Reduction of superintegrable systems: The anisotropic harmonic oscillator

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We introduce a $2N$-parametric family of maximally superintegrable systems in N dimensions, obtained as a reduction of an anisotropic harmonic oscillator in a $2N$-dimensional configuration space. These systems possess closed bounded orbits and integrals of motion which are polynomial in the momenta. They generalize known examples of superintegrable models in the Euclidean plane.

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I. INTRODUCTION

The aim of this paper is to introduce a class of maximally superintegrable systems that are obtained as a symplectic reduction of the anisotropic harmonic oscillator. These systems depend on a set of N real and N integer parameters and possess integrals of motion polynomial in the momenta. The Hamiltonian defining this family is

$$H_N = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \frac{1}{2}\sum_{i=1}^{N} \frac{k_i}{x_i^2} + \frac{\omega^2}{2}\sum_{i=1}^{N} n_i^2 x_i^2. \tag{1}$$

We recall that in classical mechanics superintegrable (also known as noncommutatively integrable [1]) systems are characterized by the fact that they possess more than N functionally independent integrals of motion, globally defined in a $2N$-dimensional phase space. In particular, when the number of integrals is $2N-1$, the systems are said to be maximally superintegrable. The dynamics of these systems is particularly interesting: all bounded orbits are closed and periodic. This issue, for the spherically symmetric potentials, was first noticed by Bertrand [2]. The phase space topology is also very rich: it has the structure of a symplectic bifoliation, consisting of the usual Liouville-Arnold invariant fibration by Lagrangian tori and of a (coisotropic) polar foliation [3,4]. Apart from the harmonic oscillator and the Kepler potential, many other potentials turn out to be superintegrable, like the Calogero-Moser potential, the Smorodinsky-Winternitz system, the Euler top, etc.

A considerable effort has recently been devoted to the search for superintegrable systems as well as to the study of the algebraic and analytic properties of these models. For a recent review of the topic, see [5].

The notion of superintegrability possesses an interesting analog in quantum mechanics. Sommerfeld and Bohr were the first to notice that systems allowing separation of variables in more than one coordinate system may admit additional integrals of motion. Superintegrable systems show accidental degeneracy of the energy levels, which can be removed by taking into account the quantum numbers associated to the additional integrals of motion. One of the best examples of this phenomenon is provided by the Coulomb atom [6–8], which is superintegrable in N dimensions [9,10]. A systematic search for quantum mechanical potentials exhibiting the property of superintegrability was started in [11–13]. These models in many cases are also exactly solvable, i.e., they possess a spectrum generating algebra, which allows to compute the whole energy spectrum essentially by algebraic manipulations [14]. In classical mechanics, the multiseparability of the Hamilton-Jacobi equation implies that there should exist at least two different sets of N quadratic integrals of motion in involution. Reduction techniques in both classical and quantum mechanics are well known (see, for instance, [15]). Essentially, the common idea of several of the existing approaches is to start from a free motion Hamiltonian defined in a suitable higher-dimensional space and to project it down into an appropriate subspace. In this way, one gets a reduced Hamiltonian that is no longer free: an integrable potential appears in the lower-dimensional space [16]. A different point of view, that we adopt here, is to start instead directly from a nontrivial (i.e., not free) dynamical system in a given phase space and to reduce it to a proper subspace, in such a way that the superintegrability of the considered system is inherited by the reduced one.

In this work, we study the reduction of an anisotropic harmonic oscillator, defined in a $2N$-dimensional classical configuration space. This system is maximally superintegrable. It is described by the Hamiltonian

$$H_{2N} = \frac{1}{2}\sum_{i=1}^{2N} \hat{p}_i^2 + \frac{\omega^2}{2}\sum_{i=1}^{2N} n_i^2 y_i^2. \tag{2}$$

We will prove that it can be suitably reduced to the system (1), and that this system is still maximally superintegrable, with integrals of motion inherited from the system (2). This goal is achieved under the assumption $n_1=n_2,\ldots,n_{2N-1}=n_{2N}$. From a geometrical point of view, the approach we adopt reposes on the Marsden-Weinstein symplectic reduction scheme [17–19]. Given a symplectic mani-

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fold (M,Ω), let $K_1,\ldots,K_k$ be k functions in involution:

$$\{K_i,K_j\}=0, \quad i,j=1,\ldots,k. \tag{3}$$

Assume also that dK_i are independent at each point. Since the flows of the associated Hamiltonian vector fields $X_{K_1},\ldots,X_{K_k}$ commute, they can be used to define a symplectic action of $G=\mathbb{R}^k$ on the manifold. Let J be the momentum map of this action, and μ be a regular value for J. Then we can conclude that $P_\mu := J^{-1}(\mu)/G$ is still a symplectic manifold, of dimension $\dim M-2k$, called the reduced phase space. In our case, K_i, $i=1,\ldots,N$, are components of the angular momentum, $J=K_1\times\cdots\times K_N$ is the momentum map, $G=SO(2)\times SO(2)\times\cdots\times SO(2)$ (N times), and the reduced space is $P_\mu=J^{-1}(\mu)/T^N$, where T^N is the N-dimensional torus, and $\dim P_\mu=2N$. This procedure is a generalization of what in celestial mechanics, since the work of Jacobi, is called "elimination of the nodes" (see [17], Chap. IX for details). The reduced Hamiltonian is reminiscent of the structure of the original Hamiltonian, defined in the $4N$-dimensional phase space, but also possesses a Rosochatius-type term [20,21], involving parameters k_i corresponding to the variables that become ignorable, in addition to the harmonic part. Therefore, using the reduction procedure, we obtain the parametric family of Hamiltonian systems (1), defined on a reduced phase P_μ.

The transformations we consider, although very simple, are nontrivial, since the reduced Hamiltonian is not shape invariant. Nevertheless, since the reduced system turns out to be maximally superintegrable, bounded orbits still remain closed in the reduced space.

For $N=2$ maximal superintegrability [11,13] and exact solvability [14] of the system (1) was already established for $n_1=n_2=1$ and for $n_1=1$, $n_2=2$, $k_2=0$. The integrals of motion in these cases are second order in the momenta.

Here we will show that in the general case (n_i and N arbitrary positive integers and k_i arbitrary real numbers) N integrals can be chosen to be of order two, the other $N-1$ functionally independent ones of order n_i+n_N or n_i+n_N-1. Other systems possessing third- and higher-order integrals have been studied in the literature [22–27].

This paper is directly related to the recent interesting work by Verrier and Evans [28], who performed a similar reducing transformation for the Kepler potential. They found a superintegrable system in three dimensions possessing a quartic integral. They also conjectured that the system (1) in three space dimensions should be maximally superintegrable, although the explicit expression of the integrals remained to be determined. In the following, we will prove this conjecture, and also we will establish that the system (1) is maximally superintegrable in full generality, i.e., for N arbitrary, providing explicitly the corresponding set of integrals of the motion.

We learned recently of an article by Evans and Verrier [29] in which the authors also establish the superintegrability of the system (1) for $N=3$. Their results are compatible with ours, though they express the integrals in terms of Chebyshev polynomials. Moreover, they also treat the quantum analog of system (1) and establish the degeneracy of the energy levels related to the representation theory of the group SU(3).

The paper is organized as follows. In Sec. II, the main properties of the anisotropic oscillator are briefly reviewed. Then its reduction to the planar case is studied in detail. We will show how superintegrability is preserved under a multipolar change of variables and subsequent reduction. In Sec. III, the same problem is treated and solved in full generality. Some open problems are discussed in the final section.

II. REDUCTION OF THE ANISOTROPIC OSCILLATOR

The anisotropic oscillator in the two-dimensional case in both classical and quantum mechanics was discussed by Jauch and Hill [8,30,31]. The system (2) is also known to be superintegrable in any dimension, if the ratios of the frequencies are rational. Let us consider a $2N$-dimensional space and assume

$$\frac{\omega_1}{n_1}=\frac{\omega_2}{n_2}=\cdots=\frac{\omega_{2N}}{n_{2N}}=\omega, \quad n_i\in\mathbb{N}. \tag{4}$$

Following [8], we define the set of invariants in an auxiliary complex phase space, with coordinates $z_i,\bar{z}_i$, $i=1,\ldots,2N$. Precisely,

$$z_j=\hat{p}_j-in_j\omega y_j, \quad \bar{z}_j=\hat{p}_j+in_j\omega y_j. \tag{5}$$

It is easily checked that the expressions

$$c_{jk}=z_j^{n_k}\bar{z}_k^{n_j} \tag{6}$$

provide integrals of motion. They can be also arranged in a real-valued form, as the combinations $(1/2)(c_{ij}+\bar{c}_{ij})$ and $(1/2i)(c_{ij}-\bar{c}_{ij})$. In particular, among these integrals we have the angular momenta

$$L_{ik}=y_i\hat{p}_k-y_k\hat{p}_i \tag{7}$$

(when $n_i=n_k$) and the tensor

$$T_{ik}=\hat{p}_i\hat{p}_k+n_in_k\omega^2y_iy_k. \tag{8}$$

We will now study reductions of the anisotropic oscillator (2) and establish the superintegrability of the corresponding dynamical systems.

A. Hamiltonian and first integrals: the planar case

We recall the definition of a momentum map. For further details, see, for instance, [17]. Let (M,Ω) be a $2n$-dimensional symplectic manifold. Suppose that a Lie group G acts on M and leaves Ω invariant. Let $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{g}^*$ its dual space, and $\langle,\rangle$ the natural pairing between the two spaces.

A momentum map for the G action on (M,Ω) is a map $J:M\to\mathfrak{g}^*$ such that, for all $X\in\mathfrak{g}$,

$$d(\langle J,X\rangle)=i_X\Omega.$$

In particular, if the manifold is exact, i.e., $\Omega=d\theta$, and the G action leaves θ invariant as well, we have

$$J_X=i_X\theta.$$

We will also assume that the map is equivariant with respect to the coadjoint action Ad^* of G on $\mathfrak{g}^*$, i.e.,

$$\langle \mathrm{Ad}^*_g \xi, X\rangle = \langle \xi, \mathrm{Ad}_{g^{-1}} X\rangle,$$

for all $g \in G$, $\xi \in \mathfrak{g}^*$, and $X \in \mathfrak{g}$.

Let us first consider a simple case, when the anisotropic oscillator is defined in a symplectic manifold M with $\dim M = 4$. So, $\Omega = \Sigma_{i=1}^4 dy^i \wedge d\hat{p}_i$. In order to make the reduction possible, we select frequencies to be equal in pairs, so that we have only two independent frequencies. Hence the system (2) takes the special form

$$H_4 = \frac{1}{2}(\hat{p}_1^2 + \hat{p}_2^2 + \hat{p}_3^2 + \hat{p}_4^2) + \frac{n_1^2\omega^2}{2}(y_1^2 + y_2^2) + \frac{n_2^2\omega^2}{2}(y_3^2 + y_4^2). \tag{9}$$

In the auxiliary coordinates $z_1, \bar{z}_1, \ldots, z_4, \bar{z}_4$, we have explicitly

$$z_1 = \hat{p}_1 - i n_1 \omega y_1, \quad z_2 = \hat{p}_2 - i n_1 \omega y_2,$$

$$z_3 = \hat{p}_3 - i n_2 \omega y_3, \quad z_4 = \hat{p}_4 - i n_2 \omega y_4. \tag{10}$$

Consequently, the Hamiltonian reads

$$H_4 = \frac{1}{2}\sum_{i=1}^4 |z_i|^2. \tag{11}$$

Put in a matrix form, the set of invariants (6) can be represented by the matrix

$$Z = \begin{pmatrix} z_1\bar{z}_1 & z_1\bar{z}_2 & z_1^{n_2}\bar{z}_3^{n_1} & z_1^{n_2}\bar{z}_4^{n_1} \\ z_2\bar{z}_1 & z_2\bar{z}_2 & z_2^{n_2}\bar{z}_3^{n_1} & z_2^{n_2}\bar{z}_4^{n_1} \\ z_3^{n_1}\bar{z}_1^{n_2} & z_3^{n_1}\bar{z}_2^{n_2} & z_3\bar{z}_3 & z_3\bar{z}_4 \\ z_4^{n_1}\bar{z}_1^{n_2} & z_4^{n_1}\bar{z}_2^{n_2} & z_4\bar{z}_3 & z_4\bar{z}_4 \end{pmatrix}. \tag{12}$$

Let us consider now the following change of coordinates:

$$y_1 = x_1 \cos x_3, \quad y_2 = x_1 \sin x_3,$$

$$y_3 = x_2 \cos x_4, \quad y_4 = x_2 \sin x_4. \tag{13}$$

The corresponding momenta read

$$\hat{p}_1 = -p_3\frac{\sin x_3}{x_1} + p_1 \cos x_3, \quad \hat{p}_2 = p_3 \frac{\cos x_3}{x_1} + p_1 \sin x_3,$$

$$\hat{p}_3 = -p_4\frac{\sin x_4}{x_2} + p_2 \cos x_4, \quad \hat{p}_4 = p_4 \frac{\cos x_4}{x_2} + p_2 \sin x_4. \tag{14}$$

The group T_2, which is the group $\mathrm{SO}(2)\times\mathrm{SO}(2)$ in the old coordinates, acts on $\mathbb{R}^4$ as follows:

$$x_1' = x_1,$$

$$x_2' = x_2,$$

$$x_3' = x_3 + a_1,$$

$$x_4' = x_4 + a_2. \tag{15}$$

This group leaves Ω invariant. The fundamental vector fields on $T^*\mathbb{R}^4$ corresponding to this action are

$$X_1 = \partial_{x_3}, \quad X_2 = \partial_{x_4}, \tag{16}$$

and, if $X = \lambda_1 X_1 + \lambda_2 X_2$, the momentum map J satisfies

$$J_{(a_1,a_2)} = \theta(\lambda_1 \partial_{x_3} + \lambda_2 \partial_{x_4}) = \lambda_1 p_3 + \lambda_2 p_4. \tag{17}$$

Let us choose a regular point in $\mathfrak{t}_2^*$ (the dual of the Lie algebra of T_2), for instance

$$p_3 = \sqrt{k_1}, \quad p_4 = \sqrt{k_2}. \tag{18}$$

The inverse image under J is

$$J^{-1}(\sqrt{k_1}, \sqrt{k_2}) = (p_1, p_2, \sqrt{k_1}, \sqrt{k_2}, x_1, x_2, x_3, x_4). \tag{19}$$

The stabilizer of this point in $\mathfrak{t}_2^*$ under the coadjoint action of T_2 is the whole group, because its action is trivial on the p coordinates.

The reduced phase space is therefore

$$J^{-1}(\sqrt{k_1}, \sqrt{k_2})/T_2 \approx \{(p_1, p_2, x_1, x_2) \in \mathbb{R}^4\} \tag{20}$$

and the reduced Hamiltonian is

$$H_2 = \frac{p_1^2}{2} + \frac{p_2^2}{2} + \frac{k_1}{2x_1^2} + \frac{k_2}{2x_2^2} + \frac{n_1^2}{2}\omega^2 x_1^2 + \frac{n_2^2}{2}\omega^2 x_2^2. \tag{21}$$

Let F be a first integral of the Hamiltonian $H_4(\hat{p}, y)$, i.e., $\{H_4, F\} = 0$. We show now how the original ring of integrals can be reduced in the low-dimensional phase space. First, we consider the restriction $\hat{F}$ of the function F to the manifold $J^{-1}(\sqrt{k_1}, \sqrt{k_2})$.

Observe that $\hat{F}$ can be defined on the quotient manifold $J^{-1}(\sqrt{k_1}, \sqrt{k_2})/T_2$, when it is constant on the equivalence classes, that is, $\hat{F}$ is independent of x_3, x_4. In this case $\hat{F}$ can be factored out in the following way:

$$\begin{tikzcd} J^{-1}(\sqrt{k_1}, \sqrt{k_2}) \arrow[r, "\hat{F}"] \arrow[d, "\pi"'] & \mathbb{R} \\ J^{-1}(\sqrt{k_1}, \sqrt{k_2})/T_2 \arrow[ur, "F_r"'] & \end{tikzcd}$$

where π is the canonical projection and

$$F_r \circ \pi = \hat{F}. \tag{22}$$

Then,

$$\{H_2, F_r\} = 0. \tag{23}$$

The integrals of the system (11) are given in the matrix (12) (although only seven of them can be functionally independent). Those that will survive the reduction (20) are the ones that are left invariant by the $\mathrm{SO}(2)\times\mathrm{SO}(2)$ rotations (15). They must Poisson commute with

$$L_{12} = \frac{i}{2n_1\omega}(z_1\bar{z}_2 - z_2\bar{z}_1) = y_1\hat{p}_2 - y_2\hat{p}_1,$$

$$L_{34} = \frac{i}{2n_2\omega}(z_3\bar{z}_4 - z_4\bar{z}_3) = y_3\hat{p}_4 - y_4\hat{p}_3. \tag{24}$$

The Poisson bracket can be written in terms of the z_i variables as

$$\{f(z_i,\bar{z}_i),g(z_i,\bar{z}_i)\} = -2i\omega\sum_{k=1}^{N}\sum_{j=2k-1}^{2k} n_k\left(\frac{\partial f}{\partial z_j}\frac{\partial g}{\partial \bar{z}_j} - \frac{\partial f}{\partial \bar{z}_j}\frac{\partial g}{\partial z_j}\right) \tag{25}$$

(in this section we have $N=2$).

Functions of $z_k,\bar{z}_k$ Poisson commuting with L_{12} and L_{34} must satisfy

$$z_2\partial_{z_1}f - z_1\partial_{z_2}f + \bar{z}_2\partial_{\bar{z}_1}f - \bar{z}_1\partial_{\bar{z}_2}f = 0,$$

$$z_4\partial_{z_3}f - z_3\partial_{z_4}f + \bar{z}_4\partial_{\bar{z}_3}f - \bar{z}_3\partial_{\bar{z}_4}f = 0. \tag{26}$$

A basis for the corresponding $SO(2)\times SO(2)$ invariants is given by

$$\xi_1 = z_1^2 + z_2^2, \quad \bar{\xi}_1 = \bar{z}_1^2 + \bar{z}_2^2, \quad \eta_1 = z_1\bar{z}_1 + z_2\bar{z}_2,$$

$$\xi_3 = z_3^2 + z_4^2, \quad \bar{\xi}_3 = \bar{z}_3^2 + \bar{z}_4^2, \quad \eta_2 = z_3\bar{z}_3 + z_4\bar{z}_4. \tag{27}$$

Finally the integrals of motion must satisfy

$$\{H_4, f(\xi_1,\bar{\xi}_1,\eta_1,\xi_3,\bar{\xi}_3,\eta_2)\} = 0. \tag{28}$$

Solutions of Eq. (28) are, for instance,

$$E_1 = \frac{1}{2}(|z_1|^2 + |z_2|^2), \qquad E_2 = \frac{1}{2}(|z_3|^2 + |z_4|^2),$$

$$Q_1 = (z_1^2 + z_2^2)^{n_2}(\bar{z}_3^2 + \bar{z}_4^2)^{n_1},$$

$$\bar{Q}_1 = (\bar{z}_1^2 + \bar{z}_2^2)^{n_2}(z_3^2 + z_4^2)^{n_1},$$

$$I_1 = (z_1^2 + z_2^2)(\bar{z}_1^2 + \bar{z}_2^2), \quad I_2 = (z_3^2 + z_4^2)(\bar{z}_3^2 + \bar{z}_4^2). \tag{29}$$

Only five of these integrals are functionally independent.

B. Reduction of the first integrals

The reduction is performed using the change of variables (13) and (14) and the convention (18). The integrals (24) reduce to constants $L_{12}=\sqrt{k_1}$, $L_{34}=\sqrt{k_2}$. The integrals (29) reduce to nontrivial integrals for the Hamiltonian in Eq. (21), namely,

$$E_1 = \frac{1}{2}p_1^2 + \frac{k_1}{2x_1^2} + \frac{1}{2}n_1^2\omega^2 x_1^2,$$

$$E_2 = \frac{1}{2}p_2^2 + \frac{k_2}{2x_2^2} + \frac{1}{2}n_2^2\omega^2 x_2^2,$$

$$Q_1 = \left(p_1^2 + \frac{k_1}{x_1^2} - n_1^2\omega^2x_1^2 - 2in_1\omega p_1 x_1\right)^{n_2} \times \left(p_2^2 + \frac{k_2}{x_2^2} - n_2^2\omega^2x_2^2 + 2in_2\omega p_2 x_2\right)^{n_1},$$

$$\bar{Q}_1 = \left(p_1^2 + \frac{k_1}{x_1^2} - n_1^2\omega^2x_1^2 + 2in_1\omega p_1 x_1\right)^{n_2} \times \left(p_2^2 + \frac{k_2}{x_2^2} - n_2^2\omega^2x_2^2 - 2in_2\omega p_2 x_2\right)^{n_1}. \tag{30}$$

The remaining two integrals in (29) give nothing new and we have

$$I_1 = 4(E_1^2 - k_1n_1^2\omega^2), \quad I_2 = 4(E_2^2 - k_2n_2^2\omega^2). \tag{31}$$

Three functionally independent real integrals of motion of the system with Hamiltonian (21) can be chosen to be

$$\left\{E_1, E_2, Q = \tfrac{1}{2}(Q_1 + \bar{Q}_1)\right\}. \tag{32}$$

They are of order 2, 2, and $2(n_1+n_2)$ in the momenta, respectively. Their existence is the proof of the maximal superintegrability of the considered system.

The integral of motion Q simplifies to give a second-order one in two cases (which were known previously [11,13]). They are as follows.

(I) $n_1=n_2=1$,

$$\frac{4E_1E_2 - Q}{2\omega^2} = (p_1x_2 - p_2x_1)^2 + \frac{k_1x_2^2}{x_1^2} + \frac{k_2x_1^2}{x_2^2}. \tag{33}$$

(II) $n_1=1$, $n_2=2$, $k_2=0$,

$$\left(\frac{8E_1^2E_2 - Q}{8\omega^2} - k_1E_2\right)^{1/2} = p_1(x_2p_1 - x_1p_2) - \omega^2x_1^2x_2 + k_1\frac{x_2}{x_1^2}. \tag{34}$$

The integrals (33) and (34) are responsible for the separation of variables in polar and parabolic coordinates, respectively. The integrals $\{E_1,E_2\}$ are responsible for the separation in Cartesian coordinates.

III. THE GENERAL CASE

Within the same approach, it is easy to extend the previous picture to the general situation of a reduction from a $2N$- to an N-dimensional configuration space:

$$H_{2N} = \frac{1}{2}\sum_{i=1}^{2N}\hat{p}_i^2 + \frac{\omega^2}{2}\sum_{j=1}^{N}n_j^2(y_{2j-1}^2 + y_{2j}^2). \tag{35}$$

Indeed, let us introduce the affine variables

$$z_k = \hat{p}_k - in_k\omega y_k, \quad k=1,\ldots,2N,$$

so that the Hamiltonian reads

$$H_{2N} = \frac{1}{2}\sum_{k=1}^{2N}|z_k|^2.$$

The Poisson bracket is defined as in Eq. (25). The invariants under the $SO(2)\times\cdots\times SO(2)$ group action generated by $L_{12},\ldots,L_{2N-1,2N}$ are

$$\xi_{2k-1} = z_{2k-1}^2 + z_{2k}^2, \quad k=1,2,\ldots,N,$$

$$\bar{\xi}_{2k-1} = \bar{z}_{2k-1}^2 + \bar{z}_{2k}^2, \quad k = 1,2,\ldots,N, \tag{36}$$

apart from the quantities $L_{12},\ldots,L_{2N-1,2N}$ and the "two-plane energies" which commute with the Hamiltonian H_{2N},

$$|z_1|^2 + |z_2|^2, \ldots, |z_{2N-1}|^2 + |z_{2N}|^2. \tag{37}$$

Imposing

$$\{H_{2N}, f(\xi,\bar{\xi})\} = 0, \tag{38}$$

where $\xi=(\xi_1,\ldots,\xi_{2N-1})$, $\bar{\xi}=(\bar{\xi}_1,\ldots,\bar{\xi}_{2N-1})$, we get the differential equation

$$\sum_{k=1}^{N} n_k\left(\xi_{2k-1}\frac{\partial}{\partial \xi_{2k-1}} - \bar{\xi}_{2k-1}\frac{\partial}{\partial \bar{\xi}_{2k-1}}\right) f = 0. \tag{39}$$

Its general solution depends on $2N-1$ invariants, which can be chosen as

$$Q_{2k-1} = (z_{2k-1}^2 + z_{2k}^2)^{n_N}(\bar{z}_{2N-1}^2 + \bar{z}_{2N}^2)^{n_k}, \quad k = 1,\ldots,N-1, \tag{40}$$

$\bar{Q}_{2k-1}$, and $I=|z_1^2+z_2^2|$.

Using the transformation (13) we now reduce the original Hamiltonian to the following one:

$$H_N = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \frac{1}{2}\sum_{i=1}^{N}\frac{k_i}{x_i^2} + \frac{\omega^2}{2}\sum_{i=1}^{N} n_i^2 x_i^2. \tag{41}$$

The corresponding reduced invariants are

$$E_l = \frac{1}{2}p_l^2 + \frac{k_l}{2x_l^2} + \frac{1}{2}n_l^2\omega^2 x_l^2, \quad l = 1,\ldots,N,$$

$$R_{2l-1} = \frac{1}{2}(Q_l + \bar{Q}_l), \quad l = 1,\ldots,N-1, \tag{42}$$

where

$$Q_l = \left(p_l^2 + \frac{k_l}{x_l^2} - n_l^2\omega^2 x_l^2 - 2in_l\omega p_l x_l\right)^{n_N} \times \left(p_N^2 + \frac{k_N}{x_N^2} - n_N^2\omega^2 x_N^2 + 2in_N\omega p_N x_N\right)^{n_l}.$$

There are $2N-1$ functionally independent integrals and consequently the system is maximally superintegrable, proving the conjecture of [28].

The main result of this paper is that we have added a further maximally superintegrable system in N dimensions to the rather short list of known ones [5,10,32–34].

IV. OPEN PROBLEMS

From the previous considerations, it emerges that it would be desirable to construct systematically transformations mapping a superintegrable system into another system, that is also superintegrable, and defined in a reduced phase space. It seems natural to associate such transformations to the rich symmetry structure possessed by superintegrable systems. For instance, changes of variables of the type (13) are clearly related to invariance properties under rotation. From this point of view, the role of higher-order groups of transformations generated by the flow associated with integrals that are polynomials in the momenta remains to be fully investigated. A quantum mechanical version of this reduction procedure is also to be understood. For $N=3$ the quantum system was treated in [29]. The reduction was performed for the classical system. The reduced system was then quantized in Cartesian coordinates.

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