# Transition to chaotic scattering: Signatures in the differential cross section

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We show that bifurcations in chaotic scattering manifest themselves through the appearance of an infinitely fine-scale structure of singularities in the cross section. These "rainbow singularities" are created in a cascade, which is closely related to the bifurcation cascade undergone by the set of trapped orbits (the chaotic saddle). This cascade provides a signature in the differential cross section of the complex pattern of bifurcations of orbits underlying the transition to chaotic scattering. We show that there is a power law with a universal coefficient governing the sequence of births of rainbow singularities and we verify this prediction by numerical simulations.

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## I. INTRODUCTION

Chaotic scattering, present in a large variety of disciplines, is an important manifestation of chaos [1]. Examples are found in classical mechanics [1], fluid dynamics [2], electronic transport in semiconductors [3], and optics [4], to name just a few. Scattering can be defined as any dynamical system with an unbounded phase space, in which the dynamics is nontrivial in a bounded region, called the "scattering region." The hallmark of chaotic scattering is the presence of a Cantor set of singularities in any (nontrivial) scattering function relating initial conditions to asymptotic variables, such as the deflection angle of a particle scattered by a classical potential [5]. These singularities correspond to initial conditions whose orbits get trapped in the scattering region for both $t \to -\infty$ and $t \to \infty$. These orbits form a very intricate fractal set in phase space, called the "chaotic saddle." Even though the chaotic saddle has zero measure, its presence has tremendous implications for the dynamics, because initial conditions close to this set will stay in the scattering region for a long time before they escape. Furthermore, trajectories that are initially very close to this set and to each other may separate rapidly, and reach wildly different asymptotic states. This results in a very sharp sensitivity to initial conditions, one of the most important features of chaos.

In experimental situations, it is often impossible to have access to direct information concerning individual trajectories. In a typical situation of experimental relevance, a beam of particles is incident on the scattering region, and the intensity (the flux) of particles that comes out as a function of the angle is measured. This intensity is measured by the differential cross-section function. In this paper, we address the following fundamental questions, of direct relevance to measurements: How are bifurcations of orbits in the chaotic saddle reflected on the differential cross section? In particular, how is the transition from regular to chaotic scattering manifested in the cross section? We herewith establish that the birth of new orbits due to bifurcations gives rise to cascades of singularities in the differential cross section. These singularities correspond to directions in which the flux of particles diverges, and they are called rainbow singularities [6,7]. Moreover, we establish that the manifestation of the transition from regular to chaotic scattering on the cross section is through an infinitely fine sequence of births of singularities in the cross section. As the scattering progresses from the regular to the chaotic regime, rainbow singularities are successively created in a series of cascades, which are closely related to the corresponding bifurcation cascades undergone by the chaotic saddle during the transition. Furthermore, we derive an analytical result showing that the intervals of the bifurcation parameter separating successive births of rainbow singularities preceding the appearance of a new periodic orbit in the chaotic saddle decrease following a power law, with a universal coefficient of $-3/2$.

The paper is organized as follows. In Sec. II, we define a simple two-dimensional model used to illustrate our reasoning and results, and we discuss the bifurcation scenario leading to chaotic scattering. In Sec. III, we examine how the bifurcation cascade undergone by the chaotic saddle is manifested in the differential cross section through rainbow singularities. In Sec. IV, we uncover a universal power law describing the successive appearance of rainbow singularities as the system's energy is varied. Finally, we summarize our conclusions in Sec. V.

## II. MODEL

We choose to establish our results using a two-dimensional classical-mechanical point particle system. We point out, however, that our results apply to a much broader class of systems (including, for example, optical systems), as they arise from very general features of the scattering dynamics of chaotic systems.

As is conventional in classical scattering systems, we parametrize the initial conditions by the impact parameter $b$, and characterize each scattering trajectory by its scattering angle $\Phi = \Phi(b)$ [see Fig. 1(a)]. The differential cross section $d\sigma/d\Omega$, corresponding to a given direction $\theta$, is given by

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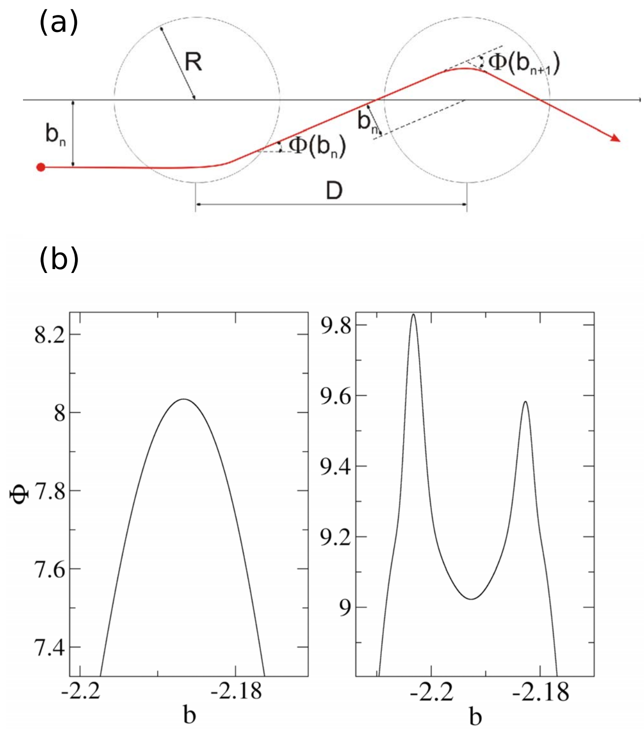


FIG. 1. (Color online) (a) Two-dimensional model. (b) Deflection function for two different $\phi_{\max}$: Creation of rainbow singularities.

$$\frac{d\sigma}{d\Omega}(\theta) = \sum_i \frac{b_i}{\sin\theta}\left|\frac{d\Phi(b_i)}{db}\right|^{-1}, \tag{1}$$

where $0 < \theta < 2\pi$. The sum is over all impact parameter $b_i$ that satisfy the relation $\Phi(b_i)+2n\pi=\theta$, with $n$ being an integer number, and it picks the contributions from all trajectories which scatter in the direction $\theta$. $n$ can be interpreted as the number of "loops" the particle makes before escaping. The differential cross section diverges when $d\Phi/db=0$, that is, whenever the deflection function $\Phi(b)$ goes through a local maximum or minimum (excluding nongeneric cases). These are the locations of the rainbow singularities. A rainbow singularity is the result of an infinite density of scattering trajectories in a given direction. In optical systems rainbow singularities appear as bright, burning spots (caustics). They are also found in atomic scattering [8], nucleus-nucleus collisions [9], and many other physical systems [10]. The other type of singularity that appears in the differential cross section is the glory singularity, when $\theta=n\pi$ with $n=0,1,2$. We shall focus only on the rainbow singularities, since glory singularities are due to kinematic effects, not related to the scattering dynamics.

The relationship between the rainbow singularities and chaotic scattering is far from trivial. It was previously thought that rainbow singularities in the cross section of a chaotic system mirror directly the fractal set of singularities in the deflection function [11]. But it is now known that this is not true in general [6]. In fact, there are systems that have chaotic scattering but still present a smooth differential cross section, without any singularities [6]. This phenomenon is called the "rainbow transition" and is related to the existence of a periodic orbit where a particle circles around a potential well.

The natural approach to study the connection between rainbow singularities and chaotic scattering, which we follow in this paper, is to investigate the sequence of bifurcations that leads to chaotic scattering by varying some relevant parameter, such as the particle's energy.

The series of bifurcations leading to chaotic scattering have been well studied, especially in classical mechanical systems [12]. During the cascade the system goes repeatedly through non-hyperbolic chaotic regimes, during which stable periodic orbits and Kolmogorov-Arnold-Moser (KAM) surfaces are present in the phase space. We now investigate how this complex sequence of bifurcations in the orbits is reflected in the differential cross-section. To focus our ideas, we study a particular case of this phenomenon. We consider a particle being acted on by a potential consisting of the superposition of two attractive circularly symmetric potential wells $V(r)$, as illustrated in Fig. 1(a). The potential of each well is assumed to be negligible beyond a given distance $R$ from its center. The centers of the two potentials are separated by a distance $D$ satisfying the non-overlapping condition $D > 2R$. This ensures that the part of the particle's trajectory outside of the two wells is a straight line. In addition, this assumption implies that, while the particle is close to one of the centers, the influence of the other potential well can be neglected. These conditions allow us to encode the scattering dynamics into a two-dimensional map, as was originally introduced in [13] and used in [14]. Each iteration of the map represents the scattering of the particle by one of the potential wells. The two variables of the map are the scattering angle $\theta_n$ and the impact parameter $b_n$ corresponding to one of the centres. Assuming for simplicity that the two potentials are identical, let $\theta_{n+1}$ be the escaping angle after the particle is scattered by one of the potentials, and let $\phi=\phi(b)$ be the angle by which the trajectory is deflected by a single scatterer. The particle, after being scattered by one of the wells, reaches the other with an impact parameter $b_{n+1}$. Assuming a straight-line trajectory between individual scattering events, we can by simple geometry find the expression for the new angle and impact parameter, as a function of their previous values: (see Fig. 1(a))

$$b_{n+1} = b_n - D\sin(\Phi_{n+1})\{\mathrm{sgn}[\cos(\Phi_{n+1})]\}, \tag{2}$$

$$\Phi_{n+1} = \Phi_n - \phi(b_n). \tag{3}$$

The particular form of the scattering angle function $\phi(b)$ depends on the detailed shape of the potential $V(r)$ and on the particle's energy. However, for all attractive circularly symmetric potential wells of the kind we assume, $\phi(b)$ has the same qualitative features for a given energy: it is an odd function, and in particular it is zero for $b=0$ (due to the circular symmetry); it approaches zero for $b\to\pm\infty$; on each side of the origin, $\phi(b)$ has a single peak, whose size $\phi_{\max}$ is the maximum deflection angle of the potential for that energy. Since we are interested in the general features of the phenomenon, we do not choose any particular potential $V(r)$ from which $\phi(b)$ is computed; instead, we prescribe directly a function $\phi(b)$ which has all the required properties described above. The results we describe below do not depend on the details of $\phi(b)$. A convenient (dimensionless) choice for the scattering function of an individual potential, which has all the above required features, is

$$\phi(b) = \phi_{\max} \exp[-2(b+1)^2] - \phi_{\max} \exp[-2(b-1)^2]. \tag{4}$$

The parameter $\phi_{\max}$ depends on the energy of the particle: the lower the energy, the greater the maximum deflection is. This is then the natural bifurcation parameter of our model. Equation (4), along with Eqs. (2) and (3), define the scattering map. A typical trajectory will eventually escape towards infinity, when the impact parameter for the next iteration $b_{n+1}$ becomes too large. We define a cutoff $R$, such that if $b_{n+1} > R$, the particle is considered to have escaped. For $b \to \pm 3$, $\phi \sim 0$ [Eq. (4)], therefore we can assume that the radius of each potential well is $R=3$. The distance $D=7$ is fixed between the two potential centers. All incident particles have initial velocities parallel to the $x$ axis and come from outside of the scattering region, see Fig. 1(a). We stress that our results are largely independent of the precise values of these parameters.

The transition from regular to chaotic scattering in this system can be understood as follows. When $\phi_{\max}$ is small (high energy), particles go through the potential with very little deflection, and escape the interaction region easily. As their energy is lowered, e.g., $\phi_{\max}$ increases, particles are more and more deflected, and the system goes through different scattering regimes. The scattering is regular if $\phi_{\max}$ is less than $\phi_c \approx \pi$ (we found $\phi_c = 3.14264478$ in our case), when the particles cannot "make the turn" around the centers, and thus all escape in finite time. Thus, for $\phi_{\max} < \phi_c$ there are no periodic orbits, and therefore no chaos. When $\phi_{\max}$ reaches $\phi_c$, the orbit with the impact parameter corresponding to $\phi_{\max}$ makes a half-turn around the center, reaching the other potential, making then another half-turn, and so on, thereby giving rise to a periodic orbit. Increasing $\phi_{\max}$ just beyond $\phi_c$, the orbit with impact parameter $b_{\max}$ corresponding to $\phi_{\max}$ is deflected by an angle greater than $\phi_c$; by the continuity of $\phi(b)$, there are two values $b_-$ and $b_+$ of $b$ such that $\phi(b_-) = \phi(b_+) = \phi_c$, with $b_- < b_{\max} < b_+$. As argued above, this means that $b_-$ and $b_+$ correspond to two periodic orbits. These new orbits bifurcated from the original one in a saddle-center bifurcation, and a hierarchy of KAM islands immediately appears in phase space. As $\phi_{\max}$ is further increased, the system goes through a complex cascade of bifurcations, which end up breaking up the KAM tori, leading to hyperbolic chaotic scattering. At this stage, after the destruction of the last torus, only the unstable periodic orbits remain [12]. An even further increase of $\phi_{\max}$ preserves the topology of the chaotic saddle (no bifurcations) within a range of $\phi_{\max}$, until the next value of $\phi_{\max}$ is reached for which the particles are able to make more turns around the centers (for example, $\phi_{\max} \approx 3\phi_c$). Then new orbits are born, new KAM islands appear, and so on. Thus, as $\phi_{\max}$ increases, the system goes through a sequence of bifurcation cascades. We next show how this route to chaotic scattering is reflected in the differential cross section.

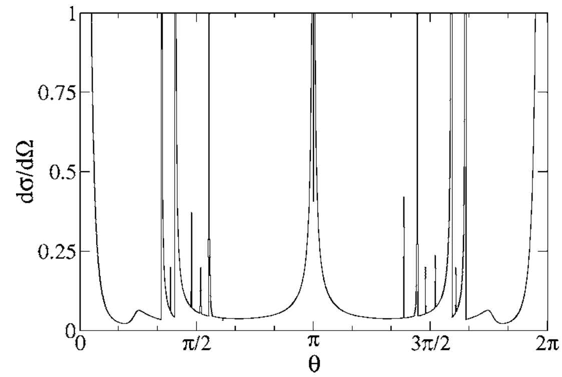


FIG. 2. The rainbows form an infinite set of singularities in the differential cross section and are distributed through all directions, even though there are some better resolved than others. The divergences around $\theta=0$, $\pi$, and $2\pi$ correspond to glory singularities.

## III. SIGNATURES IN THE DIFFERENTIAL CROSS SECTION

As discussed before, rainbow singularities are divergences in the differential cross section. Figure 2 shows the differential cross section for $\phi_{\max}=5$, when the scattering is chaotic and there are no stable periodic orbits. The rainbows are seen as sharp peaks, and although there are infinite singularities, there are some that are better resolved than others.

In order to be able to locate even the very narrow rainbow peaks, we identify the singularities by finding zeros of $d\Phi/db$. We now examine what signatures this bifurcation process imprints on the differential cross section. We start by looking closely into the process of creation of the first periodic orbit, at $\phi_{\max} = \phi_c$. While $\phi_{\max} < \phi_c$, orbits are scattered from one potential well to the other a finite number of times, before escaping towards infinity. As $\phi_{\max}$ approaches $\phi_c$ from below, orbits (with the appropriate impact parameter) are able to hop an increasing number of times between the two centers before escaping. As argued above, the complete scattering process can be regarded as the succession of individual scattering events by each potential well. Consequently, the maxima and minima of the scattering function $\Phi(b)$ of the total scattering process arise from the maxima and minima caused by each individual scattering around one

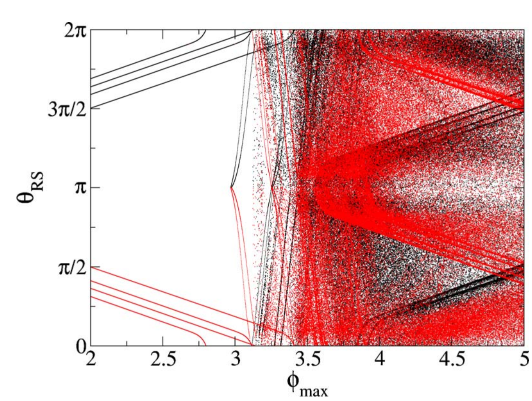


FIG. 3. (Color online) Direction $\theta_{RS}$ of the rainbow singularities as a function of $\phi_{\max}$.

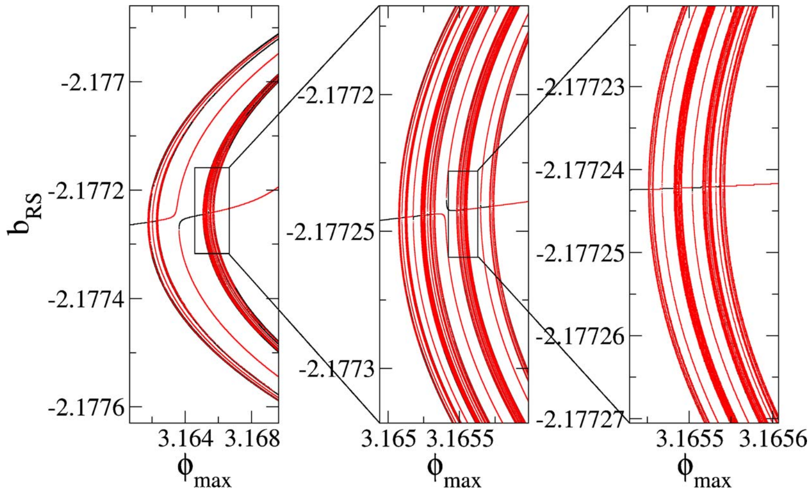


FIG. 4. (Color online) Impact parameter $b_{RS}$ of the rainbow singularities as a function of $\phi_{max}$. The successive magnifications show the fractal structure of the cascade of births of pairs of singularities.

of the wells. This means that each time $\phi_{max}$ reaches a value such that the orbits are able to hop one more time before escaping, a new maximum (or minimum) appears in $\Phi(b)$. This, by continuity, is always accompanied by the appearance of a minimum (or maximum). This process is shown in Fig. 1(b). The birth of a maximum-minimum pair implies that two new rainbow singularities appear in the differential cross section. As $\phi_{max} \to \phi_c$, this happens more and more often in a cascade that accumulates at $\phi_c$, when the periodic orbit is born. Thus, the creation of this first periodic orbit is preceded by an infinite cascade of newly created pairs of rainbow singularities.

In order to verify this reasoning, we vary the parameter $\phi_{max}$ and record the positions (values of the impact parameter) of the maxima and minima that appear in the deflection function, which is numerically calculated with the required precision for each value of $\phi_{max}$. Figures 3–5 show the results. In Fig. 3, where we show the angle of the rainbow singularities as a function of $\phi_{max}$, the transition from regular to chaotic scattering is marked by the drastic multiplication of rainbow singularities. The first periodic orbit is born at $\phi_{max} = \phi_c \approx \pi$, as just explained; the corresponding singularity cascade can be seen in the lower branch of the pattern in Fig. 4. As predicted, a cascade of creations of pairs of rainbow singularities precedes it.

As $\phi_{max}$ is increased beyond $\phi_c$, additional rainbow singularities are born in similar cascades, always in pairs corresponding to one maximum and one minimum in the scattering angle function, as shown in Fig. 1(b) for a particular case. Those other cascades correspond to further bifurcations associated with the chaotic saddle, leading to the creation of more and more periodic orbits. In this way, new orbits give rise to new rainbow singularities, which are visible in the differential cross section as high-intensity spots. Figure 4 shows the highly intricate sequence of rainbow singularities appearing in the cross section, mirroring the fractal structure of the corresponding bifurcations in the underlying chaotic set of orbits. The successive magnifications display the fractal structure of the sequence of singularity pair creations. Figure 5 shows another view of this cascade, which further demonstrates the complexity of the phenomenon. Hence, we have uncovered in the experimentally accessible scattering cross section a clear signature of bifurcations in the chaotic set of trapped orbits.

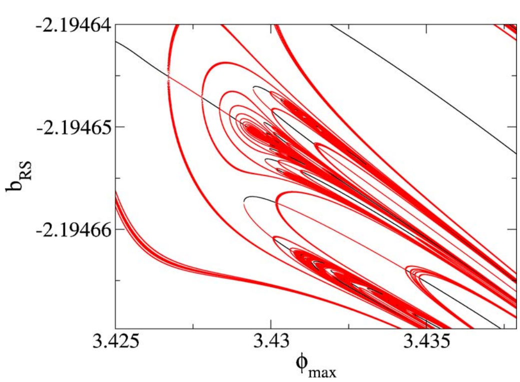


FIG. 5. (Color online) Another view of the cascade of singularities appearing in the cross section as $\phi_{max}$ increases.

We note that the creation of each pair of singularities is equivalent to a saddle-node bifurcation in a one-dimensional map. Indeed, a singularity appears when $d\Phi/db = 0$, it is a maximum if $d^2\Phi/db^2 < 0$ and a minimum if $d^2\Phi/db^2 > 0$. Defining a one-dimensional mapping $M(b)$ by $M = d\Phi/db$, the rainbow singularities correspond to fixed points of $M$, and stable fixed points are associated with maxima of $\Phi(b)$, whereas unstable fixed points are associated with minima. Dynamically, the birth of a pair of singularities is a saddle-node bifurcation in $M$, where one stable (maximum) and one unstable (minimum) fixed points are born together.

Each new pair of rainbow singularities that is born has a contribution to the total cross section $\sigma$, which is given by the integral of the differential cross section over all angles. This contribution obviously depends on the strength of the new-born singularities. Given the fractal structure of the bifurcations, it seems plausible that $\sigma$ changes with the bifurcation parameter $\phi_{max}$ similar to a Devil's staircase, because of the jumps in $\sigma$ caused by the appearance of new singularities as $\phi_{max}$ changes. This should in principle be detectable in experiments, and it would be another signature of the bifurcations of the chaotic saddle in scattering measurements. In the following, we demonstrate that the bifurcations of rainbow singularities preceding the appearance of a new unstable periodic orbit in the chaotic saddle are separated by intervals of $\phi_{max}$ which decrease as a power law.

## IV. POWER LAW IN BIFURCATION CASCADES OF RAINBOW SINGULARITIES

When the $\phi_{\max}$ of a single potential reaches certain critical values, cascades in the creation of rainbow singularities occur which precede the creation of new periodic orbits in the system. As discussed earlier, the first periodic orbit is created when $\phi_c \approx \pi$, which goes around the two centers. Another critical value $\phi_{\max}$ is related to the figure-eight orbit, and so on. There are, in fact, infinite critical values of $\phi_{\max}$.

We investigate now the asymptotic behavior of the dynamics for $\phi_{\max}$ near one of these critical values. As an example, we shall focus on the first periodic orbit, at $\phi_{\max} = \phi_c$. We assume that we have $\phi_{\max}$ slightly less than $\phi_c$, $\phi_c - \phi_{\max} \ll 1$. Let us denote the small quantity $\phi_c - \phi_{\max}$ by $\epsilon$. We then have $\epsilon \ll 1$.

If we had $\phi_{\max} = \phi_c$, we would have a periodic orbit at the corresponding impact parameter $b_{\max}$; so in this case, if we iterate the scattering map $N$ times we would come back to the same situation, where $N$ is the orbit's period. For $\phi_{\max}$ close to $\phi_c$, if we iterate the map $N$ times we will not get exactly to the same point, but we will get close to it. For simplicity we examine first the simplest case of the first periodic orbit to be formed in the system, with $\phi_{\max}$ close to $\phi_c \approx \pi$. We take a particle moving horizontally, $\Phi_1 = 0$, with an impact parameter corresponding to the maximum deflection: $b_1$ is such that $\phi(b_1) = \phi_{\max} = \phi_c - \epsilon$. Equations (2) and (3) give us then $\Phi_2 = -\pi + \epsilon$ and $b_2 = b_1 + D\sin(-\pi + \epsilon)$. For small $\epsilon$, we can approximate this by

$$b_2 \approx b_1 + D\epsilon.$$

In the next iteration, $\Phi_3$ is determined by $\phi(b_2) = \phi(b_1 + D\epsilon)$, from Eq. (3). But since $b_1$ is a maximum of the function $\phi(b)$, to first order in $\epsilon$ we have $\phi(b_2) \approx \phi(b_1)$. Using this fact, we find by following the same procedure that $\Phi_3 \approx -\pi + 2\epsilon$ and

$$b_3 \approx b_2 + 2D\epsilon.$$

In general, we find that

$$b_{n+1} \approx b_n + nD\epsilon.$$

We can expand this expression as follows:

$$\begin{aligned} b_{n+1} &= b_{n-1} + (n-1)D\epsilon + nD\epsilon \\ &= b_{n-2} + (n-2)D\epsilon + (n-1)D\epsilon + nD\epsilon \cdots \\ &= b_1 + (1 + 2 + \cdots + n)D\epsilon \\ &= b_1 + \frac{1}{4}n(n-1)D\epsilon. \end{aligned}$$

For sufficiently small $\epsilon$, the particle will go through a large number of iterations before escaping; in other words, we can assume $n \gg 1$. In this case, we have with good approximation

$$b_{n+1} \approx b_1 + \frac{1}{4}n^2 D\epsilon.$$

Note that $b$ does not increase exponentially with $n$, because we are studying trajectories in the vicinity (in parameter space) of nonhyperbolic orbits.

If $\phi_{\max}$ is in the vicinity of a different critical angle $\phi_c$, the details of the above calculation will change, because of the different geometry of the trajectories involved; but in essence the result should be the same for large $n$, since it only depends on $\phi_{\max}$ being close to $\phi_c$ (i.e., having a small $\epsilon$). The details of the different orbit geometries enter in the factor multiplying $\epsilon$ in the equation above, which will be different for different orbits. We can thus write

$$b_{n+1} \approx b_1 + CDn^2\epsilon,$$

where the factor $C$ is determined by the orbit's geometry.

The particle escapes when $b_n$ becomes sufficiently large, so that the potential can no longer deflect it towards the other centre. Let us denote this escaping impact parameter by $B$. When $b$ starts to approach $B$, we are no longer in the vicinity of the peak of $\phi(b)$, and the approximations we used to obtain the above expressions are not valid. But for $\phi_{\max}$ very close to $\phi_c$, most of the trajectory's time will be spent in the vicinity of $b_{\max}$ before it escapes, so that the escape time is dominated by this regime. This allows us to estimate the time (actually the number of iterations) $n_e$ for which the trajectory escapes, by

$$b_{n_e+1} \approx b_1 + CDn^2\epsilon = B.$$

From this we have

$$n_e^2 \approx \frac{K^2}{D\epsilon} \Rightarrow n_e \approx \frac{K}{\sqrt{D\epsilon}},$$

where $K^2 \equiv (B - b_1)/C$ is a constant. This means that the escape time scales as $(\phi_c - \phi_{\max})^{-1/2}$ as $\phi_{\max}$ approaches the critical value $\phi_c$, where it diverges.

A unit increase in $n_e$ means that trajectories are able to bounce once more between the potentials before escaping. We thus expect to have new maxima and minima in the scattering function, resulting in the appearance of new rainbow singularities. So the successive increments of $n_e$ corresponds to the birth of new singularities in the cross section, as argued in the previous section. We want to determine the interval $\Delta\epsilon$ in the bifurcation parameter $\phi_{\max}$ separating two such successive bifurcations in the cross section. Let us denote by $\epsilon$ the value of $\phi_c - \phi_{\max}$ such that $n_e = N$; then the next rainbow singularities will appear when $n_e$ increases to $N+1$, at a value of $\phi_c - \phi_{\max}$ equal to $\epsilon - \Delta\epsilon$. From the result derived above, we have

$$n_e = \frac{K}{\sqrt{D\epsilon}}, \quad n_e + 1 = \frac{K}{\sqrt{D(\epsilon - \Delta\epsilon)}}.$$

Expanding the latter expression in a Taylor series, we get

$$n_e + 1 = \frac{K}{\sqrt{D}}\left(\epsilon^{-1/2} + \frac{1}{2}\epsilon^{-3/2}\Delta\epsilon\right) = n_e + \frac{1}{2}\frac{K}{\sqrt{D}}\epsilon^{-3/2}\Delta\epsilon.$$

From this we find

$$\Delta\epsilon = \frac{2\sqrt{D}}{K}\epsilon^{3/2}. \quad (5)$$

This expression predicts that the intervals in the bifurcation parameter $\phi_{\max}$ separating two successive appearances of rainbow singularities decreases as a power law, and tends to zero as $\phi_{\max}$ approaches the critical value $\phi_c$. Of course there are many other bifurcations happening in the system, which are not accounted for in this analysis. For example, as soon as the first orbit is created at $\phi_{\max} \approx \pi$, it undergoes an extremely complicated sequence of bifurcations, typical of Hamiltonian systems, in which KAM islands arise and orbits of arbitrarily large periods appear. Many of these other bifurcations also give rise to rainbow singularities, and this may make it difficult to distinguish clearly the $\Delta\epsilon$ intervals predicted in this section. But Fig. 5 shows a clearly defined sequence of creations of rainbow singularities which is a good candidate to test our analysis. We determined numerically the position of the bifurcations which could be resolved, and estimated their accumulation point (that is, $\phi_c$), so as to calculate several consecutive values of $\Delta\epsilon$. The above equation predicts that if we plot $\Delta\epsilon$ and a function of $\epsilon$ on a log-log plot, we should get a straight line, with slope 1.5. Figure 6 shows we have a decent straight line, and the slope was found by fitting to be ~1.4, which is in reasonable agreement with the predicted value. We stress that the exponent 3/2 in the power law is entirely independent of the details of the system, and is thus a universal feature of rainbow singularity cascades, in any scattering system.

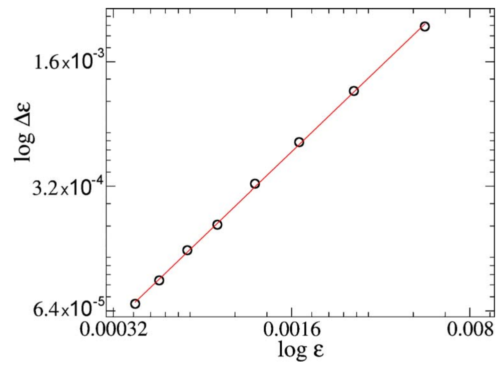


FIG. 6. (Color online) Log plot of the intervals of $\phi_{\max}$ ($\Delta\epsilon$) separating two successive rainbow singularities. The slope of the fitting is ~1.4.

## V. CONCLUSIONS

Bifurcations play a prominent role in the study of chaotic scattering, and a full understanding of how they are related to the differential cross section is still missing. In this work we showed that the differential cross section reflects the cascade of bifurcations in the chaotic saddle through a corresponding cascade of creations of rainbow singularities, and we established a universal power-law relation satisfied by the cascade of rainbow singularities which accompanies the creation of new periodic orbits in the system. This analytical result is akin to Feigenbaum's law governing the vicinity of accumulation points in period-doubling cascades in dissipative systems.

## ACKNOWLEDGMENTS

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[1] S. Bleher, C. Grebogi, and E. Ott, Physica D **46**, 87 (1990); S. Bleher, E. Ott, and C. Grebogi, Phys. Rev. Lett. **63**, 919 (1989).
[2] E. Ziemniak, C. Jung, and T. Tél, Physica D **76**, 123 (1994); A. Péntek, Z. Toroczkai, T. Tél, C. Grebogi, and J. A. Yorke, Phys. Rev. E **51**, 4076 (1995); Z. Toroczkai, G. Károlyi, A. Péntek, T. Tél, and C. Grebogi, Phys. Rev. Lett. **80**, 500 (1998).
[3] A. P. S. de Moura, Y.-C. Lai, R. Akis, J. P. Bird, and D. K. Ferry, Phys. Rev. Lett. **88**, 236804 (2002).
[4] D. Sweet D., E. Ott, and J. A. Yorke, Nature (London) **399**, 315 (1999).
[5] B. Eckhardt, J. Phys. A **20**, 5971 (1987).
[6] A. P. S. de Moura and C. Grebogi, Phys. Rev. E **65**, 035206(R) (2002).
[7] H. M. Nussenzveig, *Diffraction Effects in Semiclassical Scattering* (Cambridge University Press, Cambridge, UK, 1992).
[8] D. Beck, J. Chem. Phys. **37**, 2884 (1962); U. Buch and H. Pauli, *ibid.* **54**, 1929 (1971).
[9] M. Buenerd *et al.*, Phys. Rev. C **26**, 1299 (1982); H. G. Bohlen *et al.*, Z. Phys. A **308**, 121 (1982).
[10] P. H. Ng, M. Y. Tse, and L. K. Lee, J. Opt. Soc. Am. B **15**, 2782 (1998); C. L. Adler, J. A. Lock, and B. R. Stone, Appl. Opt. **37**, 1540 (1998).
[11] C. Jung and S. Pott, J. Phys. A **22**, 2925 (1989); C. Jung and T. Tel, *ibid.* **24**, 2793 (1991).
[12] M. Ding, C. Grebogi, E. Ott, and J. A. Yorke, Physica D **46**, 87 (1990).
[13] G. Troll and U. Smilansky, Physica D **35**, 34 (1989).
[14] Y. C. Lai and C. Grebogi, Phys. Rev. E **49**, 3761 (1994).