value problem is the following. The character of the irreducible representation corresponding to a partition $\mu$ of the symplectic Lie algebra is given by [48]

$$\mathrm{sp}_\mu(\mathbf{x}) = \frac{\det_{1 \leqslant j,k \leqslant N} [x_j^{\mu_k+N-k+1} - x_j^{-(\mu_k+N-k+1)}]}{\det_{1 \leqslant j,k \leqslant N} [x_j^{N-k+1} - x_j^{-(N-k+1)}]}. \tag{28}$$

If we set $n_j = \mu_{N-j+1} + j$, $1 \leqslant j \leqslant N$, and $x_j = 1$, $1 \leqslant j \leqslant N$, Eq. (28) gives [see, for example, Eq. (3.33) in [25] and Eq. (3.10) in [53]]

$$\mathrm{sp}_\mu(1,1,\ldots,1) = \frac{h_N^{\mathrm{C}}(\mathbf{n})}{\prod_{j=1}^N (2j)!}.$$

The above observations imply that the maximum-value problems of noncolliding diffusion problems will be related to some enumerative problems of combinatorics in the ensembles of irreducible representations of symmetry, which can be regarded as a discrete version of random matrix ensembles. The wall restriction for paths (i.e., the stay-positive condition for particles) is mapped onto the symplectic (the class C) structure in the present case. As pointed out in [32], the trivial fact $\lim_{h\to\infty}\mathbf{P}(H_N^{(N)} < h) = 1$ for (27) gives a version of Selberg integral [54,55],

$$\begin{aligned}
&\lim_{\delta \to 0} \sum_{n_1=-\infty}^{\infty} \cdots \sum_{n_N=-\infty}^{\infty} \\
&\quad \times \prod_{1 \leqslant j<k \leqslant N} \{(\delta n_j)^2 - (\delta n_k)^2\}^2 \prod_{\ell=1}^N \{(\delta n_\ell)^2 e^{-(\delta n_\ell)^2/2}\delta\} \\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{1 \leqslant j<k \leqslant N} (x_j^2 - x_k^2)^2 \prod_{\ell=1}^N \{x_\ell^2 e^{-x_\ell^2/2} dx_\ell\} \\
&= (2\pi)^{N/2} N! \prod_{j=1}^N (2j-1)!,
\end{aligned} \tag{29}$$

which is the special case with $\gamma = 1$ and $\alpha = 3/2$ of Eq. (17.6.6) in [40].

### B. Determinantal expressions and Jacobi's theta function

From (27), we can obtain the following determinantal expression for the distribution function of $H_N^{(N)}$.

*Proposition 2*. For $N \geqslant 1$, $h > 0$,

$$\begin{aligned}
\mathbf{P}(H_N^{(N)} < h) &= c_N h^{-N(2N+1)} \det_{1 \leqslant j,k \leqslant N}\left[ \sum_{n=1}^{\infty} n^{2(j+k-1)} e^{-\pi^2 n^2/(2h^2)} \right] \\
&= \frac{(-1)^N 2^{-N/2}\pi^{N(2N+1)/2}}{\prod_{j=1}^N (2j-1)!} h^{-N(2N+1)} \tau_N\left(\frac{\pi}{2h^2}\right),
\end{aligned} \tag{30}$$

where

$$\tau_N(u) = \det_{1 \leqslant j,k \leqslant N}\left[ \frac{\partial^{j+k-1}}{\partial u^{j+k-1}} \theta(u) \right] \quad \text{with } \theta(u) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 u}.$$

The proof is given in Appendix B.

Now we consider a version of Jacobi's theta function

$$\vartheta_3(x,y) = \sum_{n=-\infty}^{\infty} q^{n^2} z^{2n} = \sum_{n=-\infty}^{\infty} e^{2\pi i x n + \pi i y n^2}, \quad \mathrm{Im}\, y > 0, \tag{31}$$

where we have set $z = e^{x\pi i}$ and $q = e^{y\pi i}$. The following functional equation is satisfied (see Sec. 10.12 in [56], Sec. A.3.1 in [29]):

$$\vartheta_3(x,y) = \vartheta_3(x/y, -1/y) e^{-\pi i x^2/y} \sqrt{\frac{i}{y}}. \tag{32}$$

From this equation, we will obtain the following equalities.

*Lemma 3*. For $\alpha = 0,1,2,\ldots$,

$$\sum_{n=-\infty}^{\infty} n^\alpha e^{-\pi n^2/\eta^2 + 2\pi i \xi n/\eta^2} = \frac{i^\alpha \eta^{\alpha+1}}{2^\alpha \pi^{\alpha/2}} \sum_{n=-\infty}^{\infty} H_\alpha(\sqrt{\pi} n \eta + \sqrt{\pi}\xi/\eta) \times e^{-(\sqrt{\pi} n \eta + \sqrt{\pi}\xi/\eta)^2}, \tag{33}$$

where $H_\alpha(x)$ is the $\alpha$th Hermite polynomial defined by (5).

*Proof.* Setting $x = i\xi + \eta^2 u/(2\pi)$, $y = i\eta^2$, $\xi, \eta \in \mathbf{R}$, in (32) gives

$$\vartheta_3\left(i\xi + \frac{\eta^2 u}{2\pi}, i\eta^2\right) = \vartheta_3\left(\frac{\xi}{\eta^2} - \frac{iu}{2\pi}, \frac{i}{\eta^2}\right) e^{\pi[\xi/\eta - i\eta u/(2\pi)]^2} \frac{1}{\eta}.$$

By definition (31), this implies

$$\sum_{n=-\infty}^{\infty} e^{-\pi n^2/\eta^2 + (2\pi i \xi/\eta^2 + u)n} = \eta \sum_{n=-\infty}^{\infty} e^{-[\sqrt{\pi}\eta n + \sqrt{\pi}\xi/\eta - i\eta u/(2\sqrt{\pi})]^2}. \tag{34}$$

We note that the generating function of Hermite polynomial is given as

$$e^{2ux-u^2} = \sum_{n=0}^{\infty} H_n(x) \frac{u^n}{n!}.$$

Then (34) becomes

$$\sum_{\alpha=0}^{\infty}\sum_{n=-\infty}^{\infty}\frac{(nu)^{\alpha}}{\alpha!}e^{-\pi n^2/\eta^2+2\pi i\xi n/\eta^2}$$

$$=\eta\sum_{\alpha=0}^{\infty}\sum_{n=-\infty}^{\infty}H_{\alpha}(\sqrt{\pi}n\eta+\sqrt{\pi}\xi/\eta)$$

$$\times\left(\frac{i\eta u}{2\sqrt{\pi}}\right)^{\alpha}\frac{1}{\alpha!}e^{-(\sqrt{\pi}n\eta+\sqrt{\pi}\xi/\eta)^2}.$$

Since this equality holds for any value of $u$, Lemma 3 is proved.

By setting $\xi=0$, $\eta=h\sqrt{2/\pi}$ and $\alpha=2\ell$ in Eq. (33) of Lemma 3, we have the equalities

$$\sum_{n=1}^{\infty}n^{2\ell}e^{-\pi^2n^2/(2h^2)}=\frac{(-1)^{\ell}}{2^{\ell+1/2}\pi^{2\ell+1/2}}h^{2\ell+1}\sum_{n=-\infty}^{\infty}H_{2\ell}(\sqrt{2}nh)e^{-2n^2h^2} \tag{35}$$

for $\ell=0,1,2,\ldots$. Then Eq. (30) is transformed into the following other determinantal expression, which was announced in Introduction.

*Theorem 4*. For $N\geqslant 1$, $h>0$,

$$\mathbf{P}(H_N^{(N)}<h)=\frac{(-1)^N}{2^{N^2}\prod_{j=1}^{N}(2j-1)!}\times\det_{1\leqslant j,k\leqslant N}\left[\sum_{n=-\infty}^{\infty}H_{2(j+k-1)}(\sqrt{2}nh)e^{-2n^2h^2}\right]. \tag{36}$$

As special cases of (36) with $N=1$ and $N=2$, we have

$$\mathbf{P}(H_1^{(1)}<h)=-\frac{1}{2}\sum_{n=-\infty}^{\infty}H_2(\sqrt{2}nh)e^{-2h^2n^2}$$

and

$$\mathbf{P}(H_2^{(2)}<h)=\frac{1}{2^4\times 3!}\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}e^{-2h^2(n_1^2+n_2^2)}\times\det\begin{bmatrix}H_2(\sqrt{2}n_1h) & H_4(\sqrt{2}n_1h)\\ H_4(\sqrt{2}n_2h) & H_6(\sqrt{2}n_2h)\end{bmatrix}.$$

Since $H_2(x)=4x^2-2$, $H_4(x)=16x^4-48x^2+12$, and $H_6(x)=64x^6-480x^4+720x^2-120$, they give

$$\mathbf{P}(H_1^{(1)}<h)=\sum_{n=-\infty}^{\infty}(1-4h^2n^2)e^{-2h^2n^2} \tag{37}$$

and

$$\mathbf{P}(H_2^{(2)}<h)=\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}e^{-2h^2(n_1^2+n_2^2)}\left\{1-16h^2n_1^2+24h^4n_1^4+24h^4n_1^2n_2^2-\frac{32}{3}h^6n_1^6-32h^6n_1^4n_2^2+\frac{128}{3}h^8n_1^6n_2^2-\frac{128}{3}h^8n_1^4n_2^4\right\}. \tag{38}$$

TABLE I. Comparison of the values of $m_N^{(N)}$ and $v_N^{(N)}$ obtained by the present numerical method and the exact values. The values with "F" are read from Table 1 in [31] and those with "KIK" are from Table 1 in [13].

| $N$ | $m_N^{(N)}$ | $v_N^{(N)}$ | F $m_N^{(N)}$ | F $v_N^{(N)}$ | KIK $m_N^{(N)}$ | KIK $v_N^{(N)}$ |
|---|---|---|---|---|---|---|
| 1 | 1.251 | 0.0774 | 1.2533 | 0.0737 | 1.253314 | 0.074138 |
| 2 | 1.819 | 0.0732 | 1.8222 | 0.0746 | 1.822625 | 0.073194 |
| 3 | 2.262 | 0.0704 | 2.2677 | 0.0720 | | |
| 4 | 2.641 | 0.0664 | 2.6460 | 0.0692 | | |
| 5 | 2.979 | 0.0640 | 2.9805 | 0.0656 | | |
| 6 | 3.280 | 0.0624 | | | | |
| 7 | 3.558 | 0.0600 | | | | |
| 8 | 3.817 | 0.0556 | | | | |
| 9 | 4.057 | 0.0570 | | | | |
| 10 | 4.291 | 0.0543 | | | | |
| 20 | 6.146 | 0.0409 | | | | |
| 30 | 7.597 | 0.0396 | | | | |
| 40 | 8.790 | 0.0384 | | | | |
| 50 | 9.841 | 0.0364 | | | | |
| 60 | 10.806 | 0.0354 | | | | |
| 70 | 11.678 | 0.0310 | | | | |

From (37) we will obtain (2). Equation (38) is exactly the same as the result reported as Lemma 3.1 in our previous paper [13].

*Remark 2*. Since the derivative of the distribution function $\mathbf{P}(H_N^{(N)}<h)$ with respect to $h$ gives the probability density for $H_N^{(N)}\in dh$, the $s$th moment of $H_N^{(N)}$, $s=1,2,\ldots$, is calculated as

$$\langle (H_N^{(N)})^s\rangle=\int_0^{\infty}h^s\left(\frac{d}{dh}\mathbf{P}(H_N^{(N)}<h)\right)dh=s\int_0^{\infty}h^{s-1}\{1-\mathbf{P}(H_N^{(N)}<h)\}dh, \tag{39}$$

where the integral by part was done. If we insert the expression (30) into (39), we have a determinantal expression for

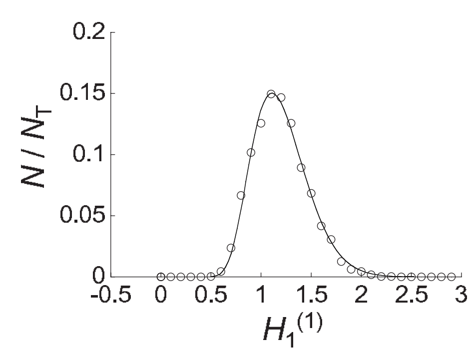


FIG. 4. Comparison of the distribution of the maximum values $H_1^{(1)}$ of the three-dimensional Bessel bridge between the numerical simulation (plotted by circles) and the theoretical values (given by curve).

the $s$th moment. It is essentially the same as the expression of Feierl given to the dominant term in the long-step asymptotics of the moment of the height distribution of watermelons with a wall. See the function $\kappa_s^{(p)}$ given in Theorem 1 of [31]. Moreover, the determinantal expression for the central limit theorem of the height distribution of the watermelons with a wall given by Feierl [Eq. (28) in Theorem 2] [31] can be identified with our second determinantal expression (36), since the functions $\phi_k(z)$ used there are nothing but the Hermite polynomials. We have enjoyed the perfect coincidence of the results obtained by the two different routes to the problem. See [13] for more a detailed discussion on the relationship between our treatment and that by Fulmek [29] and Feierl [30,31,57].

## IV. NUMERICAL STUDY

### A. Bessel bridge realized by eigenvalue process

We have prepared a computer program to generate samples of paths of $b(t)$, $0 \le t \le 1$, the Brownian bridge with duration 1 starting from the origin, in which each sample path is approximated by random walk with 10000 steps.

Assume that we have generated three independent Brownian bridges, $b_j(t)$, $0 \le t \le 1$, $j=1,2,3$, by this computer program. Then we consider the $2 \times 2$ matrix-valued process

$$C^{(1)}(t) = \begin{pmatrix} b_1(t) & b_2(t) + ib_3(t) \\ b_2 - ib_3(t) & -b_1(t) \end{pmatrix}. \tag{40}$$

It is easy to see that the eigenvalue process of $C^{(1)}(t)$ is given by the positive and negative pairs of eigenvalues $\boldsymbol{\lambda}^{(2)}(t) = (\lambda_1^{(1)}(t), -\lambda_1^{(1)}(t))$ with

$$\lambda_1^{(1)}(t) = \sqrt{[b_1(t)]^2 + [b_2(t)]^2 + [b_3(t)]^2}. \tag{41}$$

The positive eigenvalue process (41) gives the numerical realization of $r(t)$, $0 \le t \le 1$, the three-dimensional Bessel bridge. Actually the sample of path of $r(t)$ shown in Fig. 1 given in the Introduction was numerically drawn by this method. In order to check the validity of the present numerical method to simulate the Bessel bridge, we have generated 1000 samples of paths and studied the distribution of their maximum values $H_1^{(1)} = \max_{0<t<1} r(t)$ numerically. The obtained result is plotted in Fig. 4. There the exact curve obtained from Eq. (2) is also shown. The coincidence is excellent.

### B. Means and variances of $H_N^{(N)}$'s

Note that the $2 \times 2$ matrix (40) can be considered as the special case of (19) with $N=1$. If we prepare ten independent Brownian bridges, $\{b_j(t)\}_{j=1}^{10}$, numerically, we can simulate the $4 \times 4$ matrix-valued Brownian bridge in $\mathcal{H}^{\mathrm{C}}(4)$,

$$C^{(2)}(t) = \begin{pmatrix} b_1(t) & \frac{1}{\sqrt{2}}[b_2(t) + ib_3(t)] & b_5(t) + ib_6(t) & \frac{1}{\sqrt{2}}[b_7(t) + ib_8(t)] \\ \frac{1}{\sqrt{2}}[b_2(t) - ib_3(t)] & b_4(t) & \frac{1}{\sqrt{2}}[b_7(t) + ib_8(t)] & b_9(t) + ib_{10}(t) \\ b_5(t) - ib_6(t) & \frac{1}{\sqrt{2}}[b_7(t) - ib_8(t)] & -b_1 & -\frac{1}{\sqrt{2}}[b_2(t) - ib_3(t)] \\ \frac{1}{\sqrt{2}}[b_7(t) - ib_8(t)] & b_9(t) - ib_{10}(t) & -\frac{1}{\sqrt{2}}[b_2(t) + ib_3(t)] & -b_4(t) \end{pmatrix},$$

which is the $N=2$ case of (19). By tracing the two positive eigenvalues $\boldsymbol{\lambda}_+^{(2)}(t) = (\lambda_1^{(2)}(t), \lambda_2^{(2)}(t))$, $0 < \lambda_1^{(2)}(t) < \lambda_2^{(2)}(t)$, $0 \le t \le 1$, we can simulate the noncolliding paths of two Bessel bridges, $\mathbf{r}^{(2)}(t) = (r_1^{(2)}(t), r_2^{(2)}(t))$, and statistical data of $H_2^{(2)} = \max_{0<t<1} r_2^{(2)}(t)$ can be obtained. In general, we can simulate the $N$-noncolliding Bessel bridges $\mathbf{r}^{(N)}(t)$, $0 \le t \le 1$, by using numerical data of independently generated $N(2N+1)$ Brownian bridges $\{b_j(t)\}_{j=1}^{N(2N+1)}$ and by tracing the $N$ positive eigenvalues of the $2N \times 2N$ matrix.

From now on, we use the notations

$$m_k^{(N)} = \langle H_k^{(N)} \rangle,$$

$$v_k^{(N)} = \mathrm{var}(H_k^{(N)}) = \langle (H_k^{(N)} - m_k^{(N)})^2 \rangle, \quad 1 \le k \le N, \tag{42}$$

for the means and variances of maximum values of paths. Table I shows the numerical results for the outermost paths $k=N$ up to $N=70$, where averages have been calculated over 1000 samples. The present numerical results are consistent with the exact values, which can be read from the previous papers by Feierl [31] and by the present authors [13].

### C. $N \to \infty$ asymptotics of the outermost paths

Now we study the $N \to \infty$ asymptotics of $m_N^{(N)}$. We assume the form

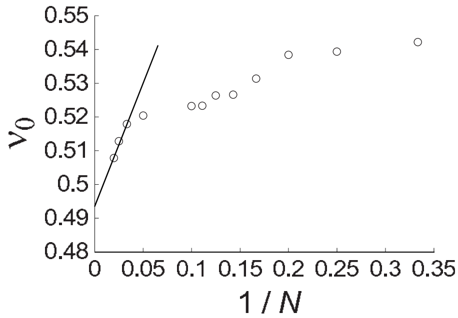


FIG. 5. $1/N$ plot of the estimated values of $\nu_0$.

$$m_N^{(N)} = c_0 N^{\nu_0}, \quad N \gg 1. \tag{43}$$

We have prepared a set of successive five results of the computer simulation $(m_{N_1}^{(N_1)}, m_{N_2}^{(N_2)}, m_{N_3}^{(N_3)}, m_{N_4}^{(N_4)}, m_{N_5}^{(N_5)})$ and fitted them to the relation (43) to estimate the exponent $\nu_0$ and the coefficient $c_0$ by a least-squares fitting. We have observed that the estimated values of $\nu_0$ and $c_0$ change rather systematically as the values $N_j$, $1 \leq j \leq 5$, increase. Figures 5 and 6 show the dependence of the estimated values of $\nu_0$ and $c_0$ on $N = N_3$. In these $1/N$ plots [58] of the estimated values, we made a linear fitting of the largest three plots as shown by the lines in the figures and obtained the values $\nu_0 = 0.493$ and $c_0 = 1.42$. They are consistent with Eq. (7): $\nu_0 = 1/2$ and $c_0 = \sqrt{2} = 1.414\ldots$. In Fig. 6 we can see that the plots in the intermediate region $0.1 < 1/N < 0.2$ give the values $1.25 < c_0 < 1.3$. They may correspond to the value $\sqrt{1.67} \simeq 1.29$ found in the estimate (6) by Bonichon and Mosbah [46], which was claimed by Schehr *et al.* as the preasymptotic behavior [32]. Figure 7 shows the log-log plot of $m_N^{(N)}$ versus $N$, where the curve is obtained by fitting the function

$$m_N^{(N)} = \sqrt{2N} + c_1 N^{-\nu_1} \quad \text{with } \nu_1 = \frac{1}{6} \tag{44}$$

to the data. The fitting parameter $c_1$ is determined as $c_1 \simeq 0.253$.

We have also studied the asymptotics of variance $v_N^{(N)}$ in $N \to \infty$. Figure 8 shows the log-log plot. By the least-squares fitting, we have obtained

$$v_N^{(N)} \simeq 0.09 N^{-0.23}. \tag{45}$$

The negative exponent implies $v_N^{(N)} \to 0$ in $N \to \infty$.

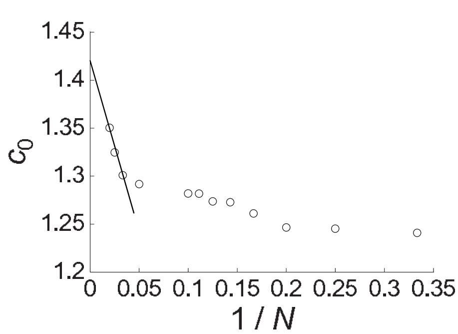


FIG. 6. $1/N$ plot of the estimated values of $c_0$.

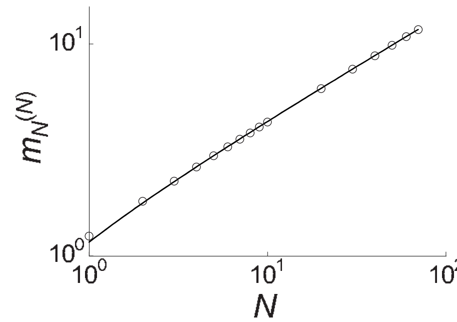


FIG. 7. Log-log plot of $m_N^{(N)}$ vs $N$ for $N$-noncolliding Bessel bridges.

### D. On inner paths

Using the numerical data of paths of $N$-noncolliding Bessel bridges generated by the method mentioned in Sec. IV B, we can examine the statistics of the maximum values attained in the time interval (0, 1) of not only the outermost paths, but also all inner paths. For example, the means $m_k^{(N)}$ and the variances $v_k^{(N)}$ for all paths $1 \leq k \leq N$ are given in Table II for $N = 10$.

Now we report our observation of the $N$ dependence for the maximum values of inner paths. Figures 9 and 10 show the $N$ dependence of the means and variances for the maximum values of the innermost path, $H_1^{(N)}$. By the log-log plots, we have obtained the following power-law behavior of the $N$ dependence:

$$m_1^{(N)} \sim N^{-0.38}, \quad v_1^{(N)} \sim N^{-1.17}. \tag{46}$$

Figure 11 shows the dependence of $m_k^{(N)}$ on $k$, which is the index of paths counting from inner to outer. We can see that all plotted curves in Fig. 11 have a common feature as a function of $k$. This fact is clarified by Fig. 12, in which the data collapse [5] is shown by plotting the quantity $m_k^{(N)}/m_N^{(N)}$ against $k/N$. The result implies that there is a universal function $f(x)$ such that the following relation holds for sufficiently large $N$,

TABLE II. Numerical values of $m_k^{(10)}$ and $v_k^{(10)}$ evaluated by computer simulations.

| $k$ | $m_k^{(10)}$ | $v_k^{(10)}$ |
|---|---|---|
| 1 | 0.547 | 0.00528 |
| 2 | 0.891 | 0.00778 |
| 3 | 1.240 | 0.00971 |
| 4 | 1.598 | 0.0126 |
| 5 | 1.958 | 0.0147 |
| 6 | 2.337 | 0.0170 |
| 7 | 2.735 | 0.0209 |
| 8 | 3.168 | 0.0262 |
| 9 | 3.659 | 0.0320 |
| 10 | 4.291 | 0.0553 |

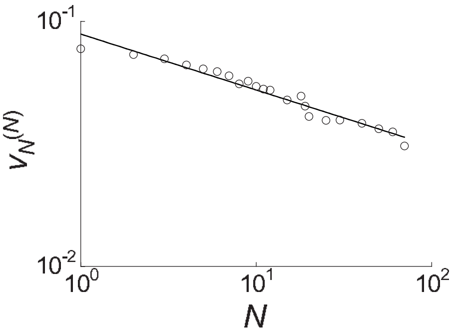


FIG. 8. Log-log plot of $v_N^{(N)}$ vs $N$ for $N$-noncolliding Bessel bridges.

$$\frac{m_k^{(N)}}{m_N^{(N)}} \sim f\left(\frac{k}{N}\right). \tag{47}$$

The scaling function $f(x)$ has two regions separated by a crossover point $x^*$. For small $x \ll x^*$, the scaling function $f(x)$ behaves as a linear function. On the other hand, for large $x \gg x^*$, $f(x)$ does not behave as a simple linear function. We have estimated the following by numerical fitting:

$$f(x) \sim \begin{cases} x & (x \ll x^*), \\ a^x & (x \gg x^*) \quad \text{with } a \simeq 5.1. \end{cases} \tag{48}$$

## V. CONCLUDING REMARKS

In this paper we have reported the exact and the numerical results on the maximum-value distributions of paths in the $N$-noncolliding Bessel bridges. We have shown that the present maximum-value problem for a version of vicious walk model of statistical physics is related to random matrix theory, the representation theory of symmetry, and number theory. There are a lot of open problems. We will list some of them here.

In expression (21) of the transition probability density $p_{\text{abs}}^h$ of the absorbing BM in an interval $(0,h)$, a variable $n \in \mathbf{N} = \{1,2,3,\ldots\}$ was introduced to indicate modes in the Fourier expansion. When we consider the $N$-path systems, a set of $N$ discrete variables $\mathbf{n}=(n_1,n_2,\ldots,n_N) \in \mathbf{N}^N$ is introduced. Though the variables $\mathbf{n}$ are auxiliary, since the physical quantities are given by the summations over $\mathbf{n}$'s as shown in (21), (27), (30), and (36), we have seen in the derivation of Lemma 1 given in Appendix A that the discrete variables $\mathbf{n}$ behave as duals of the continuous variables $\mathbf{x} = (x_1,x_2,\ldots,x_N)$ and $\mathbf{y}=(y_1,y_2,\ldots,y_N)$, which are physical variables indicating the positions of particles. The correspondence between the probability density of paths (14) given in the form of that of eigenvalues of random matrices in class C and the distribution function of the maximum value (27) given in the form of the "partition function" of discrete variables implies some duality relation. The maximum- and minimum-value problems of watermelons *without wall* recently studied by Feierl [57] and by Schehr *et al.* [32] are very interesting. Systematic study will be desired to clarify the duality between the noncolliding path problems [18] and their extreme-value problems not only for bridges (i.e., excursions, watermelon configurations) [44,45], but also for meanders (i.e., star configurations) [43].

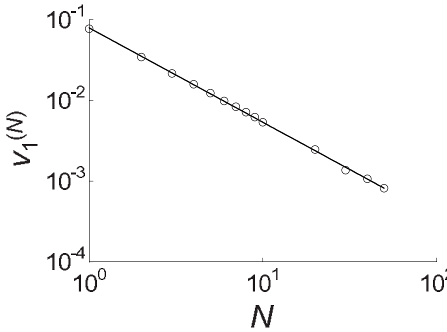


FIG. 10. Log-log plot of $v_1^{(N)}$ vs $N$ for $N$-noncolliding Bessel bridges.

As shown in the Introduction, the distribution of the maximum value of the Bessel bridge $H_1^{(1)}=\max_{0<t<1} r(t)$ and that of the position of Bessel bridge $r(1/2)$ at time $t=1/2$ are quite different from each other, because of large fluctuations of the path. For example, the mean value of the maximum $\langle H_1^{(1)}\rangle = \sqrt{\pi/2} \simeq 1.2533$ is much bigger than $\langle r(1/2)\rangle = \sqrt{2/\pi} \simeq 0.7979$. As shown by Fig. 8, however, the variance $v_N^{(N)}$ will vanish in $N\to\infty$ as Eq. (45). Then we expect

$$\langle H_N^{(N)}\rangle \simeq \langle r_N^{(N)}(1/2)\rangle \quad \text{in } N\to\infty.$$

It is known that

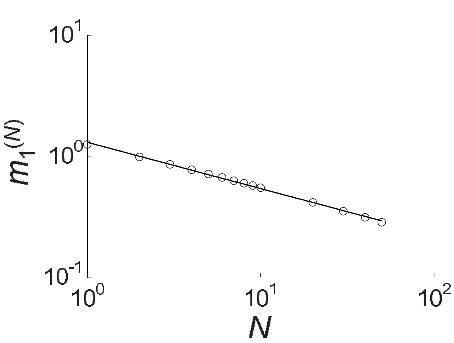


FIG. 9. Log-log plot of $m_1^{(N)}$ vs $N$ for $N$-noncolliding Bessel bridges.

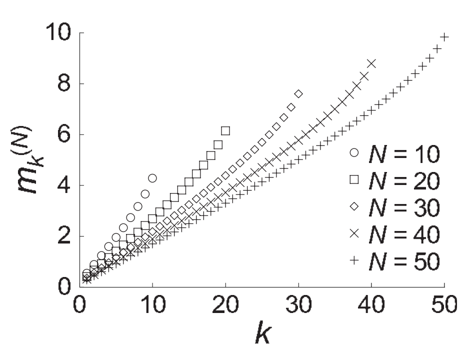


FIG. 11. Dependence of $m_k^{(N)}$ on $k$ shown for various $N$.

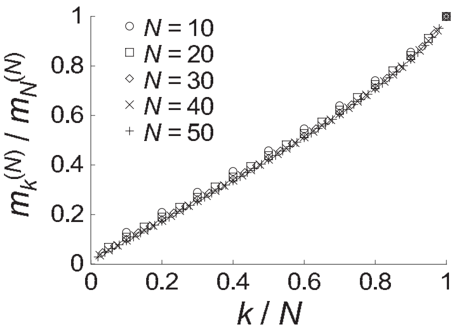


FIG. 12. Data collapse for the maximum values of inner particles for $N$-noncolliding Bessel bridges.

$$\langle r_N^{(N)}(t)\rangle \simeq 2\sqrt{2Nt(1-t)}, \quad 0 \le t \le 1 \quad \text{in } N \to \infty$$

[see, for example, Eq. (2.26) in [45], in which we should put $a=0$ and use the relation $\sqrt{x}=\lim_{N\to\infty}\langle r_N^{(N)}(t)\rangle/\sqrt{2N}$]. Then Schehr *et al.* concluded Eq. (7). By taking the proper scaling limit associated with $N\to\infty$, the fluctuation of the outermost path $r_N^{(N)}(t)$, $0\le t\le 1$, defines the Airy process [44,59–61], which obeys the Tracy-Widom distribution [62] at each time $0<t<1$. The present study suggests to us to study the maximum-value distribution of the Airy process.

We have used the exact expressions (27), (30), and (36) for distributions of $H_N^{(N)}$ given in Sec. III to check the validity of numerical calculations by our computer programs. As mentioned in Remark 1, expression (27) seems to be a "partition function" of some discrete statistical-mechanical model with the Boltzmann weight $\exp[-\pi^2|\mathbf{n}|^2/(2h^2)]$. Expression (30) reminds us of the bidirectional Wronskian solutions of nonlinear equations. And as demonstrated below Theorem 4, expression (36) is useful to reproduce the previous results reported in [12,13,29] for small $N$. Deeper understanding of these expressions in physics is desired. At the present stage it is not obvious how to discuss the asymptotics of these expressions in $N\to\infty$. We hope that our determinantal expressions (30) and (36) will be useful, since determinantal formulas play important roles in random matrix theory in analyzing the large-matrix limit [65], when they are equipped with proper mathematical tools—e.g., orthogonal polynomials, Fredholm determinants, and so on [18,40].

It will be a challenging open problem to analyze the maximum-value distributions for the inner paths in the present systems.

## ACKNOWLEDGMENTS

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## APPENDIX A: DERIVATION OF THE ASYMPTOTICS (25) and (26)

By the multilinearity of the determinant,

$$q^{(N)}(1,\mathbf{y}|\mathbf{x}) = \det_{1\le j,k\le N}\left[\frac{1}{\sqrt{2\pi}}\{e^{-(y_j-x_k)^2/2}-e^{-(y_j+x_k)^2/2}\}\right]$$
$$= \frac{1}{(2\pi)^{N/2}}e^{-(|\mathbf{x}|^2+|\mathbf{y}|^2)/2}\det_{1\le j,k\le N}[e^{y_jx_k}-e^{-y_jx_k}].$$

Here

$$\det_{1\le j,k\le N}[e^{y_jx_k}-e^{-y_jx_k}] = 2^N\prod_{j=1}^{N}(x_jy_j)\sum_{0\le m_1<m_2<\cdots<m_N}\prod_{j=1}^{N}\frac{1}{(2m_j+1)!}\det_{1\le j,k\le N}[y_j^{2m_k}]\det_{1\le j,k\le N}[x_j^{2m_k}].$$

Now we change the variables in summation from $m_j$ to $\mu_j$ by $\mu_j=m_{N-j+1}-N+j$, $1\le j\le N$, and introduce the Schur function [48,63,64]

$$s_\mu(\mathbf{x}) = \frac{\det_{1\le j,k\le N}[x_j^{\mu_k+N-k}]}{\det_{1\le j,k\le N}[x_j^{N-k}]}, \tag{A1}$$

where the denominator is the Vandermonde determinant

$$\det_{1\le j,k\le N}[x_j^{N-k}] = \prod_{1\le j,k\le N}(x_j-x_k). \tag{A2}$$

The Schur function expansion is readily performed as (see [18,39])

$$q^{(N)}(1,\mathbf{y}|\mathbf{x}) = \left(\frac{2}{\pi}\right)^{N/2} e^{-(|\mathbf{x}|^2+|\mathbf{y}|^2)/2}\prod_{j=1}^{N}(x_j y_j)\prod_{1\le j,k\le N}\{(x_j^2-x_k^2)(y_j^2-y_k^2)\}\sum_{\mu:\ell(\mu)\le N}\prod_{j=1}^{N}\frac{1}{(2\mu_{N-j+1}+2j-1)!}s_\mu(\{x_j^2\})s_\mu(\{y_j^2\}),$$

where $\ell(\mu)$ denote the number of parts of the partition $\mu$ (that is, the number of nonzero $\mu_j$, $1\le j\le N$). Since $s_\mu(\mathbf{0})=0$ unless $\mu=\mathbf{0}\equiv(0,\ldots,0)\in\mathbf{N}_0^N$, and $s_0(\mathbf{0})=1$, the asymptotics (25) is obtained.

Next we consider

$$\begin{aligned}
q_h^{(N)}(1,\mathbf{y}|\mathbf{x}) &= \left(\frac{2}{h}\right)^N\sum_{\mathbf{n}\in\mathbf{N}^N}\exp\left(-\frac{\pi^2}{2h^2}|\mathbf{n}|^2\right)\det_{1\le j,k\le N}\left[\sin\left(\frac{\pi}{h}n_j y_j\right)\sin\left(\frac{\pi}{h}n_j x_k\right)\right]\\
&= \left(\frac{2}{h}\right)^N\sum_{\mathbf{n}\in\mathbf{N}^N}\exp\left(-\frac{\pi^2}{2h^2}|\mathbf{n}|^2\right)\frac{1}{N!}\sum_{\sigma\in S_N}\det_{1\le j,k\le N}\left[\sin\left(\frac{\pi}{h}n_{\sigma(j)}y_j\right)\sin\left(\frac{\pi}{h}n_{\sigma(j)}x_k\right)\right]\\
&= \frac{1}{N!}\left(\frac{2}{h}\right)^N\sum_{\mathbf{n}\in\mathbf{N}^N}\exp\left(-\frac{\pi^2}{2h^2}|\mathbf{n}|^2\right)\det_{1\le j,k\le N}\left[\sin\left(\frac{\pi}{h}y_j n_k\right)\right]\det_{1\le j,k\le N}\left[\sin\left(\frac{\pi}{h}x_j n_k\right)\right],
\end{aligned}$$

where $\mathbf{N}=\{1,2,\ldots\}$ and $S_N$ is the set of all permutations of $N$ items $\{1,2,\ldots,N\}$. Here

$$\begin{aligned}
\det_{1\le j,k\le N}\left[\sin\left(\frac{\pi}{h}x_j n_k\right)\right] &= \det_{1\le j,k\le N}\left[\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}\left(\frac{\pi}{h}x_j n_k\right)^{2m+1}\right]\\
&= \sum_{\mathbf{m}\in\mathbf{N}_0^N}\prod_{j=1}^{N}\left\{\frac{(-1)^{m_j}}{(2m_j+1)!}x_j n_j\right\}\left(\frac{\pi}{h}\right)^{2\sum_{j=1}^{N}m_j+N}\det_{1\le j,k\le N}[(x_j n_k)^{2m_j}]\\
&= \prod_{j=1}^{N}(x_j n_j)\sum_{\mathbf{m}\in\mathbf{N}_0^N}\left(\frac{\pi}{h}\right)^{2\sum_{j=1}^{N}m_j+N}\prod_{j=1}^{N}\frac{(-1)^{m_j}}{(2m_j+1)!}\frac{1}{N!}\det_{1\le j,k\le N}[x_j^{2m_k}]\det_{1\le j,k\le N}[n_j^{2m_k}].
\end{aligned}$$

Therefore

$$\begin{aligned}
q_h^{(N)}(1,\mathbf{y}|\mathbf{x}) &= \frac{1}{N!}\left(\frac{2}{h}\right)^N\prod_{j=1}^{N}(x_j y_j)\sum_{\mathbf{n}\in\mathbf{N}^N}\exp\left(-\frac{\pi^2}{2h^2}|\mathbf{n}|^2\right)\prod_{j=1}^{N}n_j^2\\
&\quad\times\sum_{0\le m_1<m_2<\cdots<m_N}\left(\frac{\pi}{h}\right)^{2\sum_{j=1}^{N}m_j+N}\prod_{j=1}^{N}\frac{(-1)^{m_j}}{(2m_j+1)!}\det_{1\le j,k\le N}[x_j^{2m_k}]\det_{1\le j,k\le N}[n_j^{2m_k}]\\
&\quad\times\sum_{0\le \ell_1<\ell_2<\cdots<\ell_N}\left(\frac{\pi}{h}\right)^{2\sum_{j=1}^{N}\ell_j+N}\prod_{j=1}^{N}\frac{(-1)^{\ell_j}}{(2\ell_j+1)!}\det_{1\le j,k\le N}[y_j^{2\ell_k}]\det_{1\le j,k\le N}[n_j^{2\ell_k}].
\end{aligned}$$

Note that $\det_{1\le j,k\le N}[y_j^{2\ell_k}]=(-1)^{N(N-1)/2}\det_{1\le j,k\le N}[y_j^{2\ell_{N-k+1}}]$ and set $\mu_k=m_{N-k+1}-N+k$, $\nu_k=\ell_{N-k+1}-N+k$. Then $2\Sigma_{j=1}^{N}m_j+N=2|\mu|+N^2$ with $|\mu|\equiv\Sigma_{j=1}^{N}\mu_j$, and we have

$$\begin{aligned}
q_h^{(N)}(1,\mathbf{y}|\mathbf{x}) &= \frac{1}{N!}\left(\frac{2}{h}\right)^N\left(\frac{\pi}{h}\right)^{2N^2}\prod_{j=1}^{N}(x_j y_j)\prod_{1\le j<k\le N}\{(x_j^2-x_k^2)(y_j^2-y_k^2)\}\times\sum_{n\in\mathbf{N}^N}\exp\left(-\frac{\pi^2}{2h^2}|\mathbf{n}|^2\right)\prod_{j=1}^{N}n_j^2\\
&\quad\times\sum_{\mu:\ell(\mu)\le N}\sum_{\nu:\ell(\nu)\le N}\left(\frac{\pi}{h}\right)^{2(|\mu|+|\nu|)}\prod_{j=1}^{N}\left\{\frac{(-1)^{\mu_j+\nu_j}}{(2\mu_{N-j+1}+2j-1)!(2\nu_{N-j+1}+2j-1)!}\right\}\\
&\quad\times\det_{1\le j,k\le N}[n_j^{2(\mu_k+N-k)}]\det_{1\le j,k\le N}[n_j^{2(\nu_k+N-k)}]s_\mu(\{x_j^2\})s_\nu(\{y_j^2\}).
\end{aligned}$$

This gives (26).

## APPENDIX B: PROOF OF PROPOSITION 2

From (27), using (A2),

$$\mathbf{P}(H_N^{(N)} < h) = c_N h^{-N(2N+1)} \frac{1}{N!} \sum_{\mathbf{n}\in\mathbf{N}^N} \exp\left(-\frac{\pi^2}{2h^2}|\mathbf{n}|^2\right) \prod_{j=1}^N n_j^2 \det_{1\le j,k\le N}[n_j^{2(k-1)}] \det_{1\le \ell,m\le N}[n_\ell^{2(m-1)}]$$

$$= c_N h^{-N(2N+1)} \frac{1}{N!} \sum_{\mathbf{n}\in\mathbf{N}^N} \prod_{j=1}^N \left\{ n_j^2 \exp\left(-\frac{\pi^2}{2h^2} n_j^2\right) \right\} \sum_{\sigma\in S_N} \mathrm{sgn}(\sigma) \sum_{\rho\in S_N} \mathrm{sgn}(\rho) \prod_{\ell=1}^N n_\ell^{2(\sigma(\ell)+\rho(\ell)-2)}$$

$$= c_N h^{-N(2N+1)} \frac{1}{N!} \sum_{\sigma\in S_N} \sum_{\rho\in S_N} \mathrm{sgn}(\sigma)\mathrm{sgn}(\rho) \prod_{j=1}^N \left\{ \sum_{n_j=1}^\infty n_j^{2\sigma(j)+2\rho(j)-2} \exp\left(-\frac{\pi^2}{2h^2} n_j^2\right) \right\}.$$

Then

$$\mathbf{P}(H_N^{(N)} < h) = c_N h^{-N(2N+1)} \frac{1}{N!} \sum_{\mathbf{n}\in\mathbf{N}^N} e^{-\pi^2|\mathbf{n}|^2/(2h^2)} \sum_{\sigma\in S_N} \sum_{\rho\in S_N} \mathrm{sgn}(\sigma)\mathrm{sgn}(\rho) \prod_{j=1}^N n_j^{2\sigma(j)+2\rho(j)-2}$$

$$= c_N h^{-N(2N+1)} \frac{1}{N!} \sum_{\mathbf{n}\in\mathbf{N}^N} e^{-\pi^2|\mathbf{n}|^2/(2h^2)} \sum_{\sigma\in S_N} \sum_{\rho\in S_N} \mathrm{sgn}(\sigma)\mathrm{sgn}(\rho) \prod_{k=1}^N n_{\sigma^{-1}(k)}^{2k+2\rho(\sigma^{-1}(k))-2},$$

where we have set $k=\sigma(j)$. Since the summation $\Sigma_{\mathbf{n}\in\mathbf{N}^N}$ is taken, the above is equal to

$$c_N h^{-N(2N+1)} \frac{1}{N!} \sum_{\mathbf{n}\in\mathbf{N}^N} e^{-2\pi|\mathbf{n}|^2/(2h^2)} \sum_{\sigma\in S_N} \sum_{\rho\in S_N} \mathrm{sgn}(\sigma)\mathrm{sgn}(\rho) \prod_{k=1}^N n_k^{2k+2\rho(\sigma^{-1}(k))-2}$$

$$= c_N h^{-N(2N+1)} \frac{1}{N!} \sum_{\mathbf{n}\in\mathbf{N}^N} e^{-\pi^2|\mathbf{n}|^2/(2h^2)} \sum_{\sigma\in S_N} \sum_{\tau\in S_N} \mathrm{sgn}(\tau) \prod_{k=1}^N n_k^{2k+2\tau(k)-2} = c_N h^{-N(2N+1)} \sum_{\mathbf{n}\in\mathbf{N}^N} e^{-\pi^2|\mathbf{n}|^2/(2h^2)} \sum_{\tau\in S_N} \mathrm{sgn}(\tau) \prod_{j=1}^N n_j^{2j+2\tau(j)-2}$$

$$= c_N h^{-N(2N+1)} \sum_{\mathbf{n}\in\mathbf{N}^N} e^{-\pi^2|\mathbf{n}|^2/(2h^2)} \det_{1\le j,k\le N}[n_j^{2j+2k-2}] = c_N h^{-N(2N+1)} \det_{1\le j,k\le N}\left[\sum_{n=1}^\infty n^{2j+2k-2} e^{-\pi^2 n^2/(2h^2)}\right],$$

where $\tau=\rho\circ\sigma^{-1}$ and the relation $\mathrm{sgn}(\sigma)\mathrm{sgn}(\rho)=\mathrm{sgn}(\tau)$ was used. Then the first equality of (30) is proved. It is easy to confirm the second equality.


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