Random input problem for the nonlinear Schrödinger equation

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(Received 25 June 2008; revised manuscript received 8 September 2008; published 31 October 2008)

We consider the random input problem for a nonlinear system modeled by the integrable one-dimensional self-focusing nonlinear Schrödinger equation (NLSE). We concentrate on the properties obtained from the direct scattering problem associated with the NLSE. We discuss some general issues regarding soliton creation from random input. We also study the averaged spectral density of random quasilinear waves generated in the NLSE channel for two models of the disordered input field profile. The first model is symmetric complex Gaussian white noise and the second one is a real dichotomous (telegraph) process. For the former model, the closed-form expression for the averaged spectral density is obtained, while for the dichotomous real input we present the small noise perturbative expansion for the same quantity. In the case of the dichotomous input, we also obtain the distribution of minimal pulse width required for a soliton generation. The obtained results can be applied to a multitude of problems including random nonlinear Fraunhoffer diffraction, transmission properties of randomly apodized long period Fiber Bragg gratings, and the propagation of incoherent pulses in optical fibers.

DOI: 10.1103/PhysRevE.78.046610 PACS number(s): 42.65.Tg, 05.40.−a, 73.20.Fz

I. INTRODUCTION

The problems concerning the interplay between nonlinearity and randomness can be broadly divided into two principally different categories [1,2]. The first category comprises the problems where a pulse (wave) propagates in a nonlinear medium experiencing simultaneous action from some sources of randomness. These sources include, e.g., randomly distributed imperfections of the media [1,3] or active noisy inline elements, such as amplifiers in the optical fibers [4,5]. Such systems are typically described by nonlinear evolutionary equations driven by random noise sources. A typical example of such a problem is soliton propagation in the one-dimensional nonlinear Schrödinger equation (1D NLSE) channel with additive noise occurring for instance in the optical fiber communications [5]. Such a perturbation of the NLSE leads to random walks (jitters) of the soliton parameters [5,6]. Another celebrated example is the deviation from the Anderson law of localization for the stationary nonlinear transmission [7] (for additional references, see the aforementioned reviewing Refs. [1,2]).

The second class is occupied by *random input* problems where the evolutionary equation itself is deterministic but the randomness is introduced by the initial field distribution. One distinctive class of such problems (which is the subject of the current study) concerns the propagation of random waves in an *integrable* dynamical system, of which the NLSE is an avid example. Here the inverse scattering transform technique (IST) [8–10] provides the means of exploring the properties of the emerging solutions by considering an auxiliary direct scattering problem (an analogue of the forward Fourier transform for linear systems). Some important characteristics of the solution, such as, e.g., the spectral density of quasilinear radiation and parameters of the emerging solitons, can be determined by the solution of this linear direct scattering problem only. For the NLSE, the linear eigenproblem was established by Zakharov and Shabat [8] while a more general class of integrable systems yields an Ablowitz-Kaup-Newell-Segur (AKNS) hierarchy [10]. The propagation of noncoherent pulses in the NLSE has long been a subject of investigation in different contexts [11,12], including random phase modulation of temporal solitons in optical fibers [13–16], random generation of dark solitons [17], and nonlinear Fraunhofer diffraction of random fields [2,18].

In the current paper, we will deal with the second class of random nonlinear problems and restrict ourselves to the propagation of signals originated from a random input for the focusing NLSE (which in the context of optical communications corresponds to pulse propagation in an optical fiber in the regime of anomalous dispersion). As noted above within the framework of IST, one eventually arrives at a linear eigenproblem [Zakharov-Shabat spectral problem (ZSSP)] where the initial (random) field distribution plays the role of a potential [2]. For the focusing NLSE, the ZSSP is non-self-adjoint giving rise to the complex discrete spectrum, which is responsible for the parameters of emerging solitons. The self-adjoint version of the same eigenproblem occurs, for instance, when considering the defocusing NLSE and is mathematically equivalent to the 1D Dirac equation in the presence of disorder [17,19]. In the self-adjoint reduction of the ZSSP, many properties of the eigenvalue spectra remind those of the linear Schrödinger equation [20] including, e.g., Anderson localization, which manifests itself as an exponential decrease of the transmissivity with the growth of the region occupied by the disordered potential. It must also be noted that both "focusing" and "defocusing" reductions of the corresponding ZSSP are mathematically equivalent to a coupled-mode theory approach used, for instance, for analyzing the reflection and transmission spectra of transmission (long period) and reflection fiber Bragg gratings [21]. Here

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we will only consider the case of focusing NLSE (non-self-adjoint ZSSP).

We will consider here two rather different models for our random input: (i) the symmetric complex Gaussian white noise (WGN) and (ii) a special case of dichotomous (two-level) process [20,22,23] (sometimes known as the telegraph process). The word "symmetric" in case (i) signifies that its real and imaginary parts are uncorrelated and possess the same Gaussian statistics. Such a model provides a generic example of a fully incoherent random input pulse. On the other hand, the dichotomous input, (ii), considered in our study, is a piecewise real function with the length of each segment sampled from an exponential distribution. We will opt here for a positive dichotomous process with the lower level being equal to zero. The relative simplicity of the model allows one to study analytically the effect of the finite correlation time on the properties of the system [20,22,23]. It also possesses various asymptotic limits leading to regimes of real WGN as well as a shot noise [20,22].

The paper is organized as follows. In the next section, we will briefly present the basics of the direct spectral problem (ZSSP) corresponding to the focusing NLSE. Then we discuss a general topic of soliton creation from initial pulses of finite support and recall known results obtained by virtue of the invariant imbedding method, in particular we revisit the stochastic Langevin equations for the reflection coefficient of the ZSSP. We also demonstrate that within the framework of the invariant imbedding method, the problem of random soliton creation corresponds to a well-known first time passage problem. The subsequent sections are devoted to the direct study of two aforementioned models for the random input profile. Section IV deals with complex WGN—here we derive the expression for the averaged spectral density of quasilinear radiation propagating in the NLSE channel. Section V is devoted to the asymptotic derivation of the averaged spectral density for the dichotomous process in the limit of strongly different correlation times in the distributions for two entering states. In Sec. VI, we study the problem of soliton formation for the real dichotomous input profile. The results are summarized in the conclusion. The Appendixes provide all the necessary technical details of our derivations.

II. ZAKHAROV-SHABAT EIGENVALUE PROBLEM

The focusing 1D NLSE (written here in normalized units) has the form

$$\frac{\partial Q}{\partial z} + \frac{i}{2}\frac{\partial^2 Q}{\partial t^2} + i|Q|^2 Q = 0, \tag{1}$$

where $Q(t,z)$ is the field variable. In the context of nonlinear optics, Eq. (1) describes the propagation of the optical pulse envelope in a single-mode silica fiber [4,5]. In this case, z stands for the propagation distance and t is the time in the frame comoving with the group velocity of the envelope. When considering the nonlinear diffraction problems, both z and t have to be understood as spatial coordinates in the transverse directions. In other (more conventional) applications, one must simply substitute time t for z and coordinate x for t in Eq. (1) and all subsequent formulas of the paper.

Given an arbitrary localized initial profile $Q(t)|_{z=0}$, the direct scattering problem (ZSSP) associated with Eq. (1) reads [8,9]

$$\begin{aligned} i\partial\psi_1/\partial t + Q(t)\psi_2 &= \zeta\psi_1, \\ -i\partial\psi_2/\partial t - Q^*(t)\psi_1 &= \zeta\psi_2, \end{aligned} \tag{2}$$

where the asterisk designates complex conjugation. In Eq. (2), $\psi_{1,2}$ are the components of a vector eigenfunction, and ζ is the (generally complex) eigenvalue. Scattering problem (2) is studied by introducing the notion of the Jost functions. These are linearly independent solutions of Eq. (2) with the following asymptotic behavior [8,9]:

$$\Psi(t;\zeta) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\zeta t}, \quad t \to +\infty, \tag{3a}$$

$$\Phi(t;\zeta) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\zeta t}, \quad t \to -\infty. \tag{3b}$$

In addition to function $\Psi = (\psi_1, \psi_2)$, it is convenient to introduce an involuted function $\tilde{\Psi} = (\psi_2^*, -\psi_1^*)$. Then the following relation is valid:

$$\Phi(t;\zeta) = a(\zeta)\tilde{\Psi}(t;\zeta) + b(\zeta)\Psi(t;\zeta), \tag{4}$$

where $a(\zeta)$ and $b(\zeta)$ are the first and second Jost coefficients (scattering coefficients), respectively, and it is the properties of these coefficients that constitutes the basis of the direct scattering problem. The Jost coefficients satisfy the "normalization condition" $|a|^2 + |b|^2 = 1$ (see [9]).

III. THE INVARIANT IMBEDDING APPROACH FOR THE ZSSP EIGENVALUE PROBLEM ON A FINITE INTERVAL

In this paper, we will deal with random initial pulses $Q(t)$ of finite support. For such a class of random (and deterministic) problems, there exists a powerful tool of analysis, namely the invariant imbedding method [1,2,22]. Its main idea is to reformulate a boundary problem like Eqs. (2) and (3) as an equivalent initial-value problem and then make use of the Markov properties of this initial-value problem. Random Markov initial-value problems can then be analyzed using standard methods of stochastic analysis (see, e.g., [22,24]). We will briefly recall the formulation of the invariant imbedding method for Eq. (2) [1,2].

We will assume that our ZSSP potential $Q(t)$ vanishes outside the interval $[0,T]$. Let us consider the solution $\Phi(t;\zeta)$ of Eq. (2) specified by its boundary value at $t = -\infty$ [Eq. (3b)]. In view of the finite width of Q and Eqs. (3) and (4), one has the following effective boundary conditions:

$$\Phi(0;\zeta) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$\Phi(T;\zeta) = a(\zeta)e^{-i\zeta T}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b(\zeta)e^{i\zeta T}\begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{5}$$

In the invariant imbedding framework, one considers the finite width, T, of the potential as a new dynamical variable and analyzes how the properties of Jost coefficients (and hence the scattering data) depend on T. Indeed, from Eqs. (2) and boundary conditions (5), one gains two first-order ODEs providing the evolution of both Jost coefficients as functions of the width T,

$$\frac{\partial a(\zeta;T)}{\partial T} = ib(\zeta;T)e^{2i\zeta T}Q(T), \quad a(\zeta,0)=1,$$

$$\frac{\partial b(\zeta;T)}{\partial T} = ia(\zeta;T)e^{-2i\zeta T}Q^{*}(T), \quad b(\zeta,0)=0. \qquad (6)$$

This system automatically preserves the normalization condition $|a(\zeta;T)|^2+|b(\zeta,T)|^2=1$. Equations (6) can be analytically extended to the upper complex plane $\mathrm{Im}\,\zeta \geqslant 0$ due to the analyticity of the coefficient a [9].

A. General remarks on soliton creation

Let us now discuss briefly the problem of soliton emergence. It is known from the IST theory (see [9]) that the Jost coefficient $a(\zeta;T)$, defined by Eq. (4) and evolving according to Eq. (6), has a countable number of simple zeros $\zeta_k=\xi_k+i\eta_k$ in the upper complex plane $\mathrm{Im}\,\zeta \geqslant 0$. Each zero corresponds to a discrete eigenvalue of ZSSP or equivalently to a soliton of the NLSE. The real part ξ_k of the eigenvalue provides the velocity of each soliton while the imaginary η_k is related to its amplitude. The continuous spectrum of the ZSSP is real and is responsible for the nonsoliton or radiative part of the potential. Calculation of the number of solitons amounts now to counting the number of zeros of the coefficient $a(\zeta;T)$ that will of course depend on the width T. Since $a(\zeta;T)$ is analytic in the upper half-plane, the number of its zeros can be determined from the *argument principle*,

$$N = \frac{1}{2\pi i}\oint_C \frac{1}{a}\frac{\partial a}{\partial \zeta}d\zeta, \qquad (7)$$

where the loop C comprises the lower real axis $C_-:\zeta=\xi-i0$ and the infinite semi-arc in the upper half-plane. Since the asymptotic behavior of the coefficient a is $a=1+O[1/\zeta]$ for $|\zeta|\gg 1$ [9], the integral over the upper part of the loop is zero. Next, if we introduce the modulus and phase of a as $a=\rho(\zeta;T)\exp[i\alpha(\zeta;T)]$, then the number of solitons is given by

$$N(T) = \frac{1}{2\pi}\int_{C_-}\frac{\partial \alpha(\zeta;T)}{\partial \zeta}d\zeta. \qquad (8)$$

Now let a set of values $T_k:0<T_1<\cdots<T_M<T$ define the values of the pulse width when the number of solitons changes by a finite amount n_k. Positive n_k correspond to soliton creation while negative n_k correspond to annihilation. We allow for multiple soliton creation so that n_k can be arbitrary integers. The rate of change of the number of solitons with pulse width is given by

$$\frac{dN}{dT} = \sum_k n_k \delta(T-T_k).$$

Then from Eq. (8) there follows the necessary condition for soliton creation or annihilation at $T=T_k$, namely the rate of change of the total phase shift, $\int_{C_-}\partial\alpha(\zeta;T)/\partial\zeta d\zeta$, evaluated at T_k should be nonzero (singular in fact). But from the first equation in Eqs. (6), it follows that

$$\partial\alpha(\zeta;T)/\partial T = [\rho(\zeta;T)]^{-1}\times \mathrm{Re}[b(\zeta;T)Q(T)e^{2i\zeta T - i\alpha(\zeta;T)}]. \qquad (9)$$

Now unless the function $\rho(\zeta;T_k)$ has at least one zero at the real axis, the contour C_- can be analytically deformed into the real axis $\zeta=\xi$. Then the right-hand side (r.h.s.) of the above equation is a continuous, finite function of ξ and the rate of change of the total phase shift is just the difference of the values of the r.h.s. evaluated at $\xi=+\infty$ and $-\infty$. But the latter is zero in view of the asymptotic behavior of a at large ζ and the normalization condition. This means that the necessary condition for the number of solitons to change at $T=T_k$ is the existence of at least one real solution ξ_k of the equation $a(\xi_k;T_k)=0$. But this criterion is also sufficient: indeed since all the discrete eigenvalues are the solutions of the equation $a(\zeta_k;T)=0$, the condition $a(\xi_k;T_k)=0$ means that a soliton (or several solitons if the solution ξ_k is not unique) with zero amplitude is created (or destroyed) at $T=T_k$ and the number of solitons N changes.

This criterion has a very simple physical interpretation. When one starts from an infinitesimally narrow potential, $T\to 0$, there are no eigenstates (solitons) present and $a\to 1$. As one increases the width, the Jost coefficients begin to evolve according to Eq. (6), and eventually as the first threshold, T_1, is reached the first soliton (or a couple of solitons if a has multiple zeros) is created with zero amplitude and velocity (velocities) given by $a(\xi_1;T_1)=0$. As the potential widens, the first discrete eigenvalue shifts toward the upper half-plane until the next eigenvalue(s) appears at $a(\xi_2;T_2)=0$, etc. It can also happen that some of the emerging eigenvalues can return to the real axis and annihilate—that would correspond to the destruction of solitons.

The condition $a(\xi_k;T_k)=0$ alone is insufficient to determine whether a soliton is created or annihilated at $T=T_k$. In order to elucidate this, one has to know how existing eigenvalues evolve with T. The corresponding procedure is very similar to that of the IST-based soliton perturbation theory [25]. The rate of change of an arbitrary eigenvalue ζ_k is given by

$$\frac{d\zeta_k}{dT} = \frac{\partial a(\zeta_k;T)}{\partial T}\left.\left(\frac{\partial a(\zeta;T)}{\partial \zeta}\right)^{-1}\right|_{\zeta=\zeta_k}.$$

Substituting the derivative $\partial a/\partial T$ from the first Eq. (6) and assuming that the eigenvalue is real, $\zeta_k=\xi_k$, we arrive at

$$\frac{d\zeta_k}{dT} = iC(\zeta_k;T)e^{2i\zeta_k T}Q(T), \qquad (10)$$

where $C(\zeta_k;T)=b(\zeta_k;T)(\partial a(\zeta;T)/\partial\zeta)^{-1}|_{\zeta=\zeta_k}$.

Now in order to determine whether a soliton is created or destroyed at $T=T_k$, one simply has to check the sign of the derivative $d\eta_k/dT$, that is to say, the sign of a quantity $\mathrm{Re}[C(\xi_k;T_k)\exp(2i\xi_k T_k)Q(T_k)]$. If it is positive, a soliton is being created; if negative, a soliton is being destroyed. In the case of a general complex potential, it is difficult to apply this criterion directly because neither of the quantities T_k, ξ_k, and C is known analytically (apart from a very limited number of solvable cases).

There are situations, however, when one can carry out the analysis further. Such is the case of creation or annihilation of the localized solitons with zero velocities so that $\mathrm{Re}(\zeta_k)=0$ always. Let us demonstrate, for instance, that for real, positive pulses, the localized eigenstates can only be created and not destroyed. In order to do so, we must determine the set of coefficients $C(0;T_k)=b(0;T_k)/a_\zeta(0;T_k)$ where we have used a shorthand notation a_ζ for the derivative of $\partial a/\partial\zeta$. System (6) with real positive Q and $\zeta=0$ can be easily solved, yielding

$$a(0;T)=\cos\left(\int_0^T Q(t)dt\right),$$

$$b(0;T)=i\sin\left(\int_0^T Q(t)dt\right).$$

The derivative $a_\zeta(0;T)$ can also be determined without considerable difficulties. Assuming that T takes one of the threshold values T_k (easily obtained from the expression above), we arrive at the following result:

$$C(0;T_k)=\left[2\int_0^{T_k}\sin^2\left(\int_0^t Q(t')dt'\right)Q(t)tdt\right]^{-1}>0, \tag{11}$$

and the rate of change of the infinitesimally small soliton amplitude is given by

$$\frac{d\eta_k}{dT}=C(0;T_k)Q(T_k)>0. \tag{12}$$

This relation proves that the localized states can only be created when increasing the width of the potential.

B. Basic equations

The invariant imbedding in terms of the Jost coefficients a and b provides exhaustive information about spectral properties of the Zakharov-Shabat eigenvalue problem. However, it turns out that a lot of the information can also be established by analyzing the dynamics of a single complex variable, namely a reflection coefficient, given by the ratio $r_T=b/a$ on a real axis $\zeta=\xi$. To be more specific, let us introduce the following function: $r(t)=\Phi_2(t)/\Phi_1(t)$, where the components $\Phi_{1,2}(t)$ are defined as previously. From Eq. (2), it follows that inside the interval $[0,T]$ it obeys the following complex Riccati equation [2,17,18] (note that ξ is real):

$$\frac{dr}{dt}=2i\xi r-iQr^2+iQ^*. \tag{13}$$

From boundary conditions Eq. (5) one gains $r(0)=0$ and $r(T)=r_T e^{2i\xi T}$. This is of course also in full accordance with Eqs. (6). Therefore, the reflection coefficient r_T can be obtained from the solution of a single ODE Eq. (13), satisfying the initial condition $r(0)=0$, taken at $t=T$. This is a conventional invariant imbedding approach for eigenvalue problems of the type (2) and it comes in particularly handy when analyzing the spectral data of stochastic potentials, which is the case considered in this paper.

Since the normalization condition implies boundness of both Jost coefficients $a(\xi;T)$ and $b(\xi;T)$, one observes that the condition of soliton creation or annihilation $a(\xi_k;T_k)=0$ corresponds to a finite-time singularity in the solution of Riccati equation (13) observed for a given value of spectral parameter $\xi=\xi_k$. The problem of determining the thresholds of soliton creation or annihilation (these for a random potential are stochastic quantities) amounts now to determining the statistics of the blowup times T_k of Riccati equation (13). Therefore, in the remainder of the paper we will work with a single stochastic equation (13) rather than with the system (6).

In order to study the random dynamics given by Eq. (13), it is convenient to introduce the following parametrization of the complex quantity $r(t)$

$$r(t)=\cot(\theta/2)e^{i\varphi(t)}, \tag{14}$$

where θ and φ are real angular variables, the former responsible for the amplitude of the reflection coefficient while the latter is the phase. Then it is easy to demonstrate that Eq. (13) is equivalent to a system of two ordinary stochastic differential equations (SDE) defined inside a finite box,

$$\frac{d\theta}{dt}=-2\,\mathrm{Im}[Qe^{i\varphi}],\quad 0\leqslant\theta\leqslant\pi, \tag{15a}$$

$$\frac{d\varphi}{dt}=2\xi-2\cot\theta\,\mathrm{Re}[Qe^{i\varphi}],\quad 0\leqslant\varphi\leqslant 2\pi. \tag{15b}$$

Therefore, the problem under investigation is equivalent to studying a stochastic motion of a "particle" (which we will subsequently call a "probability particle") inside a finite box under the prescribed boundary conditions. One always launches this particle at one side of the box $\theta_0=\pi$, which corresponds to the initial condition $r(0)=0$. In φ, the boundary conditions are periodic, i.e., a probability particle disappearing at the $\varphi=0$ or $\varphi=2\pi$ reemerges at the opposite boundary. At the boundary $\theta=0$, a singularity in r occurs and a new soliton is born.

A special case of real, positive potential considered above now corresponds to a system (15) with real positive Q and $\xi=0$. One can readily see that the only finite solution adhering to the initial condition $\theta_0=\pi$ must have the phase $\varphi(t)=(\pi/2)(2n+1)$ with $n=0,1$ and $\varphi_0=\pi/2$. The phase φ retains its value between the points $\theta=0$ and $\theta=\pi$ and switches to another state each time a particle is reflected from either boundary in θ. Because the potential is positive, the motion of the probability particle is monotonic during each stage of dynamical evolution and one can work out exactly the number of times the particle is reflected from the boundary $\theta=0$, which gives a well-known criterion of Ref. [26]. In particular, the problem of creation of a single soliton state in real positive stochastic potential coincides in this formalism with the well-known problem of the first passage time distribution for a "particle" to reach the opposite boundary. This circumstance will be used in Sec. VI when deriving the distribution of minimal potential width required for the soliton creation.

At the end of this section, a few remarks are in order pertaining to the stochastic soliton creation. Because the Jost coefficients and the reflection coefficient $r(\xi;t)$ are stochastic functions, their zeros (or singularities) are also distributed randomly. It may happen so that if a random potential contains certain symmetries (as in the case of real positive potential from Sec. VI), one knows exactly the location of possible singularities, ξ_k, of $r(\xi;t)$, that is to say, if a soliton is created at all it is created with a certain known value of $\xi=\xi_k$ with probability 1. There are situations, however, like the one considered in the next section, when such a probability of soliton being created with the exact velocity $\xi=\xi_k$ is zero for any given ξ_k. This does not prove, however, that solitons are not created at all; one rather has to think in terms of probability density $P(\xi)$ of a soliton being created in an infinitesimally small interval $[\xi,\xi+d\xi]$. Such a probability density can in fact be nonzero even if the probability of having a soliton emerging at the exact spot ξ is zero. We shall discuss this issue further at the end of Sec. IV.

IV. THE CASE OF COMPLEX δ-CORRELATED POTENTIAL

In this section, we will consider the case of initial input as symmetric complex white Gaussian noise (WGN) $Q(t)=\eta_1(t)+i\eta_2(t)$ with the correlation properties

$$\langle \eta_i(t)\eta_j(t')\rangle = D\delta_{ij}\delta(t-t'), \tag{16}$$

i.e., both $\eta_{1,2}$ are real independent WGNs and the process is confined within the interval $[0,T]$. The resulting Langevin system (15) [or Eq. (13)] contains multiplicative white noise and therefore a proper temporal regularization is required yielding either Ito or Stratonovich representation [24]. Here we assume that the potential Q is a limit of a process with a narrow but symmetrical correlation function and thus the Stratonovich interpretation applies.

Using the standard procedure (described, e.g., in [24]) we can write down the Fokker-Planck equation (FPE) for the probability density function (PDF) $P(\theta,\varphi;t)$,

$$\frac{\partial P}{\partial t} = -2D\frac{\partial}{\partial\theta}(\cot\theta P) - 2\xi\frac{\partial P}{\partial\varphi} + 2D\frac{\partial^2 P}{\partial\theta^2} + 2D\cot^2\theta\frac{\partial^2 P}{\partial\varphi^2}. \tag{17}$$

Using periodic boundary conditions in φ, we can obtain the corresponding FPE for the marginal probability density $P(\theta;t)$,

$$\frac{\partial P}{\partial t} = -2D\frac{\partial}{\partial\theta}(\cot\theta P) + 2D\frac{\partial^2 P}{\partial\theta^2}. \tag{18}$$

Now recall that since we have $\theta_0=\pi$ at the initial moment $t=0$, the initial condition for the PDF must be $P(\theta;0)=\delta(\theta-\pi)$. The boundary conditions at $\theta=0,\pi$ are specified by supplying the probability flux defined from Eq. (18) as

$$J_\theta = 2D\left[\cot\theta P - \frac{\partial P}{\partial\theta}\right] \tag{19}$$

with the limits

$$J_0 = \lim_{\theta\to 0} J_\theta, \quad J_\pi = \lim_{\theta\to\pi} J_\theta. \tag{20}$$

If one is to allow a nonzero probability of a finite-time singularity, one must have $J_0<0$ (the probability particles diffusing out of the system from the edge $\theta=0$).

It is convenient to proceed to a new function $Y(\theta;t)$ instead of former $P(\theta;t)$ by making the substitution $Y(\theta;t)\sin(\theta)=P(\theta;t)$. This substitution brings Eq. (18) to the following form:

$$\frac{\partial Y}{\partial t} = 2D(\sin\theta)^{-1}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right), \tag{21}$$

and the boundary condition for fluxes, Eqs. (20), written explicitly in terms of new function $Y(\theta)$ now reads

$$-2D\lim_{\theta\to 0,\pi} Y'(\theta)\sin\theta = J_{0,\pi}. \tag{22}$$

Equation (21) with natural boundary conditions (zero boundary flux) was studied in Ref. [19] in a different context. Such boundary conditions correspond to the case in which no solitons can be created and the solution of NLSE (1) contains a radiative part only. However, here we would like to elucidate the situation in general and so present the complete analysis for the solution of the boundary problem defined by Eqs. (21) and (22) (the technical details are given in Appendix A). The results of our analysis are as follows:

First, the symmetric white Gaussian noise, Eq. (16), cannot produce a finite-time singularity at each given spectral point ξ. The physical situation can be explained using the following arguments. The FPE given by Eq. (18) corresponds to the following Langevin system: $\dot{\theta}=2D\cot\theta+\eta$, where $\eta(t)$ is the additive WGN. One can see that as a probability particle approaches the soliton creation boundary, which is $\theta=0$, the velocity of the particle becomes infinite and positive. This means that as the particle diffuses toward the boundary, it faces an ever-increasing advection that precludes the collision with the wall. In this case, the advection always wins against the diffusion and the probability of soliton creation in a finite time slot $[0;T]$ is exactly zero. A more rigorous proof of this statement is given in Appendix A.

Second, since no finite-time singularities emerge, the natural zero-flux boundary conditions should apply and under such conditions the solution of Eq. (21) is found to be

$$P(\theta;t) = \frac{1}{2}\sin(\theta)\sum_{n=0}^{\infty}(-1)^n(2n+1)\mathrm{P}_n(\cos\theta) \times \exp[-2Dn(n+1)t], \quad (23)$$

where $\mathrm{P}_n(\cos\theta)$ are the Legendre polynomials of the nth order.

The result above can be used for calculating the average spectral density of radiation, $\bar{n}(\xi,T)$, propagating in the nonlinear channel [9],

$$\bar{n}(\xi,T) = \frac{1}{\pi}\langle \ln(1+|r|^2)\rangle, \quad (24)$$

where $\langle\cdots\rangle$ means the ensemble averaging. Evaluating the integral, one gains

$$\bar{n}(T) = \frac{1}{\pi}\left[1 + \sum_{n=1}^{\infty}(-1)^n \frac{2n+1}{n(n+1)} e^{-2Dn(n+1)T}\right]. \quad (25)$$

This can be compared with formula (3.14) of Ref. [18] obtained asymptotically for $\xi T \ll 1$ and for the real form of the initial Gaussian δ-correlated potential $Q(t)$. Note that the case of complex symmetric noise allows the *exact* analytical treatment due to the unitary symmetry of the complex WGN in Eqs. (15). Also $\bar{n}(T)$ given by Eq. (25) is independent of the spectral parameter ξ, which is a consequence of the δ-correlated nature of noise, and so the radiation in the NLSE channel has a flat spectrum.

The infinite summation in Eq. (25) can be expressed in a closed form via the elliptic θ functions, but the resulting expression is still quite cumbersome and we do not give it here. Instead we write down the following asymptotic expansions for the averaged spectral density:

(i) The linear limit, $DT \ll 1$, yields

$$\bar{n}(T) = \frac{2DT}{\pi}. \quad (26)$$

(ii) In the limiting case of strong incoherence, $DT \gg 1$, the exponential terms in Eq. (25) are negligible and we have a constant asymptotic of the averaged spectral density,

$$\bar{n}(T) = \frac{1}{\pi}. \quad (27)$$

Similar asymptotes were obtained in Ref. [18]. In Fig. 1, we depict the general form of the averaged radiation density $\bar{n}$ versus the parameter DT as given by Eq. (25).

As mentioned at the end of Sec. III, having a zero probability of a soliton being created at a given spectral value $\xi = \xi_k$ does not necessarily mean that no solitons are created at all. And indeed in the very recent Ref. [27], the authors also considered a model of symmetric Gaussian white noise and calculated a semiclassical density of soliton states in the limit of infinitely long potential $T\to\infty$. They used the generalization of the Thouless formula applied to non-self-adjoint ZSSP, which relates the semiclassical density of eigenstates (DOS) with the Lyapunov exponent of system (2). Their results seem to suggest that there is in fact a nonzero DOS, generated by an infinitely long white complex Gaussian potential. While there are some technical queries regarding the validity of their approach (the white noise does not vanish at infinity, which makes the usual machinery of IST inapplicable in the strict mathematical sense), the finiteness of DOS looks plausible regarding the fact that under similar circumstances a Hermitian Schrödinger equation does have a finite DOS limit [20]. Unfortunately, the Lyapunov exponent method is only applicable to the long enough (formally infinitely long) potentials (which for the white noise amounts to the semiclassical treatment) and cannot be applied to our problem statement in which T is relatively small and only a few (possibly none) solitons (eigenstates) are created. Therefore, the problem of actual soliton creation by complex WGN requires further investigation.

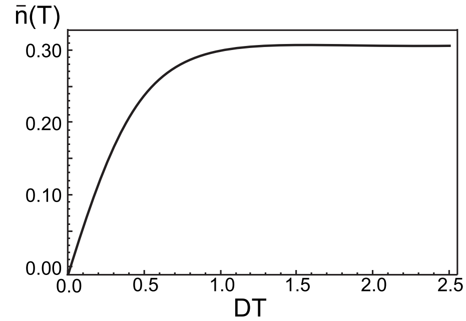

FIG. 1. The plot of the average spectral density $\bar{n}(T)$ vs DT for a complex symmetric WGN input as defined by Eq. (25).

V. THE RADIATION DENSITY: REAL DICHOTOMOUS PROCESS

In this section, we study the real asymmetric dichotomous process as an initial input for our NLSE channel. We introduce a real positive telegraph signal $s(t)$ that takes the values either $s_0=0$ or $s_1=1$ at the sojourn intervals whose lengths are random and sampled from the independent exponential PDFs: $p_s(t)=c_s\exp(-c_s t)$, $s=0,1$. These distributions correspond to the two different Poissonian distributions for the number of hops from s_0 to s_1 and back; the corresponding average numbers of hops are $c_s t$. Our potential $Q_0(t)$ can be set to $s(t)$ if one rescales both time and spectral parameter ξ in Eqs. (2), (13), and (15): $\xi/Q_0\to\xi$, $2Q_0 t\to t$. The constants $c_{0,1}$ define the correlation time of this process; see Eq. (29) below.

Let $P(t,s)$ be the probability of $s(t)$ to be equal to the value s at time t (recall that $s=0,1$). The guiding equation for $P(t,s)$ (an analogue of the FPE) is [20,23,24]

$$\frac{dP(t,s)}{dt} = -c_s P(t,s) + c_{1-s}P(t,1-s). \quad (28)$$

The general solution for the initial-value problem Eq. (28) is given, e.g., in monographs [20,24], and the steady state is characterized by a nonzero mean $\langle s(t)\rangle|_S=c_0/(c_0+c_1)$ as well as a binary stationary correlator,

$$\langle s(t)s(t')\rangle|_S = \frac{c_0 c_1}{(c_0+c_1)^2}\exp[-(c_0+c_1)|t-t'|], \quad (29)$$

so that $(c_0+c_1)^{-1}$ serves as a finite correlation time for our random process.

Returning to the original problem statement, Eqs. (15a) and (15b), where we have two state variables, θ and φ, the probability $P(s,t)$ is expressed as $P(t,s) = \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi P_s(t,\theta,\varphi)$. Here $P_s(t,\theta,\varphi)$ are the PDFs for the state variables θ and φ and the discrete variable s.

Now we assume that our initial profile $Q(t)$ is real and equal to $s(t)$. The set of two coupled FPEs for the conditional PDFs $P_s(t,\theta,\varphi)$ can be derived from the set of Langevin equations (15b) and (15a), where we now set $Q(t)\equiv s(t)$ (and rescale time correspondingly),

$$\frac{\partial P_s}{\partial t} = -\frac{\partial}{\partial q_i} g_i^s(q) P_s - c_s P_s + c_{1-s} P_{1-s}, \quad (30)$$

where $P_s = P(t,\theta,\varphi,s)$, $q=\{\theta,\varphi\}$, and $g_i^s(q)$ are written as

$$g_\theta^s(\theta,\varphi) = -s\sin\varphi,$$

$$g_\varphi^s(\theta,\varphi) = \xi - s\cot\theta\cos\varphi. \quad (31)$$

The probability flows in two orthogonal directions by virtue of Eq. (30) are found as

$$J_i(t,\theta,\varphi) = \sum_{s=0,1} g_i^s(\theta,\varphi) P_s(t,\theta,\varphi). \quad (32)$$

As the general solution of Eq. (30) with expressions (31) inserted cannot be obtained in a closed analytical form, we consider here only the cases of strongly different hopping rates c_0 and c_1, i.e., a strongly intermittent process. Then one can resort to an iterative procedure for finding the solution of Eq. (30), and repeating this procedure one can obtain the desired expressions for PDFs P_0 and P_1 in the form of ordinary expansion up to a desired order.

A. The case $c_0 \ll c_1$

Let us assume that the inequality $c_0 \ll c_1$ for the hopping rates between our two sojourn intervals is fulfilled. Then most of the time our potential is zero, i.e., "off." It is convenient to introduce the small parameter $c_0/c_1 = \epsilon \ll 1$, and also the rescaled time $\tau = c_1 t$ and spectral parameter $\tilde{\xi} = c_1^{-1}\xi$. We assume here that our PDFs can be presented in the form of regular expansions in terms of ϵ: $P_s = \sum_k \epsilon^k P_s^k$. Now we substitute these expansions in Eq. (30) and equalize the terms in each order in ϵ. This procedure allows one to decouple and solve separately the equations in each order and thus to construct the approximate solution up to a desired precision.

It is also convenient to introduce the partial averages of the spectral density $\bar{n}$, given by Eq. (24),

$$\bar{n}_s^k(\tau,\tilde{\xi}) = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta\, n(\varphi,\theta) P_s^k(\tau,\varphi,\theta), \quad (33)$$

so that we can write the full expansion of the average as $\bar{n}(\tau,\tilde{\xi}) = \sum_k \sum_s \epsilon^k \bar{n}_s^k(\tau,\tilde{\xi})$.

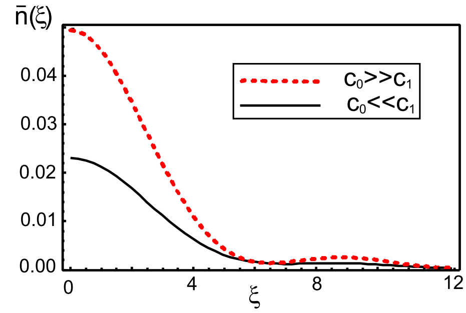

FIG. 2. (Color online) The plot of $\bar{n}(\xi)$ vs ξ for the dichotomous input with a fixed support of length $T=1$. Solid line is the case $c_0 \ll c_1$, where we set $c_1=1$, $c_0=0.1$; dashed line is the same for the case $c_0 \gg c_1$, where we set $c_0=1$, $c_1=0.1$. The initial probabilities $P_0(0)=P_1(0)=1/2$.

The details of the iterative procedure used for constructing the solution for PDFs $P_s^k(\tau,\varphi,\theta)$ are given in Appendix B. Note that for finding the explicit expressions given below, formula (B13) comes handy.

In the zeroth order by virtue of Eqs. (B8) and (B9) from Appendix B, we have the following expressions:

$$\bar{n}_1^0(\tau,\tilde{\xi}) = P_1(0) n(\tau) e^{-\tau}, \quad (34)$$

$$\bar{n}_0^0(\tau,\tilde{\xi}) = P_1(0) \int_0^{\tau} d\tau' n(\tau') e^{-\tau'}, \quad (35)$$

where $n(\tau) = (1/\pi)\ln(1+|r(\tau)|^2)$, and $r(\tau)$ is given by

$$r(\tau) = \frac{i c_1^{-1}}{\kappa \cot[\kappa\tau/2] - i\tilde{\xi}}, \quad (36)$$

with $\kappa = \sqrt{c_1^{-2} + \tilde{\xi}^2}$. In Eq. (36), $r(\tau)$ is merely the solution (B6) taken along the trajectory with $\theta_0 = \pi$, $\varphi_0 = \pi/2$, i.e., with $r_0 = 0$.

In the next order by the use of Eqs. (B11) and (B12) after some straightforward calculations one finds

$$\bar{n}_1^1(\tau,\tilde{\xi}) = \int_0^{\tau} d\tau' e^{\tau'-\tau} \left\{ P_0(0) n(\tau-\tau') + P_1(0) \times \int_0^{\tau'} d\tau'' n(\tau-\tau'+\tau'') e^{-\tau''} \right\}, \quad (37)$$

$$\bar{n}_0^1(\tau,\tilde{\xi}) = \int_0^{\tau} [\bar{n}_1^1(\tau';\tilde{\xi}) - \bar{n}_0^0(\tau';\tilde{\xi})] d\tau'. \quad (38)$$

Note that since the definition of $n(\tau)$ involves only $|r|^2$ and also due to Eq. (36), the average radiation density is the function of ξ^2 only and is thus an even function of ξ. The corresponding plot for the average spectral density of linear radiation $\bar{n}(\xi)$ calculated up to the first order in ϵ is given in Fig. 2 (solid line).

B. The case $c_1 \ll c_0$

Now let us turn to the case in which the opposite inequality holds, i.e., $c_1 \ll c_0$. In this case, the potential is "on" most of the time. We again introduce a small parameter $c_1/c_0=\varepsilon \ll 1$, and scaled quantities $\tau=c_0 t$, $\tilde{\xi}=c_0^{-1}\xi$. The solution of Eq. (30) is again represented as a regular perturbative expansion: $P_s=\Sigma_k \varepsilon^k P_s^k$. The details of this procedure are given in Appendix B in outline, the procedure itself being quite similar to that used for the opposite case.

The corresponding partial averages in the zeroth order in ε by virtue of Eqs. (B15) and (B16) are found as

$$\bar{n}_0^0(\tau,\tilde{\xi})=0,$$

$$\bar{n}_1^0(\tau,\tilde{\xi})=P_1(0)n(\tau)+P_0(0)\int_0^\tau d\tau' e^{-\tau'} n(\tau-\tau'). \quad (39)$$

In the first order in ε using formulas (B18) and (B19), we have the following expressions for the averages:

$$\bar{n}_0^1(\tau,\tilde{\xi})=e^{\tau}\int_0^\tau d\tau' e^{-\tau'}\bar{n}_1^0(\tau',\tilde{\xi}), \quad (40)$$

$$\begin{aligned}\bar{n}_1^1(\tau,\tilde{\xi})=P_0(0)\Bigg\{&\int_0^\tau d\tau' e^{\tau'}\int_0^{\tau'} d\tau'' e^{-\tau''}\\ &\times\int_0^{\tau''} d\tau''' e^{-\tau'''} n(\tau-\tau'+\tau''+\tau''')\\ &-\int_0^\tau d\tau'\int_0^{\tau'} d\tau'' e^{-\tau''} n(\tau-\tau'+\tau'')\Bigg\}\\ &+P_1(0)\Bigg\{\int_0^\tau d\tau'\int_0^{\tau'} d\tau'' e^{-\tau''} n(\tau-\tau'+\tau'')\\ &-\int_0^\tau d\tau' n(\tau-\tau')\Bigg\}. \end{aligned} \quad (41)$$

Formulas (39)–(41) use the same expression for $n(\tau)$ as given by Eq. (36), where one has to substitute c_0 for c_1.

Again, we have $\bar{n}(\xi,T)$ as a function of ξ^2 and thus an even function of ξ parabolic at the top. The corresponding plot for the average spectral density of linear radiation $n(\tilde{\xi})$ calculated up to the first order of ε is also given in Fig. 2 (dashed line).

VI. CREATION OF SOLITONS FOR THE REAL POSITIVE DICHOTOMOUS INPUT

As mentioned in Sec. III, the creation of a soliton from a real positive potential corresponds to the singularity achieved when a probability particle starting at the point $\varphi=\pi/2$, $\theta=\pi$ reaches the boundary $\theta=0$. It was also mentioned that mathematically this corresponds to a well-known first passage problem (or exit time problem); see, e.g., [24]. Here we will use this fact and derive analytically the PDF $P(T_0)$ of the exit times (i.e., PDF for the pulse width necessary for creating a soliton). We will assume that solitons are created with zero velocities, i.e., we assume that $\xi=0$. If one takes $\xi=0$, then system (15) simplifies considerably since the only solution for the phase $\varphi(t)$ satisfying the initial condition is $\varphi(t)=\text{const}=\pi/2$. We are then left with a single equation for the variable θ. The attached FPEs read

$$\frac{\partial P_0}{\partial t}=-c_0P_0+c_1P_1, \quad (42a)$$

$$\frac{\partial P_1}{\partial t}=2Q_0\frac{\partial P_1}{\partial \theta}-c_1P_1+c_0P_0, \quad (42b)$$

with the initial condition $P_s(\theta;0)=P_s(0)\delta(\theta-\pi)$. Introducing Laplace transforms for both components $\tilde{P}_s=\int_0^\infty P_s(t)\exp[-\lambda t]dt$, substituting $\tilde{P}_0$ from the first equation in Eqs. (42), and inserting it into the second equation of our FPE system, we arrive at the ODE,

$$\frac{\partial \tilde{P}_1}{\partial \theta}-\frac{\lambda}{2Q_0}\frac{\lambda+c_1+c_0}{\lambda+c_0}\tilde{P}_1=-\frac{1}{2Q_0}\left[\frac{c_0+\lambda P_1(0)}{c_0+\lambda}\right]\delta(\theta-\pi). \quad (43)$$

Its solution is

$$\tilde{P}_1(\theta;\lambda)=\frac{1}{2Q_0}\left[\frac{c_0+\lambda P_1(0)}{c_0+\lambda}\right]\times\exp\left[\frac{\lambda}{2Q_0}\frac{\lambda+c_0+c_1}{c_0+\lambda}(\theta-\pi)\right]. \quad (44)$$

According to [24], the PDF for the exit time (first soliton creation time) $P(T_0)$ is related to the probability flux, J, at the boundary $\theta=0$,

$$P(T_0)=\frac{J(T_0)}{\int_0^\infty J(t)dt}=\frac{P_1(T_0;0)}{\int_0^\infty P_1(0;t)dt}. \quad (45)$$

Therefore its Laplace transform, $\tilde{P}(\lambda)$, is given by

$$\tilde{P}(\lambda)=\left[\frac{c_0+\lambda P_1(0)}{c_0+\lambda}\right]\exp\left[-\frac{\pi\lambda}{2Q_0}\frac{\lambda+c_0+c_1}{c_0+\lambda}\right].$$

From the physical point of view it is clear that the PDF $P(T_0)$ is nonzero only if $T_0>T_*=\pi/(2Q_0)$, where T_* is the minimal possible exit time achieved for the realization where $s=1$ all the time. Introducing time shift $\Delta T=T_0-T_*$ and a function $g(\Delta T)=P(\Delta T)\exp[c_0(T_*+\Delta T)]$, we obtain the formula for the Laplace transform of function g,

$$\tilde{g}(\lambda)=\exp[(c_0-c_1)T_*]\left\{\frac{c_0P_0(0)}{\lambda}\exp\left[\frac{c_0c_1T_*}{\lambda}\right]+P_1(0)\times\left(\exp\left[\frac{c_0c_1T_*}{\lambda}\right]1\right)+P_1(0)\right\}.$$

The inverse transforms of all three terms in brackets are known [28] and we finally arrive at the following result for the desired PDF for the single soliton creation time, $P(\Delta T)$:

$$P(\Delta T) = e^{-c_1 T_*} e^{-c_0 \Delta T} \Bigg\{ P_1(0)\,\delta(\Delta T) + P_1(0)\sqrt{\frac{c_0 c_1 T_*}{\Delta T}}\mathrm{I}_1(2\sqrt{c_0 c_1 \Delta T}) + c_0 P_0(0)\mathrm{I}_0(2\sqrt{c_0 c_1 T_* \Delta T})\Bigg\}. \tag{46}$$

Here $\mathrm{I}_1(x)$ and $\mathrm{I}_0(x)$ are the modified Bessel functions of the first kind. The first (singular) term on the r.h.s. of Eq. (46) corresponds to the contribution of the deterministic trajectory where $s=1$ always, and the latter two describe the diffusive corrections. One can also find a mean first passage time (i.e., the mean pulse width sufficient for creating a soliton) as

$$\langle T_0 \rangle = -\left.\frac{\partial \tilde{P}}{\partial \lambda}\right|_{\lambda=0} = T_*\left(1 + \frac{c_1}{c_0}\right) + \frac{P_0(0)}{c_0}. \tag{47}$$

The mean first passage time diverges as c_0 goes to zero because the realizations where our potential hops to zero make an infinite contribution to the average, since once arriving at the zero state the system spends an infinite amount of time there.

VII. CONCLUSION

In this paper, we have studied the properties of quasilinear radiation and soliton states generated in the NLSE channel for two types of random input: the symmetric WGN and the real dichotomous "on-off" process. The integrability of NLSE allows one to reformulate the problem in terms of the linear non-self-adjoint direct scattering problem, where the random input profile serves as a potential.

For the first case of complex WGN input, we obtained the exact expression for the averaged spectral density of quasilinear radiation propagating in the channel in dependence on the extent of the input signal. This spectrum appeared to be flat, which is a consequence of the δ-correlated model for the random input.

For the second type of input—the real dichotomous process—we derived an exact PDF for the minimal signal width required for soliton creation. When considering the density of quasilinear radiation, we were able to study perturbatively the vicinities of two limiting regimes (always "on" and always "off"), which is perhaps the simplest model case of an intermittent random signal. The asymptotic expressions for the spectral density of quasilinear radiation were obtained in these limits. We showed that for such an input, the averaged spectrum is no longer flat due to finite correlation time of the potential. The spectrum exhibits decaying oscillation tails, with the shape depending on the parameters of initial dichotomous distribution.

ACKNOWLEDGMENTS

We would like to thank Sergei Turitsyn and Leonid Pastur for valuable comments and interest in our work. This study was supported by the EPSRC and the Royal Society.

APPENDIX A: THE SOLUTION OF FPE [Eq. (21)]

We need to solve boundary problem (21) with the initial condition $Y(\theta;0)=\delta(\theta-\theta_0)/\sin\theta_0$. Here we do it by means of Laplace transform.

Introducing the Laplace transformed solution $\tilde{Y}(\theta,\theta_0;s)$, one arrives at the following equation:

$$-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\tilde{Y}}{\partial\theta}\right) + \frac{s}{2D}\tilde{Y} = \frac{1}{2D\sin\theta}\delta(\theta-\theta_0), \tag{A1}$$

with the same boundary conditions Eq. (22). The solution is merely a time Laplace transform of the retarded Green function for the Fokker-Planck operator, which can be presented as

$$\tilde{Y}(\theta,\theta_0;s) = -\frac{1}{2D\sin\theta_0}\frac{1}{W(\theta_0)} \times \begin{cases} f_1(\theta,s)f_2(\theta_0,s), & \theta<\theta_0, \\ f_2(\theta,s)f_1(\theta_0,s), & \theta>\theta_0. \end{cases} \tag{A2}$$

Here $f_{1,2}(\theta,s)$ are the two linearly independent solutions of the Legendre equation,

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + l(l+1)f = 0,$$

$$l(l+1) = -\frac{s}{2D}, \tag{A3}$$

with $f_1(\theta,s)$ satisfying the boundary condition at $\theta=0$ and $f_2(\theta,s)$ satisfying the boundary condition at the opposite end $\theta=\pi$. By $W(\theta)$ we denote the Wronskian of these two solutions. Introduce the two functions $l_\pm(s)$ given by

$$l_+(s) = -\frac{1}{2} + \frac{i}{2\sqrt{s_*}}\sqrt{s-s_*},$$

$$l_-(s) = -\frac{1}{2} - \frac{i}{2\sqrt{s_*}}\sqrt{s-s_*},$$

where $s_*=D/2$, and the branch of the square root is fixed by assuming that it is positive for the real positive values of the argument.

When choosing the corresponding pair of solutions, we must make sure that they take real and positive values at $s>0$. Therefore, we will seek both solutions $f_{1,2}$ as linear combinations of the functions $\mathcal{P}(\theta)$ and $\mathcal{Q}(\theta)$, which are closely related to the conical functions,

$$\mathcal{P}(\theta) = \frac{1}{2}[\mathrm{P}_{l_+}(\cos\theta) + \mathrm{P}_{l_-}(\cos\theta)] = \mathrm{P}_{l_\pm}(\cos\theta),$$

$$\mathcal{Q}(\theta) = \frac{1}{2}[\mathrm{Q}_{l_+}(\cos\theta) + \mathrm{Q}_{l_-}(\cos\theta)],$$

where $\mathrm{P}_l(x)$ and $\mathrm{Q}_l(x)$ are the Legendre functions of the first and second kind. These functions are real and positive provided that $s>0$. Next one can make use of the asymptotic expansions of both Legendre functions at $x\to\pm 1$ [29] to show that the solutions satisfying the boundary conditions Eq. (22) can be chosen in the following form:

$$f_1(\theta,s)=\mathcal{P}(\theta)+C_1\mathcal{Q}(\theta),$$

$$f_2(\theta,s)=C_2\mathcal{P}(\theta)+\mathcal{Q}(\theta),$$

with the arbitrary constants $C_{1,2}$. Inserting these into Eq. (A2), we arrive at

$$\tilde{Y}(\theta,\theta_0;s) = \frac{1}{2D(1-C_1C_2)} \times\begin{cases}[C_1\mathcal{Q}(\theta)+\mathcal{P}(\theta)][\mathcal{Q}(\theta_0)+C_2\mathcal{P}(\theta_0)], & \theta<\theta_0,\\ [\mathcal{Q}(\theta)+C_2\mathcal{P}(\theta)][C_1\mathcal{Q}(\theta_0)+\mathcal{P}(\theta_0)], & \theta>\theta_0,\end{cases}$$

where the constants $C_{1,2}$ have to be chosen in such a way as to provide the positiveness of the solution $\tilde{Y}$ while keeping the negative value of flux J_0. In our problem, we always start at the point $\theta_0=\pi$, where $\mathcal{P}(\theta_0)$ turns into infinity. Therefore, to have a finite solution, we must assume that $C_2=0$, which immediately yields $J_\pi=0$. On the other hand, the flux of this solution at $\theta=0$ is equal to $J_0=C_1\mathcal{Q}(0)$. This flux must be negative to allow soliton creation. Since $\mathcal{Q}(\theta)$ itself is always positive, we must have $C_1<0$. On the other hand, the solution itself must be positive for all values of θ. Therefore, $C_1\mathcal{Q}(\theta)+\mathcal{P}(\theta)$ must be positive for $\theta<\theta_0$. In the vicinity of $\theta=0$, function $\mathcal{P}(\theta)$ is close to 1 but $\mathcal{Q}(\theta)$ goes to plus infinity. Therefore, for a negative value of C_1 we cannot have a positive solution in the vicinity of $\theta=0$. This proves the *nonexistence* of the solution with the flux conditions corresponding to the soliton creation with a given spectral value ξ.

Thus it turns out that the only positive, finite solution $\tilde{Y}(\theta,\theta_0;s)$ is the one corresponding to the natural (zero-flux) boundary conditions $J_0=J_\pi=0$ studied in [19]. It has the form

$$\tilde{Y}(\theta;s)=\lim_{\theta_0\to\pi}\tilde{Y}(\theta,\theta_0;s)=\frac{1}{2D}\mathcal{P}(\theta)\lim_{\theta_0\to\pi}\mathcal{Q}(\theta_0) = \frac{1}{2D}\frac{\pi}{2}\mathrm{P}_{l_\pm}(\cos\theta)\mathrm{sech}\left[\frac{\pi}{2\sqrt{s_*}}\sqrt{s-s_*}\right]. \quad \text{(A4)}$$

So we can now restore the original solution in time domain using Mellin's inverse formula and eventually arrive at Eq. (23), a result that complies with that given in Ref. [19].

APPENDIX B: PERTURBATIVE SOLUTION OF Eq. (30)

Assuming the regular expansions for PDFs, $P_s=\sum_k\epsilon^kP_s^k$, rescaling time, and ξ as described in Sec. V A in the leading order in ϵ from Eq. (30), one gains the following system:

$$\frac{\partial P_0^0}{\partial\tau}=P_1^0, \quad \text{(B1a)}$$

$$\frac{\partial P_1^0}{\partial\tau}=c_1^{-1}\sin(\varphi+\tilde{\xi}\tau)\frac{\partial P_1^0}{\partial\theta}+c_1^{-1}\cot\theta\frac{\partial}{\partial\varphi}\cos(\varphi+\tilde{\xi}\tau)P_1^0-P_1^0, \quad \text{(B1b)}$$

where we proceeded to the rotating frame, i.e., we made a substitution $\varphi\to\varphi-\tilde{\xi}\tau$. Now Eq. (B1b) is autonomous and can be solved exactly given the initial condition $P_1^0(0;\varphi,\theta)=P_1(0)\delta(\varphi-\frac{\pi}{2})\delta(\theta-\pi)$. Let us rewrite Eq. (B1b) in a canonical form of a quasilinear first-order PDE,

$$a_\theta(\tau,\theta,\varphi)\frac{\partial P_1^0}{\partial\theta}+a_\varphi(\tau,\theta,\varphi)\frac{\partial P_1^0}{\partial\varphi}+\frac{\partial P_1^0}{\partial\tau}=K(\tau,\theta,\varphi)P_1^0, \quad \text{(B2)}$$

where the explicit form of the coefficients is

$$a_\theta(\tau,\theta,\varphi)=-c_1^{-1}\sin(\varphi+\tilde{\xi}\tau),$$

$$a_\varphi(\tau,\theta,\varphi)=-c_1^{-1}\cos(\varphi+\tilde{\xi}\tau)\cot\theta,$$

$$K(\tau,\theta,\varphi)=-c_1^{-1}\sin(\varphi+\tilde{\xi}\tau)\cot\theta-1.$$

Then applying the method of characteristics, we have the set of equations defining the dynamics of $\varphi(\tau)$ and $\theta(\tau)$,

$$\frac{d\theta}{d\tau}=a_\theta(\tau,\varphi,\theta)=-c_1^{-1}\sin(\varphi+\tilde{\xi}\tau),$$

$$\frac{d\varphi}{d\tau}=a_\varphi(\tau,\varphi,\theta)=-c_1^{-1}\cot\theta\cos(\varphi+\tilde{\xi}\tau), \quad \text{(B3)}$$

with general initial conditions for the dependent variables $\varphi(0)=\varphi_0$, $\theta(0)=\theta_0$. The solution of Eqs. (B3) can be most readily obtained if we restore the original complex parametrization of the reflection coefficient, see Eq. (14), i.e., we return back to a complex function,

$$\tilde{r}(\tau)=\cot\frac{\theta(\tau)}{2}\exp\{i[\varphi(\tau)+\tilde{\xi}\tau]\}. \quad \text{(B4)}$$

Then one can see that a system of characteristic equations (B3) is equivalent to a single Ricatti equation for the complex function $\tilde{r}$ [cf. Eq. (13)],

$$\frac{d\tilde{r}}{d\tau}=i\tilde{\xi}\tilde{r}+\frac{i}{2c_1}(1-\tilde{r}^2), \quad \tilde{r}(0)=r_0. \quad \text{(B5)}$$

Its solution is given by

$$\tilde{r}(\tau)=\frac{r_0\kappa\cos[\kappa\tau/2]+i(c_1^{-1}+r_0\tilde{\xi})\sin[\kappa\tau/2]}{\kappa\cos[\kappa\tau/2]+i(c_1^{-1}r_0-\tilde{\xi})\sin[\kappa\tau/2]}, \quad \text{(B6)}$$

where $\kappa=\sqrt{c_1^{-2}+\tilde{\xi}^2}$. Note that the initial values φ_0,θ_0 play the role of hydrodynamic Lagrangian coordinates of probability particles following the probability flow in the phase space (φ,θ). We shall make extensive use of this hydrodynamic analogy in the following derivations. To simplify the notations somewhat, we use the shorthand designation for the vectors in the probability space $q_0=\{\varphi_0,\theta_0\}$, $q=\{\varphi,\theta\}$. Exploiting this hydrodynamic analogy, one can introduce a

Lagrangian PDF $\mathbf{P}_1^0(\tau|q_0)=P_1^0(\tau;q(\tau,q_0))$, which is just a PDF as seen by a Lagrangian probability particle moving with the probability flow. Further on, we will adopt bold roman letters to mark the Lagrangian quantities. The Lagrangian PDF obeys an ODE,

$$\frac{d\mathbf{P}_1^0}{d\tau}=K(\tau,q(\tau|q_0))\mathbf{P}_1^0. \tag{B7}$$

The solution of Eq. (B7) satisfying the initial conditions is

$$\mathbf{P}_1^0(\tau|q_0)=P_1(0)\,\delta(\varphi_0-\pi/2)\,\delta(\theta_0-\pi)\frac{\sin\,\theta(\tau|q_0)}{\sin\,\theta_0}e^{-\tau}. \tag{B8}$$

The original (i.e., Eulerian) PDF $P_1^0(\tau;q)$ can then be restored from its Lagrangian counterpart by inverting the vector dependency $q=q(\tau;q_0)$ [or alternatively $\tilde{r}=\tilde{r}(\tau;r_0)$] and substituting the resulting functions $q_0(\tau;q)$ into the Lagrangian PDF $\mathbf{P}_1^0(\tau|q_0)$ (see, e.g., [22]).

As for Eq. (B1a), it must be considered in the Eulerian frame and its solution is

$$P_0^0(\tau;q)=P_0(0)\,\delta(\varphi-\pi/2)\,\delta(\theta-\pi)+\int_0^\tau P_1^0(\tau';q)d\tau'. \tag{B9}$$

The first order in ϵ yields the inhomogeneous set of equations,

$$\frac{\partial P_0^1}{\partial\tau}=-P_0^0+P_1^1, \tag{B10a}$$

$$\frac{\partial P_1^1}{\partial\tau}=c_1^{-1}\sin(\varphi+\tilde{\xi}\tau)\frac{\partial P_1^1}{\partial\theta}+c_1^{-1}\cot\,\theta\frac{\partial}{\partial\varphi}\cos(\varphi+\tilde{\xi}\tau)P_1^1-P_1^1+P_0^0. \tag{B10b}$$

Proceeding in the same way as described above and taking into account the initial conditions for the first-order PDF corrections, $P_s^1(0;\varphi,\theta)=0$, we arrive at

$$\mathbf{P}_1^1(\tau|q_0)=e^{-\tau}\sin\,\theta(\tau|q_0)\times\int_0^\tau\frac{e^{\tau'}}{\sin\,\theta(\tau'|q_0)}P_0^0(\tau';q(\tau'|q_0))d\tau', \tag{B11}$$

$$P_0^1(\tau;q)=\int_0^\tau[P_1^1(\tau';q)-P_0^0(\tau';q)]d\tau'. \tag{B12}$$

Since the expressions for P_1^k are written in Lagrangian frame, it is often convenient to perform averaging in the Lagrangian frame,

$$\begin{aligned}\bar{n}_1^k&=\int_0^{2\pi}d\varphi\int_0^\pi d\theta n_1^k(q)P_1^k(\tau;q)\\&=\int_0^{2\pi}d\varphi_0\int_0^\pi d\theta_0\frac{\partial q}{\partial q_0}\mathbf{n}_1^k(\tau|q_0)\mathbf{P}_1^k(\tau|q_0)\\&=\int_0^{2\pi}d\varphi_0\int_0^\pi d\theta_0\frac{\sin\,\theta_0}{\sin\,\theta(\tau|q_0)}\mathbf{n}_1^k(\tau|q_0)\mathbf{P}_1^k(\tau|q_0),\end{aligned} \tag{B13}$$

where $\mathbf{n}_1^k(\tau|q_0)\equiv n_1^k[q(\tau|q_0)]$ is the radiation density in the Lagrangian frame.

Now let us turn to Eq. (30) in the case $c_0\gg c_1$. The considerations carried out below do not differ essentially from the previous case, so we omit most of the details. Assuming the regular expansions for PDFs, $P_s=\Sigma_k\varepsilon^k P_s^k$, from Eq. (30) in the zeroth order in ε one arrives at the system

$$\frac{\partial P_0^0}{\partial\tau}=-P_0^0, \tag{B14a}$$

$$\frac{\partial P_1}{\partial\tau}=c_0^{-1}\sin(\varphi+\tilde{\xi}\tau)\frac{\partial P_1^0}{\partial\theta}+c_0^{-1}\cot\,\theta\frac{\partial}{\partial\varphi}\cos(\varphi+\tilde{\xi}\tau)P_1^0+P_0^0, \tag{B14b}$$

where we proceeded to the rotating frame. Taking into account the initial conditions, the solutions are found to be

$$P_0^0(\tau;q)=P_0(0)\,\delta(\varphi-\pi/2)\,\delta(\theta-\pi)e^{-\tau}, \tag{B15}$$

$$\mathbf{P}_1^0(\tau|q_0)=P_1(0)\,\delta(\varphi_0-\pi/2)\,\delta(\theta_0-\pi)\frac{\sin\,\theta(\tau|q_0)}{\sin\,\theta_0}+\sin\,\theta(\tau|q_0)\int_0^\tau\frac{P_0^0(\tau';q(\tau'|q_0))}{\sin\,\theta(\tau'|q_0)}d\tau'. \tag{B16}$$

The dependencies $q=q(\tau|q_0)$ are determined by expression Eq. (B6), where one must substitute c_0 for c_1. In the next order in ε, we have

$$\frac{\partial P_0^1}{\partial \tau} = -P_0^1 + P_1^0 \tag{B17a}$$

$$\frac{\partial P_1^1}{\partial \tau} = c_0^{-1}\sin(\varphi + \tilde{\xi}\tau)\frac{\partial P_1^1}{\partial \theta} + c_0^{-1}\cot\theta \frac{\partial}{\partial \varphi}\cos(\varphi + \tilde{\xi}\tau)P_1^1 - P_1^0 + P_0^1, \tag{B17b}$$

with the solutions

$$P_0^1(\tau;q) = e^{\tau}\int_0^{\tau} e^{-\tau'}P_1^0(\tau';q)d\tau', \tag{B18}$$

$$\mathbf{P}_1^1(\tau|q_0) = \sin\theta(\tau|q_0) \times \int_0^{\tau} \frac{P_0^1(\tau';q(\tau'|q_0)) - P_1^0(\tau';q(\tau'|q_0))}{\sin\theta(\tau'|q_0)}d\tau'. \tag{B19}$$

Again when switching from Eulerian to Lagrangian averaging, Eq. (B13) is pertinent.

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