# Explosive dynamics and localization of wave triads in a coupled magnetoelastic system

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The dynamics of three-wave parametric coupling is considered on an example of nonequilibrium magnetostrictive medium under electromagnetic pumping. Subthreshold and supercritical mode of three-wave excitation are described analytically and simulated numerically for a triad of magnetoelastic waves. Theoretical analysis of the supercritical mode shows that space-time development of three-wave excitation in such a nonequilibrium system has the character of explosive instability and localization of "positive energy" waves.

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## I. INTRODUCTION

Interaction of elastic vibrations with an extremely nonlinear magnetic subsystem of solid media provides favorable conditions for modeling and experimental studies of strongly nonlinear dynamic phenomena in distributed coupled systems [1]. Controllability of magnetoelastic coupling by external magnetic field opens the wide range of supercritical parametric effects such as elastic wave phase conjugation (WPC) and front reversal [2–4], double bistability of parametric magnetoelastic resonance [1,5], three-phonon coupled parametric excitation [6], self-pulsing and excitability in a magnetoelastic resonator [7,8], etc. In the present paper, we report the results of an analytical study and a numerical simulation of explosive instability and spatial localization mode of parametrically coupled triads of hybridized magnetoelastic waves.

Explosive dynamics in nonlinear wave systems is usually associated with the presence of so-called "negative energy waves" interacting with normal positive waves [9–11]. The energy of a negative wave decreases when its amplitude is increasing. As a result the amplitudes of positive and negative waves can increase instantaneously due to their interaction and energy exchange. In spite of a number of theoretical works describing the mathematical models of negative and positive wave interaction, only a few experimental results on the observation of explosive instability in nonlinear wave systems were reported, mainly in the plasma physics field [12–14]. Taking into account the energetic reason, the common opinion is that explosive instability is impossible in the systems with only positive or only negative waves. In this paper the explosive instability for the system of purely positive energy waves is demonstrated on an example of parametric coupling of three magnetoelastic waves in magnetic crystal under homogeneous electromagnetic pumping. The threshold of the instability development will be derived. Moreover, it will be shown that at the supercritical mode of pumping the explosive increase of amplitudes of the traveling wave triad is accompanied by spatial localization of the deformation field, similarly to the "peaking and localization" phenomena in physics of combustion and explosion [15] or in laser thermochemistry [16].

## II. DYNAMIC EQUATIONS OF THREE-WAVE PARAMETRIC COUPLING

We consider the nonlinear interaction of magnetoelastic waves with a transversal alternative magnetic field $h_\perp(t)$ on an example of antiferromagnetic crystal of symmetry group $D_{3d}^6$, with a magnetic anisotropy of "easy plane" type (AFEP). Such crystals are known as model objects for nonlinear acoustics and magnetoelastic dynamics of solids. Their magnetic dynamics can be described by only one dynamic variable $\varphi(t,\mathbf{r})$, corresponding to the angle of rotation of the antiferromagnetic vector in the base plane, at frequencies much lower than the microwave resonance frequency of the antiferromagnetic mode [1]. The magnetoelastic Lagrangian density $L$ of the crystal consists of elastic $L_e$, magnetic $L_m$, and magnetoelastic $L_{ME}$ components

$$L = L_e(\dot{\vec{u}},\hat{u}) + L_m(\dot{\varphi},\varphi,\nabla\varphi) + L_{ME}(\hat{u},\varphi), \tag{1}$$

$$L_e = \frac{1}{2}\rho\dot{\vec{u}}^2 - \frac{1}{2}\hat{C}^{(2)}\hat{u}^2, \tag{2}$$

$$L_m = \frac{M_0}{2H_E}\{\gamma^{-2}[\dot{\varphi}^2 - v_m^2(\nabla\varphi)^2] + [H_D + H\cos\varphi + h_\perp(t)\sin\varphi]^2\}, \tag{3}$$

$$L_{ME} = (\hat{B}_1\hat{u})\cos 2\varphi + (\hat{B}_2\hat{u})\sin 2\varphi, \tag{4}$$

where $\rho$ is the crystal density, $\hat{C}^{(2)}$ is the second order elastic modulus tensor, $\hat{u}$ is the matrix of elastic deformations, $\hat{B}_{1,2}$ are the matrixes of magnetostriction, $\gamma$ is the magnetomechanical ratio, $v_m$ is the spin wave velocity, $H$, $H_E$, $H_D$ are external bias, exchange, and Dzyaloshinsky-Moria fields, respectively, and $M_0$ is the magnetic sublattice magnetization. The details on notations can be found in Ref. [1].

The Lagrangian (1) creates the system of coupled equations of motion

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$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{\xi}}} - \frac{\delta}{\delta \vec{\xi}} \int d\vec{r} L = 0, \tag{5}$$

where $\vec{\xi}=(\vec{u},\varphi)$ is the four-dimensional dynamic variable, and $\delta/\delta\vec{\xi}$ means a variational derivative.

For quasistatic reaction of magnetic subsystem on alternative deformations at frequencies much lower than the resonance frequency of quasiferromagnetic mode $\omega \ll \omega_{fà} = \gamma\sqrt{H(H+H_D)+2H_E H_{ms}}$, where $H_{ms}$ is the effective field of magnetoelastic anisotropy, the system (5) can be resolved relatively to the magnetic variable $\varphi=\varphi[\hat{u},h_\perp(t)]$. The system (5) in such cases is reduced to one equation for hybridized mode of magnetoelastic waves

$$\rho\ddot{\vec{u}} = -\frac{\delta}{\delta\vec{u}}\int d\vec{r} F^{\text{eff}} = 0, \tag{6}$$

where the effective potential energy density describing three-wave parametric coupling is

$$F^{\text{eff}} = \frac{1}{2}\hat{C}(H)\hat{u}^2 + \left(\frac{2H_E}{M_0}\right)^2\left(\frac{\gamma}{\omega_{f0}}\right)^6 h_\perp(t) H_D (32\hat{B}_1^2 - 3\hat{B}_2^2)\hat{B}_2\hat{u}^3, \tag{7}$$

where $\hat{C}(H)$ is the effective elastic modulus tensor renormalized by magnetoelastic interaction [1]

$$\hat{C}(H) = \hat{C}^{(2)} - \frac{H_E}{M_0}\left(\frac{\gamma}{\omega_{f0}}\right)^2 (2\hat{B}_2)^2. \tag{8}$$

For the AFEP crystal placed in the bias field $\mathbf{H}\parallel\mathbf{x}$, with the $z$ axis parallel to the $C_3$ crystallographic axis, the symmetry $D_{3d}^6$ allows propagation in the $(x,z)$ plane of pure transversal waves with polarization $\mathbf{u}\parallel\mathbf{x}$.

Let us consider parametric coupling of wave triad with wave vectors $\mathbf{k}_{1,2,3}$ corresponding to the following pulse conservation law:

$$\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = \mathbf{0}. \tag{9}$$

We will also assume that the frequency of harmonic pumping $h_\perp(t)=h_0 \cos \omega_p t$ corresponds to the resonance condition

$$\omega_p = \omega_1 + \omega_2 + \omega_3. \tag{10}$$

For simplicity, we consider symmetric orientations of wave vectors $\mathbf{k}_{2,3}$ relative to the $z$ axis ($k_{2y}=-k_{3y}$) while orientation of $\mathbf{k}_1$ vector is antiparallel to the $z$ direction (see Fig. 1). For the case of one-dimensional parametric interaction in an active layer of finite length ($0<z<L$), the system of equations for slowly variable amplitudes of waves $A_{1,2,3}(t,z)$ can be derived as

$$\omega_n\left(\frac{\partial A_n}{\partial t} + v_n\frac{\partial A_n}{\partial z}\right) = \Psi\varepsilon_{nml}A_m^* A_l^*, \tag{11}$$

where $\varepsilon_{nml}$ is the completely antisymmetric tensor $(l,m,n = 1,2,3)$, $v_n$ is the projection of the group velocity of wave "$n$" on the $z$ axis

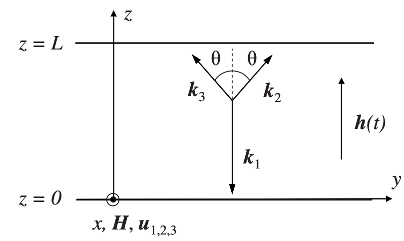


FIG. 1. Geometry and vector diagram of the law of conservation of momentum for the considered three waves (with wave vectors $\mathbf{k}_{1,2,3}$) coupled excitation in antiferromagnets under high frequency electromagnetic pumping $[\mathbf{h}(t)]$.

$$v_n = \frac{C_{44}(H)}{\rho\omega_n}k_{nz} + \frac{C_{14}(H)}{\rho\omega_n}k_{ny}, \tag{12}$$

$\omega_n$ is the wave frequency

$$\omega_n = \rho^{-1/2}[C_{44}(H)k_{nz}^2 + C_{66}(H)k_{ny}^2 + 2C_{14}(H)k_{ny}k_{nz}]^{1/2}, \tag{13}$$

and $\Psi$ is the coupling coefficient for the parametric interaction of three waves with the transverse pumping field given by

$$\Psi = \frac{9}{2\rho}h_0 H_D\left(\frac{2H_E}{M_0}\right)^2\left(\frac{\gamma}{\omega_{f0}}\right)^6 B_{14}k_{1z}\left[\frac{1}{2}(B_{11}-B_{12})k_{2y} + B_{14}k_{2z}\right]\left[\frac{1}{2}(B_{11}-B_{12})k_{3y} + B_{14}k_{3z}\right]. \tag{14}$$

Here, $B_{ij}$ are the magnetostrictive constants in the Voigt's notation [1]. The system (11) is analyzed with boundary conditions $A_1|_{z=L}=0$ and $A_{2,3}|_{z=0}=A_{0\ 2,3}$, corresponding to typical experimental configuration of input of the incident waves ($A_{2,3}$) from one side of the active zone ($z=0$) and the absence of any source of backward wave ($A_1$) from the other side ($z>L$). The feature of equations (11) is that $v_1<0$, while $v_{2,3}>0$.

## III. STATIONARY SOLUTIONS

For stationary conditions $(\partial/\partial t=0)$ the assumption $A_{02}(\omega_2 v_2)^{1/2}=A_{03}(\omega_3 v_3)^{1/2}$ allows one to reduce the system (11) to a system of two equations for the variables $A_1$ and $A_2$. Under this assumption, the solutions of this system are

$$A_1(z) = |A_{02}A_{03}|^{1/2}\eta\cosh(\alpha L)\tanh[\alpha(L-z)],$$

$$A_2(z) = A_{02}\frac{\cosh(\alpha L)}{\cosh[\alpha(L-z)]}, \tag{15}$$

where the parameter $\alpha$ is defined from the relationship

$$\frac{\alpha L}{\cosh(\alpha L)} = \left|\frac{\Psi\sqrt{A_{02}A_{03}}}{\omega_1 v_1}\right|\eta^{-1}L. \tag{16}$$

Parameter $\eta$, dependent on the angle of incidence of waves $n=2,3$, is equal to

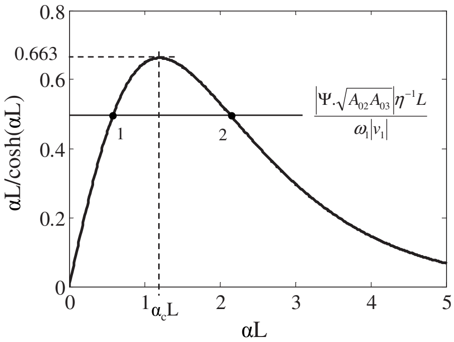


FIG. 2. Plot of the function $f(\alpha L)=(\alpha L)/\cosh(\alpha L)$ versus $\alpha L$. Point 1 corresponds to a stable solution of the system (4) and point 2 to an unstable one. Above $f(\alpha_c L)=0.633$, no stable solution can be found, and explosive instability appears.

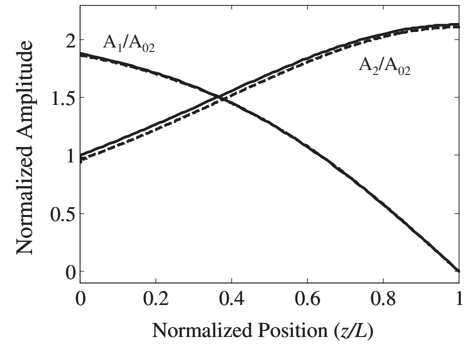


FIG. 3. Spatial distribution of the slowly variable amplitudes of waves $A_{1,2}/A_{02}$ in the steady state regime under the threshold of explosive instability. Results obtained by the analytical solution (5) are shown with solid line, and results obtained by numerical calculation with dotted line.

$$\eta = (\omega_2\omega_3 v_2 v_3)^{1/4}/|\omega_1 v_1|^{1/2}. \tag{17}$$

We note that for interaction of collinear waves we have $\eta = 1/\sqrt{2}$. Solutions (15) correspond to the phase shift of the three-wave correlator $(A_1A_2A_3)=|A_1A_2A_3|\exp(i\varphi)$, relatively to the phase of pumping, equal to $\varphi=\pi$.

The solution of Eq. (16) is illustrated by Fig. 2. Under the instability threshold, Eq. (16) has two solutions as shown on Fig. 2. But, a stability analysis shows that only the one corresponding to the lower value of $\alpha L$ (point 1) is stable, i.e., the stationary solutions (15) are stable if $\alpha<\alpha_c$, where $\alpha_c$ corresponds to the maximum on the curve of Fig. 2: $\alpha_c L = 1.2$.

The analysis shows that when the right-hand part of Eq. (16) is higher than the critical value given by $(\alpha_c L)/\cosh(\alpha_c L)=0.663$, Eq. (16) has no solution. As will be shown below by numerical simulation this condition corresponds to the explosive instability behavior of the system. Thus, the supercritical mode of such instability corresponds to the following inequality:

$$\left|\frac{\Psi\sqrt{A_{02}A_{03}}}{\omega_1 v_1}\right| \eta^{-1}L > 0.663. \tag{18}$$

The supercritical conditions can be achieved if magnetoelastic coupling, amplitude of electromagnetic pumping, and initial amplitude of direct waves are strong enough. Estimation of the threshold condition (15) for real AFEP crystal $\alpha$-$Fe_2O_3$ ($\rho=5.29$ g/cm$^3$, $C_{44}=8.5\times10^{11}$ dyn/cm$^3$, $2B_{14}=3\times10^7$ erg/cm$^3$, $B_{11}$-$B_{12}=10^7$ erg/cm$^3$, $M_0=870$ Gs, $H_E=9.2\times10^6$ Oe, $H_D=22$ kOe, $H_{me}=0.6$ Oe) of length $L\sim1$ cm, under a bias field $H=100$ Oe, for initial deformation $k_2A_{02}\sim10^{-5}$ (with a frequency $\omega/2\pi=10$ MHz) defines the critical values of pumping field strength $h_{0c}\sim1-10$ Oe available experimentally. The three-phonon single mode supercritical parametric excitations was recently observed in $\alpha$-$Fe_2O_3$ magnetoacoustic resonator in Ref. [6].

## IV. EXPLOSIVE INSTABILITY AND LOCALIZATION

Time-space pattern of the development of the explosive instability was obtained by the direct numerical solution of the system (11). For numerical integration we used a Fourier pseudospectral time domain (PSTD) code, with convolutional perfect matched layers [17] introduced at each end of the 1D domain in order to absorb outgoing waves. For time integration, a fourth-order Adams-Bashforth method is used to evaluate $A_{1,2,3}$ at each time step.

The numerical computations were made for a $L=0.03$ m length active layer, for collinear interactions ($\theta=0$ on Fig. 1) with $v_2=v_3=-v_1=3000$ m/s group velocities. The form of the slowly variable amplitudes $A_{2,3}$ on the boundary $z=0$ was chosen to be equal to $A_{2,3}(0,t)=0.5A_{02}[1-\cos(2\pi t/T)]$, where $0<t<T$ and $T=2$ ms is the duration of the amplitude envelope. The initial condition is $A_{1,2,3}(z,0)=0$.

Under the instability threshold, the spatial distribution of the slowly variable amplitudes of waves $A_{1,2}/A_{02}$ for stationary conditions can be calculated with the analytical solution (15). Comparison of this analytical solution with the direct numerical calculation is displayed on Fig. 3, for $|\Psi A_{02}/\omega_1 v_1|\eta^{-1}L=0.652$. This validates the implementation of the numerical simulation of system (11).

Now, when the instability threshold is exceeded ($|\Psi A_{02}/\omega_1 v_1|\eta^{-1}L=0.678$) the numerical calculations show that the amplitude of waves $A_{1,2}/A_{02}$ blowup in finite time. This result is in accord with the analysis of three-phonon single mode excitations for AFEP resonator in Ref. [6]. Moreover, the numerical simulation for traveling wave triad under consideration shows also the clearly expressed localization of supercritical excitation in a space region that is much smaller than the active area $0<z<L$ of the wave propagation medium (see Fig. 5).

In order to display the explosive instability and localization with finite amplitudes, the electromagnetic pumping in the numerical simulation was stopped ($\Psi$ was put to 0) at time $t_{\text{pump}}=565.83$ $\mu$s just before the blowup time. Time evolution of the amplitude $A_2/A_{02}$ at the position $z=L$ is shown on Fig. 4(a). A zoom of the time evolution of $A_2/A_{02}$ around the blowup time is shown on the inset of Fig. 4. A logarithmic representation of this amplitude evolution demonstrates that the increase of the amplitude, exponential at the beginning of the parametric process, becomes faster when the blowup time is approached. This is characteristic of explosive instability. On Fig. 5 the spatial distributions of the amplitudes $A_1/A_{02}$ (dotted line) and $A_2/A_{02}$ (solid line) at

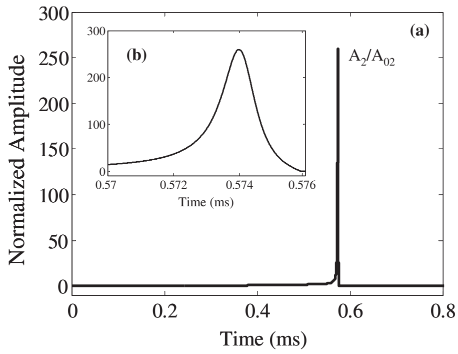


FIG. 4. (a) Time evolution of the amplitude of wave $A_2/A_{02}$ at the position $z=L$ when the explosive instability threshold is exceeded. The electromagnetic pumping is stopped at time $t_{\text{pump}}=565.83$ $\mu$s. In the inset (b), a zoom of the time evolution of $A_2/A_{02}$ demonstrates increasing of amplitude in time faster than exponential law.

time $t_{\text{pump}}$ are displayed. The amplitudes $A_1/A_{02}$ and $A_2/A_{02}$ locally increase inside the active layer both at the same position. These results clearly demonstrate simultaneous time and space localizations of phonons in the process of three-wave coupled excitation in antiferromagnets under high frequency electromagnetic pumping.

## V. CONCLUSION

Numerical simulations of coupling of three positive energy magnetoelastic waves under homogeneous supercritical electromagnetic pumping demonstrate behaviors typical for phenomena of explosive instability and localization [10,15]. In the framework of the theoretical model under consideration, the stabilization occurs when the incident wave pulse is rather short or the pumping is switched off in the beginning of the supercritical process. Other mechanisms of singularity stabilization depend on higher order nonlinear properties of the magnetic medium. As was shown in Ref. [6] for single mode three-phonon interaction in antiferromagnet, stabilization is caused by the nonlinear frequency shift of the mode. Another reason for stabilization can be the nonlinear damping of high amplitude magnetoelastic waves. Nonlinear mechanisms of stabilization are the subjects of special studies. The phenomenon of explosive instability of wave triad under electromagnetic pumping is not specific for antiferromagnetic crystals and can be observed in any magnetic medium with strong enough magnetoelastic interaction.

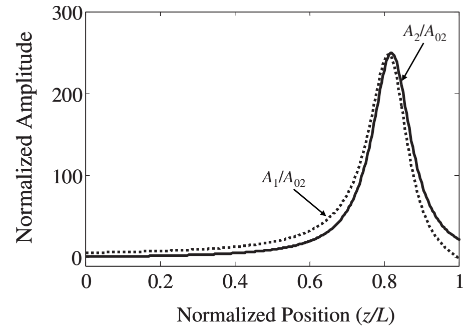


FIG. 5. Spatial distributions of the amplitudes $A_1/A_{02}$ (dotted line) and $A_2/A_{02}$ (solid line) at time $t_{\text{pump}}=565.83$ $\mu$s.

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