# Influence of the dispersive and dissipative scales $\alpha$ and $\beta$ on the energy spectrum of the Navier-Stokes $\alpha\beta$ equations

Xuemei Chen[1] and Eliot Fried[2]
[1]*Department of Engineering Science and Mechanics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24060-0002, USA*
[2]*Department of Mechanical Engineering, McGill University, Montréal, Québec, Canada H3A 2K6*
(Received 1 May 2008; published 31 October 2008)

Lundgren's vortex model for the intermittent fine structure of high-Reynolds-number turbulence is applied to the Navier-Stokes $\alpha\beta$ equations and specialized to the Navier-Stokes $\alpha$ equations. The Navier-Stokes $\alpha\beta$ equations involve dispersive and dissipative length scales $\alpha$ and $\beta$, respectively. Setting $\beta$ equal to $\alpha$ reduces the Navier-Stokes $\alpha\beta$ equations to the Navier-Stokes $\alpha$ equations. For the Navier-Stokes $\alpha$ equations, the energy spectrum is found to obey Kolmogorov's $-5/3$ law in a range of wave numbers identical to that determined by Lundgren for the Navier-Stokes equations. For the Navier-Stokes $\alpha\beta$ equations, Kolmogorov's $-5/3$ law is also recovered. However, granted that $\beta<\alpha$, the range of wave numbers for which this law holds is extended by a factor of $\alpha/\beta$. This suggests that simulations based on the Navier-Stokes $\alpha\beta$ equations may have the potential to resolve features smaller than those obtainable using the Navier-Stokes $\alpha$ equations.

DOI: 10.1103/PhysRevE.78.046317 PACS number(s): 47.27.E−, 47.27.Gs, 47.27.Jv, 89.75.Da

## I. INTRODUCTION

The Navier-Stokes $\alpha$ model for statistically homogeneous and isotropic turbulent flow yields a system of governing equations,

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathrm{grad}\,\mathbf{v})\mathbf{u} + (\mathrm{grad}\,\mathbf{u})^T\mathbf{v} = -\,\mathrm{grad}\,\frac{p}{\rho} + \nu\Delta\mathbf{v}, \tag{1a}$$

$$\mathbf{v} = (1-\alpha^2\Delta)\mathbf{u}, \tag{1b}$$

$$\mathrm{div}\,\mathbf{u} = 0, \tag{1c}$$

for filtered and unfiltered velocities $\mathbf{u}$ and $\mathbf{v}$ and a pressure $p$. Aside from the density $\rho$ and the kinematic viscosity $\nu$, (1) involves an additional parameter $\alpha>0$ carrying dimensions of length. Within the framework of Lagrangian averaging, $\alpha$ is the statistical correlation length of the excursions taken by a fluid particle away from its phase-averaged trajectory. More intuitively, $\alpha$ is often interpreted as the characteristic linear dimension of the smallest eddies that the model is capable of resolving. Like equations arising from Reynolds averaging, the Navier-Stokes $\alpha$ equations therefore provide an approximate model that resolves motions only above some critical scale, while relying on filtering to encompass effects at smaller scales.

Equations (1), with $\nu=0$, known as Euler $\alpha$ equations, were first introduced by Holm *et al.* [1,2] for the mean motion of an ideal, incompressible fluid and as a three-dimensional generalization of the Camassa-Holm [3] equation. Subsequently, Chen *et al.* [4–7] introduced the viscous term, giving rise to (1) with $\nu>0$ and explored the utility of the resulting equations as a subgrid model for turbulent flow. Their numerical results showed that the large-scale features, including statistics and structures, are preserved by the Navier-Stokes $\alpha$ equations, even at coarser levels where the fine scales are not fully resolved, and that an energy spectrum consistent with Kolmogorov's [8] $-5/3$ law can be identified in the inertial range. For $\alpha$ finite, this behavior is seen to roll off to a steeper spectrum when the wave number $k$ is such that $\alpha k \gg 1$. Later Foias *et al.* [9] showed that this model preserves Kelvin's circulation theorem with some minor modifications and, by emulating Foias's [10] work on the Navier-Stokes equations, obtained a faster roll-off energy spectrum below the length scale $\alpha$. The inertial range of the Navier-Stokes-$\alpha$ is thus shorter than that of the Navier-Stokes equations, in agreement with the direct numerical simulation results of Chen *et al.* [7]. The analysis of Foias *et al.* [9] revealed that, at a given Reynolds number Re, the number of degrees of freedom entering computations involving the Navier-Stokes $\alpha$ equations is proportional to $(L/\alpha)\mathrm{Re}^{3/2}$, with $L$ being the integral scale. This result was confirmed by Graham *et al.* [11]. Subsequently, however, Graham *et al.* [12] reported that, when $\alpha$ is no more than a few times the Kolmogorov dissipation scale, "flow polymerization" may result in a Reynolds-number-independent reduction, by factor of about 10, in the number of degrees of freedom. Foias *et al.* [13] later established the global existence and uniqueness of strong solutions for the three-dimensional Navier-Stokes $\alpha$ model, taking advantage of the regularized vorticity stretching effect. A summary of properties and advantages of the Navier-Stokes $\alpha$ equations is provided by Holm *et al.* [14].

As Chen *et al.* [4] note, eliminating the unfiltered velocity $\mathbf{v}$ between (1a) and (1b) yields a single evolution equation that can be expressed in the form

$$\rho\dot{\mathbf{u}} = \mathrm{div}\,\boldsymbol{\Sigma}, \tag{2}$$

where an overdot denotes the material time derivative determined by the filtered velocity [so that, in particular, $\dot{\mathbf{u}} = \partial\mathbf{u}/\partial t + (\mathrm{grad}\,\mathbf{u})\mathbf{u}$] and the tensor $\boldsymbol{\Sigma}$ is defined by

$$\boldsymbol{\Sigma} = -\tilde{p}\mathbf{1} + 2\rho\nu\mathbf{D} + 2\rho\alpha^2(\mathring{\mathbf{D}} - \nu\Delta\mathbf{D}), \tag{3}$$

with

 

$$\mathbf{D} = \frac{1}{2}[\text{grad}\,\mathbf{u} + (\text{grad}\,\mathbf{u})^T] \quad (4)$$

the filtered stretch rate,

$$\overset{\circ}{\mathbf{D}} = \dot{\mathbf{D}} + \mathbf{DW} - \mathbf{WD} \quad (5)$$

the corotational rate of $\mathbf{D}$ {where $\mathbf{W} = \frac{1}{2}[\text{grad}\,\mathbf{u} - (\text{grad}\,\mathbf{u})^T]$ denotes the filtered spin}, and

$$\tilde{p} = p + \frac{1}{2}\rho(|\mathbf{u}|^2 + \alpha^2|\mathbf{D}|^2) \quad (6)$$

a filtered pressure. The structure of (2) is formally suggestive of a conservation law expressing the balance of linear momentum. However, although

$$\mathbf{\Sigma} + 2\rho\alpha^2\nu\Delta\mathbf{D} = -\tilde{p}\mathbf{1} + 2\rho\nu\mathbf{D} + 2\rho\alpha^2\overset{\circ}{\mathbf{D}} \quad (7)$$

describes the stress in an incompressible Rivlin-Ericksen [15] fluid of second grade, a stress that is power conjugate to $\mathbf{D}$ and that yields non-negative dissipation per unit volume provided that $\nu > 0$, $\mathbf{\Sigma}$ is not power conjugate to $\mathbf{D}$ and therefore cannot be viewed as an embodiment of the conventional notion of Cauchy stress. Based on experience with theories for structured media, Fried and Gurtin [16] argued that the presence in $\mathbf{\Sigma}$ of a term involving the Laplacian of $\mathbf{D}$ indicates that a continuum framework emcompassing the Navier-Stokes $\alpha$ equations (1) should involve not only the classical Cauchy stress but also a hyperstress.

Building on this observation, Fried and Gurtin [16] developed a general framework for fluid-dynamical theories involving gradient dependencies. For an incompressible fluid with velocity $\mathbf{u}$, the basic equations arising within that framework are

$$\rho\dot{\mathbf{u}} = \text{div}\,\mathbf{T} + \text{curl}(\text{div}\,\mathbf{G}). \quad (8a)$$

$$\mathbf{T} = \mathbf{T}^T, \quad (8b)$$

$$\text{div}\,\mathbf{u} = 0, \quad (8c)$$

where $\mathbf{T}$ is the Cauchy stress and $\mathbf{G}$ is the hyperstress. The underlying dissipation per unit volume has the form

$$\Gamma = \mathbf{S}:\mathbf{D} + \mathbf{G}:\text{grad}\,\boldsymbol{\omega} - \rho\dot{\varphi} \geqslant 0, \quad (9)$$

where

$$\mathbf{S} = \mathbf{T} + \tilde{p}\mathbf{1} \quad (10)$$

is the extra stress,

$$\boldsymbol{\omega} = \text{curl}\,\mathbf{u} \quad (11)$$

is the vorticity associated with $\mathbf{u}$, and $\varphi$ is the internal kinetic energy, measured per unit mass.

The basic equations (8) are independent of constitutive assumptions. Granted that $\varphi$ is quadratic in $\mathbf{D}$, that $\mathbf{S}$ is linear in $\mathbf{D}$ and the corotational rate of $\mathbf{D}$, and that $\mathbf{G}$ is linear in grad $\boldsymbol{\omega}$, the stipulation that $\Gamma$ be non-negative in all processes requires that $\varphi = \alpha^2|\mathbf{D}|^2$, that

$$\mathbf{S} = 2\rho\nu\mathbf{D} + 2\rho\alpha^2\overset{\circ}{\mathbf{D}}, \quad (12)$$

and that

$$\mathbf{G} = \rho\nu\beta^2[\text{grad}\,\boldsymbol{\omega} + \boldsymbol{\gamma}(\text{grad}\,\boldsymbol{\omega})^T], \quad (13)$$

where the coefficients $\alpha$ and $\beta$ carry dimensions of length and, to ensure that the dissipation inequality (9) is satisfied in all processes, the kinematic viscosity $\nu$ and the dimensionless parameter $\gamma$ obey

$$\nu \geqslant 0 \quad \text{and}\ |\gamma| \leqslant 1. \quad (14)$$

Since, by (10), $\mathbf{T} = -\tilde{p}\mathbf{1} + \mathbf{S}$, using the particular expressions (12) and (13) for $\mathbf{S}$ and $\mathbf{G}$ in the flow equation (8a) yields the evolution equation

$$\rho\dot{\mathbf{u}} = -\,\text{grad}\,\tilde{p} + \nu(1 - \beta^2\Delta)\Delta\mathbf{u} + \rho\alpha^2\left(\overline{\Delta\mathbf{u}}^{\,\cdot} + (\text{grad}\,\mathbf{u})^T\mathbf{u} + \frac{1}{2}\,\text{grad}\,|\mathbf{D}|^2\right), \quad (15)$$

with $\overline{\Delta\mathbf{u}}^{\,\cdot} = \partial(\Delta\mathbf{u})/\partial t + [\text{grad}(\Delta\mathbf{u})]\mathbf{u}$. Alternatively, defining $\mathbf{v}$ by (1b) and $p$ by (6), we find that, when $\mathbf{S}$ and $\mathbf{G}$ are given by (12) and (13), the basic equations (8) yield the system

$$\frac{\partial\mathbf{v}}{\partial t} + (\text{grad}\,\mathbf{v})\mathbf{u} + (\text{grad}\,\mathbf{u})^T\mathbf{v} = -\,\text{grad}\,\frac{p}{\rho} + \nu(1 - \beta^2\Delta)\Delta\mathbf{u}, \quad (16a)$$

$$\mathbf{v} = (1 - \alpha^2\Delta)\mathbf{u}, \quad (16b)$$

$$\text{div}\,\mathbf{u} = 0, \quad (16c)$$

which reduces to the Navier-Stokes $\alpha$ equations (1) on equating $\beta$ to $\alpha$. On the other hand, setting $\beta = 0$ in either (15) or (16) yields the equations governing the motion of a Rivlin-Ericksen [15] fluid of second grade.

Interpreting $\mathbf{v}$ and $\mathbf{u}$ as unfiltered and filtered velocities, we refer to (16) as the Navier-Stokes $\alpha\beta$ equations.

When $\mathbf{S}$ and $\mathbf{G}$ are given by (12) and (13), the expression (9) determining the dissipation per unit volume specializes to give

$$\frac{\Gamma}{\rho\nu} = 2|\mathbf{D}|^2 + \beta^2(1 + \gamma)|\mathbf{A}|^2 + \beta^2(1 - \gamma)|\mathbf{Z}|^2 \geqslant 0, \quad (17)$$

where we have introduced $\mathbf{A} = \frac{1}{2}[\text{grad}\,\boldsymbol{\omega} + (\text{grad}\,\boldsymbol{\omega})^T]$ and $\mathbf{Z} = \frac{1}{2}[\text{grad}\,\boldsymbol{\omega} - (\text{grad}\,\boldsymbol{\omega})^T]$. Importantly, therefore, $\alpha$ does not enter the dissipation and the contribution to the extra stress involving the corotational rate of $\mathbf{D}$ is dispersive as opposed to dissipative. In contrast, the hyperstress is entirely dissipative. As Fried and Gurtin [17] observe, this identifies the length scales $\alpha$ and $\beta$ as dispersive and dissipative, respectively. One therefore expects that the dispersive length $\alpha$, which is of energetic origin, should represent a characteristic measure of the eddy scales within the inertial range whereas the dissipative length $\beta$ should represent a characteristic measure of the eddy scales within the dissipation range. Consistent with the hierarchy of eddy scales entering Richardson's [18] energy cascade, we therefore expect that

$$\beta < \alpha. \tag{18}$$

A standard step toward validating any turbulence model is to determine whether it captures Kolmogorov's [8] $-5/3$ law in the inertial range [5,9,19–21]. That the Navier-Stokes $\alpha$ model possesses this property was confirmed analytically by Foias *et al.* [9] using Kraichanan's [22,23] cascading mechanism and numerically by Chen *et al.* [7]. The analogous question for the Navier-Stokes $\alpha\beta$ model remains unanswered.

Mimicking an argument employed by Foias *et al.* [9], we may estimate the number of degrees of freedom entering computations involving the Navier-Stokes $\alpha\beta$ equations. The ratio of this estimate to that for the Navier-Stokes $\alpha$ equations is independent of the Reynolds number Re and given by $(\alpha/\beta)^3$. On this basis, the extension of the inertial range that occurs for $\beta<\alpha$ is accompanied by an increase in computational cost. The extent to which the Navier-Stokes $\alpha\beta$ allows for flow polymerization of the sort discussed by Graham *et al.* [12] has not been addressed. In particular, it would be interesting to know whether there exist choices of $\alpha$ and $\beta$, with $\beta<\alpha$, for which flow polymerization occurs and the increased cost associated with taking $\beta<\alpha$ would be mitigated.

Lundgren [24] introduced a strained spiral vortex model as an assembly of asymptotic solutions to the Navier-Stokes equations. It was found that the energy spectrum $E(k)$ of the spiral model associated with axial vorticity obeys $E(k) \sim k^{-5/3}$, which lies between the predictions obtained by the vortex tube and layer models [25,26], and more importantly, conforms to Kolmogorov's [8] similarity theory. Specifically, Lundgren's [24] model consists of axial vorticity helically wound up around the Burgers vortex tube. This model was extensively discussed subsequently by Lundgren [27], Pullin and Saffman [28], and Pullin *et al.* [29]. An experimental realization of the stretched vortex model was achieved by Cuypers *et al.* [30] and a detailed comparison between the experimental measurements and the predictions of Lundgren's [24] model was provided subsequently by Cuypers *et al.* [31].

Lundgren [24] introduced and studied the vortex model to provide an analytical connection between the heuristic scaling results of Kolmogorov and the properties of the Navier-Stokes equations and to account for the intermittent fine structure of high-Reynolds-number turbulence. Here, we apply Lundgren's [24] model to the Navier-Stokes $\alpha\beta$ equations. In so doing, our objective is to uncover the physical influence of the dispersive and dissipative length scales $\alpha$ and $\beta$ on the structure of the energy spectrum and the extent to which choosing $\beta<\alpha$ may allow for improved agreement with the spectral properties of the Navier-Stokes equations. An understanding of these influences, and any attendant limitations that they may place on the applicability of the model, is indispensible. Without it, we cannot expect to make meaningful use of the model. Further, when reduced to the Navier-Stokes $\alpha$ equations (by setting $\beta=\alpha$), we seek an independent confirmation of the analytical results of Foias *et al.* [9] and the numerical results of Chen *et al.* [7]. Our analysis is thus meant to serve as a first, but important, step toward a more generic understanding based on analysis and numerics along the lines conducted by Foias *et al.* [9] and Chen *et al.* [7] for the Navier-Stokes $\alpha$ equations.

To accomplish our goals, we work with appropriate counterparts of the Helmholtz vorticity equation. Specifically, taking the curl of (16a) and using the identities $(\text{grad}\,\mathbf{v})\mathbf{u}+(\text{grad}\,\mathbf{u})^T\mathbf{v}=\text{grad}(\mathbf{u}\cdot\mathbf{v})-\mathbf{u}\times\text{curl}\,\mathbf{v}$ and $\text{curl}(\Delta\mathbf{u}) =\Delta(\text{curl}\,\mathbf{u})$, we obtain

$$\frac{\partial \mathbf{q}}{\partial t} + (\text{grad}\,\mathbf{q})\mathbf{u} - (\text{grad}\,\mathbf{u})\mathbf{q} = \nu(1-\beta^2\Delta)\Delta\boldsymbol{\omega}, \tag{19}$$

where

$$\mathbf{q} = \text{curl}\,\mathbf{v} = (1-\alpha^2\Delta)\text{curl}\,\mathbf{u} = (1-\alpha^2\Delta)\boldsymbol{\omega} \tag{20}$$

is the unfiltered vorticity.

The paper is organized as follows. In Sec. II, we utilize dimensional arguments to show that the energy spectrum of the Navier-Stokes $\alpha\beta$ model is consistent with the $-5/3$ law and obtain a relation for the wave number at which viscous dissipation becomes dominant. In Sec. III, we apply Lundgren's [24] spiral vortex model to the Navier-Stokes $\alpha\beta$ equations. Calculations underlying the asymptotic inviscid spiral solution are presented in the Appendix.

## II. HEURISTIC DETERMINATION OF THE ENERGY SPECTRUM AND THE EXTENT OF THE INERTIAL RANGE OF THE NAVIER-STOKES $\alpha\beta$ MODEL

Fried and Gurtin [17] show that, for the Navier-Stokes $\alpha\beta$ model, the turbulent kinetic energy $\mathcal{E}_{\alpha\beta}$, per unit mass, for a periodic flow in a cubic cell $\mathcal{D}=[0,L]^3$ can be expressed as

$$\mathcal{E}_{\alpha\beta} = \frac{1}{2L^3}\int_{\mathcal{D}} (|\mathbf{u}|^2 + \alpha^2|\boldsymbol{\omega}|^2)d\mathbf{x}. \tag{21}$$

Alternatively, introducing the Fourier transform $\hat{\mathbf{u}}=\mathcal{F}\{\mathbf{u}\}$ of $\mathbf{u}$ and suppressing dependence upon $t$, we find that

$$\mathcal{E}_{\alpha\beta} = \frac{1}{2L^3}\int_{\mathcal{D}_k} (1+\alpha^2|\mathbf{k}|^2)|\hat{\mathbf{u}}(\mathbf{k})|^2 d\mathbf{k} = \int_0^\infty E_{\alpha\beta}(k)dk, \tag{22}$$

where $k=|\mathbf{k}|$ is the wave number, $\mathcal{D}_k$ is the volume in wave number space, and $E_{\alpha\beta}$ is the energy spectrum. It then follows that

$$E_{\alpha\beta}(k) = \frac{1}{2L^3}\int_{|\boldsymbol{\kappa}|=k} (1+\alpha^2|\boldsymbol{\kappa}|^2)|\hat{\mathbf{u}}(\boldsymbol{\kappa})|^2 d\boldsymbol{\kappa}. \tag{23}$$

Aside from the explicit dependence on $\alpha$, (23) shows that the energy spectrum of the Navier-Stokes $\alpha\beta$ equations generally depends on both $\alpha$ and $\beta$ through $\hat{\mathbf{u}}$.

We now explore the consequences of applying Kolmogorov's first and second similarity hypotheses to the Navier-Stokes $\alpha\beta$ model. Kolmogorov's first similarity hypothesis stipulates that, in the universal equilibrium range of all wave numbers $k$ greater than the wave number that demarks the boundary between the integral and inertial ranges, the energy spectrum is determined by a universal relation depending on all relevant physical parameters. For the Navier-Stokes $\alpha\beta$

model, dimensional considerations dictate that this universal relation has the form

$$E_{\alpha\beta}(k) = \epsilon_{\alpha\beta}^{2/3} k^{-5/3} \Phi(\eta_{\alpha\beta}k, \alpha k, \beta k), \tag{24}$$

where $\eta_{\alpha\beta}=(\nu^3/\epsilon_{\alpha\beta})^{1/4}$ is the Kolmogorov length scale for the Navier-Stokes $\alpha\beta$ model, with $\epsilon_{\alpha\beta}$ being the relevant energy-transfer rate, and $\Phi$ is dimensionless.

Kolmogorov's second similarity hypothesis stipulates that, in the inertial range, the energy spectrum is independent of viscous dissipation. The Navier-Stokes $\alpha\beta$ model includes two sources of viscous dissipation: the Newtonian contribution $2\rho\nu\mathbf{D}$ to the extra stress $\mathbf{S}$ and the hyperstress $\mathbf{G}=\rho\nu\beta^2[\text{grad}\,\boldsymbol{\omega}+\gamma(\text{grad}\,\boldsymbol{\omega})^T]$. The arguments $\eta_{\alpha\beta}k$ and $\beta k$ of $\Phi$ therefore embody dissipative mechanisms. It follows that, in the inertial range, the function $\Phi$ must be independent of $\eta_{\alpha\beta}k$ and $\beta k$, so that

$$E_{\alpha\beta}(k) = \epsilon_{\alpha\beta}^{2/3} k^{-5/3} \phi(1+\alpha^2k^2), \tag{25}$$

where, like $\Phi$, $\phi$ is dimensionless and, without loss of generality, dependence upon $\alpha k$ has been replaced by dependence on $1+\alpha^2k^2$.

We might further assume that the dependence of $\varphi$ on $1+\alpha^2k^2$ obeys a power law, in which case

$$E_{\alpha\beta}(k) = C_{\alpha\beta}\epsilon_{\alpha\beta}^{2/3} k^{-5/3}(1+\alpha^2k^2)^m, \tag{26}$$

where the exponent $m$ cannot be determined through scaling analysis and $C_{\alpha\beta}$ is the universal Kolmogorov constant relevant to the Navier-Stokes $\alpha\beta$ theory. For $\alpha k \ll 1$, $(1+\alpha^2k^2)^m \sim 1$, regardless of the value of the exponent $m$, and Kolmogorov's $-5/3$ law is recovered regardless of the value of $\beta$.

A similar scaling analysis applies to the wave number $k_{\alpha\beta}$ at which viscous dissipation becomes dominant for the Navier-Stokes $\alpha\beta$ model. In particular, using $1/\alpha$ to nondimensionalize $k_{\alpha\beta}$, we have

$$k_{\alpha\beta} = \frac{1}{\alpha}\Lambda\left(\frac{\alpha}{\beta}, \frac{\alpha}{\eta_{\alpha\beta}}\right), \tag{27}$$

where $\Lambda$ is dimensionless. If we further assume that the dependence on the dimensionless groups $\alpha/\beta$ and $\alpha/\eta_{\alpha\beta}$ obeys a power law, (27) specializes to an expression,

$$k_{\alpha\beta} = \frac{1}{\alpha}\left(\frac{\alpha}{\beta}\right)^r\left(\frac{\alpha}{\eta_{\alpha\beta}}\right)^l, \tag{28}$$

involving unknown exponents $r$ and $l$.

The foregoing discussion specializes directly to the Navier-Stokes $\alpha$ model. For that model, we denote the energy spectrum, energy-transfer rate, and Kolmogorov length scale by $E_\alpha$, $\epsilon_\alpha$, and $\eta_\alpha$, respectively. Further, we use $k_\alpha$ to denote the wave number at which viscous dissipation becomes dominant. In particular, for the Navier-Stokes $\alpha$ model (26) and (28) become

$$E_\alpha(k) = C_\alpha \epsilon_\alpha^{2/3} k^{-5/3}(1+\alpha^2k^2)^m \tag{29}$$

and

$$k_\alpha = \frac{1}{\alpha}\left(\frac{\alpha}{\eta_\alpha}\right)^l. \tag{30}$$

The energy spectrum and the cutoff wave number obtained in this way are consistent with the analytical results of Foias *et al.* [9] and the numerical results of Chen *et al.* [7] with $m=-2/3$ and $l=2/3$.

## III. VORTEX MODEL FOR NAVIER-STOKES $\alpha\beta$ EQUATIONS

### A. Problem description

To make our analysis self-contained, we first review the salient features of Lundgren's [24] model. It is assumed that there is approximate balance between the two competing mechanisms of vortex stretching and vortex amalgamation in the flow field. Coalescence causes the creation of spiral fine structure involving appreciable dissipation. In particular, this process is modeled by taking the vorticity field to consist of a number of long, slender, concentrated vortex tubes with radii of curvature large in comparison to their cross-sectional dimensions. Each of these tubes is assumed to be stretched with a transient, but spatially uniform, spiral structure, as might be generated by vortex coalescence and roll-up in a process resembling that observed in the numerical investigations of Christiansen [32] and by Zabusky *et al.* [33]. The tubes do not interact and the lengthened tubes recombine into shorter tubes with fresh spiral fine structure. The axial stretching of the vortices is accompanied by radial contraction, contraction which, in tandem with differential rotation induced by the central core, accounts for the energy cascade to smaller scales. It is assumed that, at any given instant, the vortices are at different stages of evolution and, thus, that the energy spectrum is proportional to the time-averaged spectrum of a single vortex over the course of its lifespan.

Lundgren's [24] original work made convenient use of cylindrical coordinates $(r,\theta,z)$. We retain that practice. Motivated by Lundgren's [24] description of local stretching, we consider flows for which the fluid is strained by an axisymmetric flow with velocity

$$\mathbf{u}_b(r,z) = -\frac{br}{2}\mathbf{e}_r + bz\mathbf{e}_z, \tag{31}$$

where the frequency $b$ embodies the imposed constant straining.

### B. Asymptotic two-dimensional spiral solution for the unfiltered vorticity

A solution is sought in which the only nontrivial component of the filtered vorticity is axial and independent of the axial coordinate $z$, i.e.,

$$\boldsymbol{\omega}(r,\theta,z,t) = \omega(r,\theta,t)\mathbf{e}_z. \tag{32}$$

From (20) and (32), the unfiltered vorticity is necessarily of the form

$$\mathbf{q}(r,\theta,z,t) = q(r,\theta,t)\mathbf{e}_z, \tag{33}$$

where $q$ is related to $\omega$ through

$$q = (1 - \alpha^2 \Delta)\omega. \tag{34}$$

The unfiltered and filtered vorticities are assumed to be sufficiently localized so that the induced component of the filtered velocity is negligible in comparison to the imposed straining flow at infinity. Granted also that the induced filtered velocity is independent of $z$, it follows that the net filtered velocity has the form

$$\mathbf{u}(r,\theta,z,t) = U_r(r,\theta,t)\mathbf{e}_r + U_\theta(r,\theta,t)\mathbf{e}_\theta + \mathbf{u}_b(r,z). \tag{35}$$

Using (32)–(35) in (19), we find that $q$ is governed by the equation

$$\frac{\partial q}{\partial t} + \left(U_r - \frac{br}{2}\right)\frac{\partial q}{\partial r} + \frac{U_\theta}{r}\frac{\partial q}{\partial \theta} = bq + \nu(1 - \beta^2\Delta)\Delta\omega, \tag{36}$$

which differs from Lundgren's [24] governing equation for the vorticity $\omega$ only in the second term of the right-hand side. Further, imposing the requirement (16c) that $\mathbf{u}$ be divergence-free yields

$$\frac{\partial(rU_r)}{\partial r} + \frac{\partial U_\theta}{\partial \theta} = 0. \tag{37}$$

A stream function $\psi$ may therefore be introduced such that

$$U_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \tag{38a}$$

$$U_\theta = -\frac{\partial \psi}{\partial r}, \tag{38b}$$

$$\omega = -\Delta\psi, \tag{38c}$$

$$q = -(1 - \alpha^2\Delta)\Delta\psi. \tag{38d}$$

A change of variables due to Lundgren [24] makes it possible to obtain solutions of the system (36) and (38) from solutions of the strictly two-dimensional problem that arises on setting $b=0$ in (36). Specifically, using Lundgren's [24] stretched radial and temporal variables

$$\xi = \sqrt{S(t)}r, \quad T = \int_0^t S(t')dt', \tag{39}$$

with

$$S(t) = \exp(bt), \tag{40}$$

and defining

$$\Psi(\xi,\theta,T) = \psi(r,\theta,t), \tag{41a}$$

$$Q(\xi,\theta,T) = \frac{q(r,\theta,T)}{S(t)}, \tag{41b}$$

$$\chi(\xi,\theta,T) = \frac{\omega(r,\theta,t)}{S(t)}, \tag{41c}$$

reduces (36) and (38d) to two-dimensional equations of the form

$$\frac{\partial Q}{\partial T} + \frac{1}{\xi}\left(\frac{\partial \Psi}{\partial \theta}\frac{\partial Q}{\partial \xi} - \frac{\partial \Psi}{\partial \xi}\frac{\partial Q}{\partial \theta}\right) = \nu(1 - \beta^2 S\Delta_\xi)\Delta_\xi\chi, \tag{42a}$$

$$Q = -(1 - \alpha^2 S\Delta_\xi)\Delta_\xi\Psi, \tag{42b}$$

where the subscript $\xi$ stands for the differentiation with respect to $(\xi,\theta)$ and, in terms of the variable $T$, we may write $S=1+bT$. An axially strained solution can therefore be generated from an unstrained two-dimensional solution with the same initial conditions. However, the factor $S$ in (42b) makes the unstrained two-dimensional solution depend on the straining frequency $b$. In particular, the unfiltered vorticity is given by

$$q(r,\theta,t) = S(t)Q\left(\sqrt{S(t)}r, \theta, \int_0^t S(t')dt'\right). \tag{43}$$

When the viscosity vanishes, that is, when $\nu=0$, Eq. (42a) is formally identical to the vorticity equation considered by Lundgren [24]. In the present context, the difference lies with the relation (42b) between the unfiltered vorticty and the stream function. For $\alpha=0$, (42b) specializes to the relation between the vorticity and stream function arising in Lundgren's [24] analysis. In the Appendix, we show that the form of Lundgren's [24] asymptotic inviscid spiral solution is unaltered for $\alpha > 0$. From the Appendix, we have

$$Q(\xi,\theta,T) = \sum_{n=-\infty}^{\infty} Q_n(\xi,T)\exp(in\theta), \tag{44}$$

where the Fourier coefficients are given by

$$Q_n(\xi,T) = \sum_{m=0}^{\infty} T^{-m} f_n^{(m)}(\xi)\exp(-in\Omega(\xi,T)T), \tag{45}$$

and $\Omega$ is related to the stream function via (A3).

Equation (45) shows that the zeroth and the higher harmonics of the unfiltered vorticity are of the same order of magnitude and that they persist for large $T$. However, the higher harmonics of the filtered velocity, which can be calculated using the stream function, decay as $T^{-3}$. See Lundgren's [24] original work for detailed discussion of this and related issues. Granted that $T$ is large, so that the induced component of the unfiltered velocity field is axisymmetric, and that the Reynolds number $\mathrm{Re}=L^{4/3}\epsilon^{1/3}/\nu$ (with $\epsilon$ being the energy dissipation rate for the Navier-Stokes equations) is sufficiently large, Lundgren [24] obtained viscous corrections to the asymptotic solution obtained for the Navier-Stokes equations. Under the same condition that $T$ is large, a similar approach can be taken to obtain viscous corrections to the current model. Specifically, suppose that $\chi$ also has the Fourier expansion

$$\chi(\xi,\theta,T) = \sum_{n=-\infty}^{\infty} \omega_n(\xi,T)\exp(in\theta), \tag{46}$$

with $\omega_{-n}=\omega_n^*$. Equation (34) together with (41b) and (41c) yields the following relation between the Fourier coefficients $\omega_n$ and $Q_n$:

$$\alpha^2 S\left(\frac{\partial^2 \omega_n}{\partial \xi^2}+\frac{1}{\xi}\frac{\partial \omega_n}{\partial \xi}\right)-\left(1+\frac{\alpha^2 S n^2}{\xi^2}\right)\omega_n=-Q_n. \quad (47)$$

Bearing in mind the axisymmetry of the velocity field at large time, the equation corresponding to (42a) for the Fourier coefficients reads

$$\frac{\partial Q_n}{\partial T}+in\Omega Q_n=\nu\left(\frac{\partial^2 \omega_n}{\partial \xi^2}+\frac{1}{\xi}\frac{\partial \omega_n}{\partial \xi}-\frac{n^2}{\xi^2}\omega_n\right)-\nu\beta^2 S\left(\frac{\partial^4 \omega_n}{\partial \xi^4}+\frac{2}{\xi}\frac{\partial^3 \omega_n}{\partial \xi^3}-\frac{1}{\xi^2}\frac{\partial^2 \omega_n}{\partial \xi^2}+\frac{1}{\xi^3}\frac{\partial \omega_n}{\partial \xi}+\frac{2n^2}{\xi^3}\frac{\partial \omega_n}{\partial \xi}-\frac{4n^2}{\xi^4}\omega_n-\frac{2n^2}{\xi^2}\frac{\partial^2 \omega_n}{\partial \xi^2}+\frac{n^4}{\xi^4}\omega_n\right). \quad (48)$$

Recalling (47), applying the change of variables

$$\omega_n(\xi,T)\exp(in\Omega T)=h_n(\xi,T),$$

and retaining only the terms of the most significant order in $T$ results in the equation

$$\frac{\partial h_n(\xi,T)}{\partial T}=-\nu n^2\left(\frac{\beta}{\alpha}\right)^2[\Omega'(\xi)]^2T^2, \quad (49)$$

which has the solution

$$h_n(\xi,T)=-\frac{\nu n^2}{3}\left(\frac{\beta}{\alpha}\right)^2[\Omega'(\xi)]^2T^3. \quad (50)$$

Returning to (47), we obtain a viscous solution for $Q_n$ of the form

$$Q_n(\xi,T)=f_n(\xi)\exp\left[-in\Omega(\xi)T-\frac{\nu n^2}{3}\left(\frac{\beta}{\alpha}\right)^2(\Omega'(\xi))^2T^3\right], \quad (51)$$

where the functions $f_n$ are the inviscid asymptotic coefficients. We note that (51) differs from the analogous expression obtained by Lundgren [24] for the Navier-Stokes equations only by the presence of the factor $(\beta/\alpha)^2$ of the second term in the exponential.

### C. Determination of the filtered vorticity

A general axially strained spiral solution of the system (36) and (38) may be obtained from the two-dimensional solution given by (51) using (43). This solution has the form

$$q(r,\theta,t)=\sum_{n=-\infty}^{\infty}q_n(r,t)\exp(in\theta), \quad (52)$$

with

$$q_n(r,t)=S(t)f_n(\sqrt{S(t)}r)\exp\left(\frac{in}{b}\Omega[\sqrt{S(t)}r][1-S(t)]-\frac{\nu\beta^2n^2}{3\alpha^2b^3}\{\Omega'[\sqrt{S(t)}r]\}^2[1-S(t)]^3\right), \quad (53)$$

and $S$ as defined in (40).

We now obtain the filtered vorticity $\omega$ from (34), which, rewritten as

$$\Delta\omega-\frac{1}{\alpha^2}\omega=-\frac{1}{\alpha^2}q, \quad (54)$$

is a modified two-dimensional inhomogeneous Helmholtz equation. Recasting the equation in Cartesian coordinates $(x_1,x_2)$ by defining $x_1=r\cos\theta$ and $x_2=r\sin\theta$, the fundamental solution of (54) takes the form

$$\mathcal{G}(x_1,x_2)=\frac{1}{2\pi}K_0\left(\frac{r}{\alpha}\right),$$

where $K_0$ is the modified Bessel function of the second kind. It follows (cf., e.g., Polyanin [34]) that $\omega$ can be expressed as

$$\omega(x_1,x_2,t)=\frac{1}{\alpha^2}\int_{\mathbb{R}^2}q(y_1,y_2,t)K_0\left(\frac{s}{\alpha}\right)dy_1dy_2, \quad (55)$$

with $s=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$.

### D. Calculation of the energy spectrum

Define an unfiltered vorticity scalar correlation function $R_{qq}$ by

$$R_{qq}(\mathbf{r},t):=\frac{1}{L^3}\int_{\mathcal{D}}\mathbf{q}(\mathbf{x},t)\cdot\mathbf{q}(\mathbf{x}+\mathbf{r},t)d\mathbf{x}, \quad (56)$$

where $\mathbf{r}$ is the vector directed from point $\mathbf{x}$ to point $\mathbf{y}=\mathbf{x}+\mathbf{r}$ and homogeneity has been assumed. Here, $\mathcal{D}=[0,L]^3$ is a cubic cell and, following Lundgren [24], the filtered vorticity (and, thus, the unfiltered vorticity) is artificially set equal to zero outside of $\mathcal{D}$. The power spectral density $\phi_{qq}$ of the unfiltered vorticity is defined by the Fourier integral of the correlation function $R_{qq}$, i.e.,

$$\phi_{qq}(\mathbf{k},t):=\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}R_{qq}(\mathbf{r},t)e^{-i\mathbf{k}\cdot\mathbf{r}}d\mathbf{r}, \quad (57)$$

which, by a change of variable $\mathbf{r}=\mathbf{y}-\mathbf{x}$, yields

$$\phi_{qq}(\mathbf{k},t)=\frac{1}{(2\pi L)^3}\left|\int_{\mathcal{D}}\mathbf{q}(\mathbf{x},t)e^{-i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x}\right|^2. \quad (58)$$

A three-dimensional unfiltered vorticity power spectrum function may be defined by averaging $\phi_{qq}$ over the surface of a sphere of radius $k$ in wave number space. This is

$$E_{qq}(k,t)=\int_{|\boldsymbol{\kappa}|=k}|\boldsymbol{\kappa}|^2\phi_{qq}(\boldsymbol{\kappa},t)d\hat{\boldsymbol{\kappa}}, \quad (59)$$

where $d\hat{\boldsymbol{\kappa}}=\sin\varphi_\kappa d\varphi_\kappa d\theta_\kappa$ is an element of solid angle in $\boldsymbol{\kappa}$ space and $\varphi_\kappa$ and $\theta_\kappa$ are the zenithal and azimuthal angles that $\boldsymbol{\kappa}$ forms with the local axes. From Eq. (34), it is straightforward to calculate the relation between the unfiltered vorticity power spectrum $E_{qq}$ and the filtered vorticity power spectrum (the dissipation spectrum) $E_{\omega\omega}$ for homogeneous turbulence. In fact,

$$E_{qq}(k,t)=(1+\alpha^2k^2)^2E_{\omega\omega}(k,t). \quad (60)$$

Retaining all of Lundgren's [24] assumptions, and bearing in mind that the unfiltered vorticity (51) has the same structure as the vorticity derived by Lundgren [24], the power spectrum of the unfiltered vorticity must be similar to the power spectrum of the vorticity calculated by Lundgren [24]. Recall that the energy spectrum for the Navier-Stokes equations calculated by Lundgren [24] is

$$E(k) = Ak^{-5/3}\exp\left(-\frac{2\nu k^2}{3b}\right), \tag{61}$$

where the constant $A$ is given by

$$A = \frac{4\pi}{3}\frac{l_0 N_c}{L^3}b^{1/3}\sum_{n=1}^{\infty} n^{-4/3}\int_0^{\infty}\frac{|f_n(v)|^2 v\,dv}{(-\Omega'(v))^{4/3}}, \tag{62}$$

with $l_0$ being the length (assumed to have integral scale) of the vortices at the time of their creation and $N_c$ being the constant mean rate at which vortices are created. Recall, further, that the inertial range calculated by Lundgren [24] is the collection of wave numbers $k$ for which

$$\frac{\sqrt{S_c}}{r_0} \ll k \ll \sqrt{\frac{b}{\nu}}, \tag{63}$$

where $r_0$ is the radius of the central vortex core and $S_c = S(t_c)$, with $t_c$ being the cutoff time at which the vortices are assumed to recombine to form new shorter vortices. Since (51) differs from the analogous expression obtained by Lundgren [24] for the Navier-Stokes theory only by the presence of the factor $(\beta/\alpha)^2$ in the second term in the exponential, the introduction of a dissipative length scale $\beta$ distinct from the dispersive length scale $\alpha$ functions to rescale the kinematic viscosity from $\nu$ to $(\beta/\alpha)^2\nu$. Based on this observation, we conclude that the energy spectrum for the Navier-Stokes $\alpha\beta$ model obeys

$$E_{\alpha\beta}(k) = \frac{Ak^{-5/3}}{(1+\alpha^2k^2)^2}\exp\left(-\frac{2\nu\beta^2k^2}{3\alpha^2 b}\right). \tag{64}$$

Owing to the change in the exponential term of the spectrum, the inertial range is modified to the collection of wave numbers $k$ for which

$$\frac{\sqrt{S_c}}{r_0} \ll k \ll \frac{\alpha}{\beta}\sqrt{\frac{b}{\nu}}. \tag{65}$$

Assuming that the heuristic expectation (18) holds, so that $\alpha/\beta > 1$, the upper bound of the inertial range is expanded by a factor of $\alpha/\beta$ compared to (63) for the Navier-Stokes model. Recalling Townsend's [25] assumption that the axial strain rate is proportional to the root-mean-square strain rate in turbulent flow, then $b \sim \sqrt{\epsilon_{\alpha\beta}/\nu}$. For the Navier-Stokes $\alpha\beta$ equations, this reduces $\sqrt{b/\nu}$ in (65) to the inverse of the Kolmogorov length $\eta_{\alpha\beta} = (\nu^3/\epsilon_{\alpha\beta})^{1/4}$, which confirms the heuristic result (28) with $r=1$ and $l=1$.

In the inertial range represented by (65), the contribution to the energy spectrum from the exponential factor in (64) is trivial and thus the energy spectrum $E_{\alpha\beta}$ for the Navier-Stokes $\alpha\beta$ equations obeys

$$E_{\alpha\beta}(k) = \begin{cases} Ak^{-5/3}\exp\left(-\dfrac{2\nu\beta^2k^2}{3\alpha^2 b}\right), & \alpha k \ll 1, \\ \dfrac{Ak^{-17/3}}{\alpha^4}\exp\left(-\dfrac{2\nu\beta^2k^2}{3\alpha^2 b}\right), & \alpha k \gg 1. \end{cases} \tag{66}$$

This confirms the heuristic result (26) with $m=-2$ and the $k^{-5/3}$ power of the spectrum when $\alpha k \ll 1$ is not affected by the introduction of a dissipative length scale $\beta$ distinct from the dispersive length scale $\alpha$. This is consistent with the view that the spectral decay properties in the inertial range should not be affected by dissipative mechanisms. Further, it predicts a much faster decay when $\alpha k \gg 1$. Some uncertainty remains here since $k \gg 1/\alpha$ may exceed the wavenumber demarking the boundary between the inertial and dissipation ranges.

The results obtained for the Navier-Stokes $\alpha\beta$ equations can be easily specialized to provide solutions for the Navier-Stokes $\alpha$ equations by setting $\beta$ equal to $\alpha$. In particular, for $\beta=\alpha$, (65) reduces to (63). Invoking Townsend's [25] assumption that $b \sim \sqrt{\epsilon_\alpha/\nu}$ confirms (30) with $l=1$. Also, setting $\beta=\alpha$ in (66) verifies the heuristic result (29) and yields the undetermined power $m=-2$. These values for $m$ and $l$, however, differ from the values obtained by Chen *et al.* [7] or Foias *et al.* [9], where $m=-2/3$ and $l=1$. These inconsistencies raise questions regarding the accuracy of the vortex model at very small scales. Moreover, by choosing $\beta=0$, so that the hyperviscosity vanishes, our results can be specialized to provide solutions to the equations for second-grade fluids. From (51), we see that setting $\beta=0$ with $\alpha>0$ yields a sinusoidal expression for $Q_n$. Hence, to the most significant order of time in (48), a decay of vorticity occurs only when dispersive effects are accompanied by viscous effects. This demonstrates that one virtue of the Navier-Stokes $\alpha$ and Navier-Stokes $\alpha\beta$ models is that they involve a combined dispersive and viscous regularization and that the viscous regularization functions to dampen the vorticity over time. Importantly, if $\alpha$ and $\beta$ approach zero at an identical rate, then Lundgren's [24] result for the Navier-Stokes equations is recovered.

## IV. DISCUSSION AND CONCLUSIONS

Formal scaling arguments show that, for $\alpha k \ll 1$, the Navier-Stokes $\alpha$ and Navier-Stokes $\alpha\beta$ models should exhibit energy spectra that are consistent with Kolmogorov's [8] $-5/3$ law in the inertial range. In particular, it is found that, in the inertial range, the spectrum for the Navier-Stokes $\alpha\beta$ model should be independent of the dissipative length scale $\beta$. Application of Lundgren's [24] strained spiral vortex model to the Navier-Stokes $\alpha\beta$ equations confirms these heuristic results. For the Navier-Stokes $\alpha$ equations, our findings are consistent with the numerical results of Chen *et al.* [7] and the analytical results of Foias *et al.* [9]. For the Navier-Stokes $\alpha\beta$ equations, our findings suggest that, like the other $\alpha$ models explored by Cao *et al.* [20] and Ilyin and Titi [21], the Navier-Stokes $\alpha\beta$ model may be a viable candidate for a subgrid model of turbulence.

It is important to note that the spectra (66) emerging from the vortex model differ from their heuristic counterparts (29) and (26) in the sense that the constant $A$ entering (66) is not proportional to $\epsilon_{\alpha\beta}^{3/2}$ (or $\epsilon_{\alpha}^{2/3}$ for the Navier-Stokes $\alpha$ equations). The result can be simplified following Lundgren's [24] calculation of the dissipation. Specifically, using $\epsilon'_{\alpha\beta}$ to denote the contributions to the dissipation from (64), the constant $A$ in (66) can be replaced by $C_{\alpha\beta}\epsilon_{\alpha\beta}^{2/3}$. However, the coefficients $C_{\alpha\beta}$, as given by

$$C_{\alpha\beta}=\frac{(2/3)^{2/3}\epsilon'_{\alpha\beta}}{\Gamma(2/3)\epsilon_{\alpha\beta}^{2/3}\nu^{1/3}b^{2/3}}, \tag{67}$$

where $\Gamma$ now denotes the Gamma function, are model dependent in contrast to the universal constant arising in Kolmogorov's [8] analysis.

Formal scaling arguments also yield expressions for the wave numbers at which viscous dissipation becomes dominant for the Navier-Stokes $\alpha$ and Navier-Stokes $\alpha\beta$ models. In particular, it is found that the critical wave number for the Navier-Stokes $\alpha\beta$ model should depend not only on the dispersive length scale $\alpha$—as is the case for the Navier-Stokes $\alpha$ model—but also on the dissipative length scale $\beta$. This result is also confirmed by application of Lundgren's [24] strained spiral vortex model to the Navier-Stokes $\alpha\beta$ equations (and specializing accordingly to the Navier-Stokes $\alpha$ equations).

For the interval of wave numbers contained within the inertial range, we find that the vortex model is insufficiently sensitive to distinguish between the Navier-Stokes and Navier-Stokes $\alpha$ equations: in both cases, the upper bound for the inertial range is simply $\sqrt{b/\nu}$. This result is at odds with the works of Chen *et al.* [7] and Foias *et al.* [9], who find that the inertial range for the Navier-Stokes $\alpha$ model ends at a wave number smaller than that at which the inertial range of the Navier-Stokes equations ends. On the basis of these results, the upper bound $\sqrt{b/\nu}$ for the range of inertial wave numbers determined by applying the vortex model to the Navier-Stokes $\alpha$ equations must be recognized as a possible overestimate. Furthermore, whereas the works of Chen *et al.* [7] and Foias *et al.* [9] suggest that in the range where $k\alpha \gg 1$ the Navier-Stokes $\alpha$ equations exhibit a spectrum with power of $-3$, the vortex model yields a power of $-17/3$. Obukhov [35] postulated that the flux $\epsilon$ of kinetic energy is proportional to the "available cascading kinetic energy" in the vicinity of $k$ divided by a characteristic local time of the cascade, which scales as $[k^3E(k)]^{-1/2}$. The available kinetic energy in the vicinity of $k$ can be approximated by integrating $E(k)$ in a logarithmic spectral vicinity of $k$ and is of order $kE(k)$. The rate of energy transfer from larger to smaller scales is thus given by $kE(k)/[k^3E(k)]^{-1/2}=\sqrt{k^5E^3(k)}$. Consequently, for eddy scales significantly smaller than $\alpha$, (66) predicts that, for the Navier-Stokes $\alpha$ model, the energy transfer process decays with $k^{-6}$. Recall that the energy spectrum for the vortex model considered here is calculated by dividing the power spectrum of the filtered velocity by the factor $(1+\alpha^2k^2)^2$. However, in calculating the filtered velocity, scales below $\alpha$ are not fully resolved. Specifically, closure of the Navier-Stokes $\alpha$ model relies on Taylor's [36] frozen turbulence hypothesis, which indicates that eddy scales significantly smaller than $\alpha$ phase-lock into coherent structures and are swept along by the large scales. Further, if we assume that these frozen structures behave like rigid bodies transported by the filtered velocity field, there is then no stretching of the filtered velocity field below the scale $\alpha$. The stretched spiral vortex model used here is therefore meaningful only up to a critical time corresponding to the instant when the characteristic dimension of the vortex core is comparable to the filter width $\alpha$. As a consequence of the frozen turbulence hypothesis, the Navier-Stokes $\alpha$ model reduces the magnitude of the gradient of the Lagrangian mean velocity and limits how thin vortex tubes may become as they are transported for eddy scales smaller than $\alpha$ (cf., e.g., Chen *et al.* [7] and Graham *et al.* [11]). From this perspective, the stretched spiral structure used here is inconsistent with the features of the Navier-Stokes $\alpha$ turbulence for eddy scales significantly smaller than $\alpha$. A strategy that might overcome this inconsistency would be to rescale the lifetime of the vortices to the point where the smallest length scale is greater than or equal to $\alpha$. This would set a cutoff time in the upper limit of the integral used to calculate the energy spectrum. A closed form expression for the integral with an upper cutoff time would most likely be difficult to obtain. Numerical calculations such as those performed by Pullin *et al.* [29] for the Navier-Stokes model might, however, make it possible to quantify the time range over which the strained spiral vortex model remains applicable for the Navier-Stokes $\alpha$ model. Another reason for the unusually large algebraic factor in the expression for the energy decay rate is that the set of wave numbers $k$ for which $\alpha k$ exceeds unity may lie outside of the inertial range predictable by the vortex model and, thus, may include some portion of the dissipation range. For the dissipation range, algebraic decay rates of the energy spectrum with powers as large as $-7$ have been reported by previous authors (cf., e.g., Heisenberg [37] and Chandrasekhar [38]). However, it is generally believed that the decay should be dominated by an exponential term modulated by an algebraic factor involving an exponent on the order of $-2$ (cf., e.g., Saffman [26] and Martínez *et al.* [39]). This discrepancy suggests that, outside the inertial range, the vortex model may yield predictions that are not generic features of the Navier-Stokes $\alpha$ (or the Navier-Stokes $\alpha\beta$) equations. We are currently using direct numerical simulations to investigate the spectral properties of the Navier-Stokes $\alpha\beta$ equations independent of any vortex model.

For $\beta < \alpha$, the regularizing viscous term with coefficient $\rho\nu\alpha^2$ entering the Navier-Stokes $\alpha$ equations (1) is greater than that, $\rho\nu\beta^2$, entering the Navier-Stokes $\alpha\beta$ equations (16). In view of this observation, the extended upper bound $(\alpha/\beta)\sqrt{b/\nu}$ obtained for the inertial range of the Navier-Stokes $\alpha\beta$ equations seems reasonable.

When applied to the Navier-Stokes $\alpha\beta$ equations, Lundgren's [24] dynamic vortex model yields spectra that conform to the $-5/3$ law and exhibit intermittent fine structure akin to that displayed by the Navier-Stokes equations. These results should be viewed as necessary, but not sufficient, conditions for the Navier-Stokes $\alpha\beta$ model to serve as a viable subgrid model for turbulence. For $k\alpha \ll 1$, the energy spectrum (66) differs from (61) solely in the exponential term, which indicates that, in the context of Lundgren's [24] model, the Navier-Stokes $\alpha$ and Navier-Stokes $\alpha\beta$ models yield results faithful to the Navier-Stokes equation, at least in their respective truncated inertial ranges. Importantly, the exponential terms in the spectra allow us to determine the cutoff wave numbers of the relevant inertial ranges. Comparison of these wave numbers shows that the inertial range of the Navier-Stokes $\alpha\beta$ model extends beyond that of the Navier-Stokes $\alpha$ model. This suggests that simulations based on the Navier-Stokes $\alpha\beta$ equations may have the potential to resolve features smaller than those resolvable using the Navier-Stokes $\alpha$ equations. Granted that the inertial range of the Navier-Stokes $\alpha\beta$ equations includes higher wave numbers than that of the Navier-Stokes $\alpha$ equations, there remains the question of how much smaller than $\alpha$ we can take $\beta$ before simulations become too costly. A related question concerns the extent to which it might be possible to determine a choice of $\beta/\alpha$ that is optimal with regard to the trade-off between numerical accuracy and numerical efficiency. These questions pertain to the number of degrees of freedom for the respective equations and are important issues that demand further study.

## ACKNOWLEDGMENT

We thank the U.S. Department of Energy for its support of this work.

## APPENDIX: ASYMPTOTIC INVISCID SPIRAL SOLUTION

Our approach to determining the asymptotic inviscid spiral solution follows closely that taken by Lundgren [24]. The only significant difference lies in the relation between the stream function and the unfiltered vorticity. From (42), the two-dimensional inviscid unfiltered vorticity and stream equations read

$$\frac{\partial q}{\partial t}+\frac{1}{r}\frac{\partial\psi}{\partial\theta}\frac{\partial q}{\partial r}-\frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial q}{\partial\theta}=0, \qquad \text{(A1a)}$$

$$q=-\left(\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right)+\alpha^2(1+bt)\left(\frac{\partial^4\psi}{\partial r^4}+\frac{2}{r}\frac{\partial^3\psi}{\partial r^3}-\frac{1}{r^2}\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r^3}\frac{\partial\psi}{\partial r}-\frac{2}{r^3}\frac{\partial^3\psi}{\partial r\partial\theta^2}+\frac{4}{r^4}\frac{\partial^2\psi}{\partial\theta^2}+\frac{2}{r^2}\frac{\partial^4\psi}{\partial r^2\partial\theta^2}+\frac{1}{r^4}\frac{\partial^4\psi}{\partial\theta^4}\right). \qquad \text{(A1b)}$$

Expanding the unfiltered vorticity in Fourier series $q=\sum_{n=-\infty}^{\infty}q_n\exp(in\theta)$ with $q_{-n}=q_n^*$ and using the expansion in (A1a), we obtain

$$\frac{\partial q_n}{\partial t}+in\Omega q_n-\frac{i}{r}\frac{\partial}{\partial r}\left(\sum_{j\neq 0}(n-j)\psi_j q_{n-j}\right)+\frac{in}{r}\sum_{j\neq 0}\psi_j\frac{\partial q_{n-j}}{\partial r}=0, \qquad \text{(A2)}$$

where

$$\Omega=-\frac{1}{r}\frac{\partial\psi_0}{\partial r} \qquad \text{(A3)}$$

and the terms with $j=0$ have been written separately. Introducing a Fourier expansion for the stream function $\psi=\sum_{n=-\infty}^{\infty}\psi_n\exp(in\theta)$, Eq. (A1b) in the form of Fourier coefficients reads

$$q_n=-\left(\frac{\partial^2\psi_n}{\partial r^2}+\frac{1}{r}\frac{\partial\psi_n}{\partial r}-\frac{n^2}{r^2}\psi_n\right)+\alpha^2(1+bt)\left(\frac{\partial^4\psi_n}{\partial r^4}+\frac{2}{r}\frac{\partial^3\psi_n}{\partial r^3}-\frac{1}{r^2}\frac{\partial^2\psi_n}{\partial r^2}+\frac{1}{r^3}\frac{\partial\psi_n}{\partial r}+\frac{2n^2}{r^3}\frac{\partial\psi_n}{\partial r}-\frac{4n^2}{r^4}\psi_n-\frac{2n^2}{r^2}\frac{\partial^2\psi_n}{\partial r^2}+\frac{n^4}{r^4}\psi_n\right). \qquad \text{(A4)}$$

From (A3) and (A4), we have

$$rq_0=\frac{\partial}{\partial r}(r^2\Omega)-\alpha^2(1+bt)\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r^2\Omega)\right)\right]. \qquad \text{(A5)}$$

Motivated by the forms of Eqs. (A2) and (A4), the following expansions are assumed for large $t$:

$$\Omega(r,t)=\Omega^{(0)}(r)+t^{-1}\Omega^{(1)}(r)+t^{-2}\Omega^{(2)}(r)+\cdots, \qquad \text{(A6a)}$$

$$q_n(r,t)=f_n(r,t)\exp[-in\Omega^{(0)}(r)t], \qquad \text{(A6b)}$$

$$f_n(r,t)=f_n^{(0)}(r)+t^{-1}f_n^{(1)}(r)+t^{-2}f_n^{(2)}(r)+\cdots, \qquad \text{(A6c)}$$

$$\psi_n(r,t)=g_n(r,t)\exp[-in\Omega_0(r)t], \qquad \text{(A6d)}$$

$$g_n(r,t)=t^{-5}g_n^{(5)}(r)+t^{-6}g_n^{(6)}(r)+\cdots. \qquad \text{(A6e)}$$

Substituting these series into (A4) renders the $g_n$ coefficients in terms of the $f_n$ coefficients, giving, for example,

$$g_n^{(5)}=\frac{f_n^{(0)}}{\alpha^2bn^4(d\Omega^{(0)}/dr)^4} \qquad \text{(A7)}$$

and

$$g_n^{(6)}=\frac{f_n^{(1)}-4i\alpha^2bn^3(d\Omega^{(0)}/dr)^3dg_n^{(5)}/dr}{\alpha^2bn^4(d\Omega^{(0)}/dr)^4}-\frac{\{n(d\Omega^{(0)}/dr)^2+6ibd^2\Omega^{(0)}/dr^2+(2ib/r)d\Omega^{(0)}/dr\}g_n^{(5)}}{bn(d\Omega^{(0)}/dr)^2}. \qquad \text{(A8)}$$

When $n=0$, (A3) together with (A6d) and (A6e) implies that $\Omega^{(1)}=\Omega^{(2)}=\Omega^{(3)}=\Omega^{(4)}=0$. Hence, $\Omega$ is constant to within terms of order $t^{-5}$.

When $n\neq 0$, (A2) may be written as

$$\frac{\partial}{\partial t}[q_n\exp(in\Omega^{(0)}t)]+in(\Omega-\Omega^{(0)})q_n\exp(in\Omega^{(0)}t)-\frac{i}{r}\exp(in\Omega^{(0)}t)\frac{\partial}{\partial r}\left(\sum_{j\neq 0}(n-j)\psi_j q_{n-j}\right)+\frac{in}{r}\exp(in\Omega^{(0)}t)\sum_{j\neq 0}\psi_j\frac{\partial q_{n-j}}{\partial r}=0. \qquad \text{(A9)}$$

Using (A6b) and (A6d) in (A9), we find that

$$\frac{\partial f_n}{\partial t}+in(\Omega-\Omega^{(0)})f_n-\frac{i}{r}\frac{\partial}{\partial r}\left(\sum_{j\neq 0}(n-j)g_jf_{n-j}\right)+\frac{in}{r}\sum_{j\neq 0}g_j\left(\frac{\partial f_{n-j}}{\partial r}-i(n-j)tf_{n-j}\frac{d\Omega^{(0)}}{dr}\right)=0. \tag{A10}$$

Finally, using (A6c) and (A6e) in (A10), we are able to determine the higher-order coefficients of $f_n$ through the lowest-order coefficient $f_n^{(0)}$. For instance, we find that $f_n^{(1)}=f_n^{(2)}=0$ and

$$f_n^{(3)}=\frac{n}{3r}\sum_{j\neq 0}(n-j)\frac{f_j^{(0)}f_{n-j}^{(0)}}{\alpha^2 bj^4(d\Omega^{(0)}/dr)^3}.$$

That is, all the coefficients in the asymptotic solution are determined in terms of the coefficients $f_n^{(0)}$, which are arbitrary functions of $r$ and must be found from the initial conditions of the problem. Importantly, however, this asymptotic solution differs from Lundgren's [24] in the sense that it relies on the straining frequency $b$ and, thus, is model dependent.


[1] D. D. Holm, J. E. Marsden, and T. S. Ratiu, Phys. Rev. Lett. **80**, 4173 (1998).
[2] D. D. Holm, J. E. Marsden, and T. S. Ratiu, Adv. Math. **137**, 1 (1998).
[3] R. Camassa and D. D. Holm, Phys. Rev. Lett. **71**, 1661 (1993).
[4] S. Chen, C. Foias, D. D. Holm, E. Olson, E. S. Titi, and S. Wynne, Phys. Rev. Lett. **81**, 5338 (1998).
[5] S. Chen, C. Foias, E. Olson, E. S. Titi, and S. Wynne, Physica D **133**, 49 (1999).
[6] S. Chen, C. Foias, E. Olson, E. S. Titi, and S. Wynne, Phys. Fluids **11**, 2343 (1999).
[7] S. Chen, D. D. Holm, L. G. Margolin, and R. Zhang, Physica D **133**, 66 (1999).
[8] A. N. Kolmogorov, Dokl. Akad. Nauk SSSR **30**, 301 (1941).
[9] C. Foias, D. D. Holm, and E. S. Titi, Physica D **152-153**, 505 (2001).
[10] C. Foias, in *Harmonic Analysis and Nonlinear Partial Differential Equations*, A Volume in Honor of Victor L. Shapiro, Papers from the conference held in Riverside, CA, November 1995, edited by Michel L. Lapidus, Lawrence H. Harper and Adolfo. J. Rumbos, special issue of Contemp. Math. **208**, 151 (1997).
[11] J. Pietarila Graham, D. D. Holm, P. D. Mininni, and A. Pouquet, Phys. Rev. E **76**, 056310 (2007).
[12] J. P. Graham, D. D. Holm, P. Mininni, and A. Pouquet, Phys. Fluids **20**, 035107 (2008).
[13] C. Foias, D. D. Holm, and E. S. Titi, J. Dyn. Differ. Equ. **14**, 1 (2002).
[14] D. D. Holm, C. Jeffrey, S. Kurien, D. Livescu, M. A. Taylor, and B. A. Wingate, Los Alamos Sci. **29**, 152 (2005).
[15] R. S. Rivlin and J. L. Ericksen, Arch. Ration. Mech. Anal. **4**, 323 (1955).
[16] E. Fried and M. E. Gurtin, Theor. Comput. Fluid Dyn. **22**, 433 (2008).
[17] E. Fried and M. E. Gurtin, Phys. Rev. E **75**, 056306 (2007).
[18] L. F. Richardson, *Weather Prediction by Numerical Process* (Cambridge University Press, Cambridge, U.K., 1922).
[19] K. Ohkitani, Publ. RIMS, Kyoto Univ. **40**, 1267 (2004).
[20] C. Cao, D. D. Holm, and E. S. Titi, J. Turbul. **6**, 20 (2005).
[21] A. A. Ilyin, E. M. Lunasin, and E. S. Titi, Nonlinearity **19**, 879 (2006).
[22] R. H. Kraichnan, Phys. Fluids **10**, 1417 (1967).
[23] R. H. Kraichnan, in *Statistical Mechanics: New Concepts, New Problems, New Applications*, edited by S. A. Rice, K. F. Freed, and J. C. Light, (Univ. of Chicago Press, Chicago, 1972), p. 201.
[24] T. S. Lundgren, Phys. Fluids **25**, 2193 (1982).
[25] A. A. Townsend, Proc. R. Soc. London, Ser. A **208**, 534 (1951).
[26] P. G. Saffman, in *Topics in Nonlinear Physics*, edited by N. J. Zabusky (Springer, New York, 1968), pp. 485–614.
[27] T. S. Lundgren, Phys. Fluids A **5**, 1472 (1993).
[28] D. I. Pullin and P. G. Saffman, Phys. Fluids A **5**, 126 (1993).
[29] D. I. Pullin, J. D. Buntine, and P. G. Saffman, Phys. Fluids **6**, 3010 (1994).
[30] Y. Cuypers, A. Maurel, and P. Petitjeans, Phys. Rev. Lett. **91**, 194502 (2003).
[31] Y. Cuypers, A. Maurel, and P. Petitjeans, J. Turbul. **5**, 1 (2004).
[32] J. P. Christiansen, J. Comput. Phys. **13**, 363 (1973).
[33] N. J. Zabusky, M. H. Hughes, and K. V. Roberts, J. Comput. Phys. **30**, 96 (1979).
[34] A. D. Polyanin, *Handbook of Linear Partial Differential Equations for Engineers and Scientists* (Chapman & Hall/CRC Press, Boca, Raton, FL, 2002).
[35] A. M. Oboukhov, Geo. Geophys. Ser. **5**, 453 (1941).
[36] G. I. Taylor, Proc. R. Soc. London, Ser. A **164**, 476 (1938).
[37] W. Heisenberg, Proc. R. Soc. London, Ser. A **195**, 402 (1948).
[38] S. Chandrasekhar, Proc. R. Soc. London, Ser. A **200**, 20 (1949).
[39] D. O. Martínez, S. Chen, G. D. Doolen, R. H. Kraichnan, L. P. Wang, and Y. Zhou, J. Plasma Phys. **57**, 195 (1997).