Scale-free topology-induced double resonance in networked two-state systems

Zonghua Liu[1] and Toyonori Munakata[2]
[1]*Institute of Theoretical Physics, East China Normal University, Shanghai, 200062, China*
[2]*Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, Kyoto 606-8501, Japan*
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We study numerically the effect of a scale-free topology on the signal-to-noise ratio of networked two-state systems and find a double resonance phenomenon, i.e., a resonance on coupling strength and a stochastic resonance on noise strength. This finding suggests an alternative approach of self-tuning, i.e., tuning from the scale-free topology, instead of the self-tuning of potential. A heuristic theory through a starlike network is presented to explain the double resonance.

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Two-state systems are ubiquitous in nature and many of them are connected networks. The key importance of these systems is that they can be used as models for signal detection. It has been found that there is stochastic resonance (SR) in these systems where the signal-to-noise ratio (SNR) is sensitive to the noise amplitude and can reach a maximum value at an optimal noise level [1–4]. SR has been well studied in a single or an array of two-state systems and a large number of its applications have been reported [2–9], especially in the biological systems.

SR can be also observed in other systems, such as the monostable systems [9,10] and systems without external signal [11–13], etc. For the former, Zaikin *et al.* found that the added multiplicative noise can make the monostable potential of the system become an "effective" double-well potential and then the added additive noise can cause a SR in this "effective" potential. While for the latter, the resonance is called coherence resonance so as to distinguish from the situation with signal. Except the noise induced resonance, it has been revealed that resonance can be also induced by other factors, such as the system size [14] and the coupling strength [15]. For an ensemble of noise-driven bistable overdamped oscillators, Pikovsky *et al.* shows that when a small periodic force acts on the ensemble, the linear response of the system has a maximum at a certain system size, thus called the system size resonance.

Recently, Acebron *et al.* investigates the situation of an ensemble of bistable oscillators on a scale-free (SF) network and finds that the amplitude amplification at the hub shows an effect of resonance on the coupling strength [15]. In this work, they focus on the situation of no external noise, hence the individual two-state system at a node oscillates around one of its two equilibria and have no jumping between them for a weak signal, see the left-hand panels of Fig. 1. That is, the observed amplitude amplification is not based on the jumping or firing between the two equilibria. The significance of this finding is that the SF topology can be used to sustain the high sensitivity of signaling devices.

Considering the fact that signal detection in neurons is through the firings and neurons can exhibit excitable behavior, we may simplify the behavior of neuron as two states, i.e., quiescence and firing. It has been pointed out that the residence-time distribution of a double-well system can exhibit the main features of the interspike interval histograms of firing neurons [16], thus it is necessary to know whether the effect of amplitude amplification still exist when there is firings in neurons or jumping in double-well systems. For answering this question, we here study the situation of noise and focus on how the SF topology influences the SNR of a weak signal in coupled two-states systems. Our study shows that with noise, the effect of amplitude amplification disappears but a new interesting phenomenon appears. That is, there are a SR on noise strength and another resonance on coupling strength at the hub of the SF network, whose mechanism is completely different from the case of no firing or noise. In the case of no firings, each node oscillates around one of the two equilibria, thus has memory on its initial condition. The influence of the neighboring nodes on the hub can be considered as an external noise because of their random initial conditions and thus induce a resonance. While for the case with firings, the oscillators will lose their memory on initial conditions and the resonance comes from the reducing of barrier height. On the other hand, our case is different from the situations in Refs. [9,14] in the aspect that the firing behavior in our case is sustained by an additive noise and the resonance is induced by the coupling strength.

Very interestingly, it has been reported that the visual and auditory systems of animals show high sensitivity to external signals [17–20]. The mechanism for this phenomenon has been considered to be self-tuning of barrier height for firing [17,18]. For example, consider a two-state system with potential $V(x)=a(x-1)^2(x+1)^2$. Self-tuning in a somewhat simplified form means that for a weak signal, the barrier height of $V(x)$ for firing decreases automatically through the variation of a, reflecting rare firing events [21]. Obviously, the self-tuning of parameter a and the resonance on coupling strength have the same effect: The reducing of barrier height. Thus, our finding suggests an alternative approach of self-tuning, i.e., tuning from the strength of scale-free topology, instead of the self-tuning of potential.

We first construct a SF network according to the algorithm given by Barabasi-Albert (BA) where the number of total nodes is N, the average link is $\langle k \rangle$, and the degree distribution satisfies $P(k) \sim k^{-3}$ [22,23]. Then we let each node be a double-well system, i.e., a typical two-state system, and let each link have a coupling strength λ. The networked double-well systems are as follows:

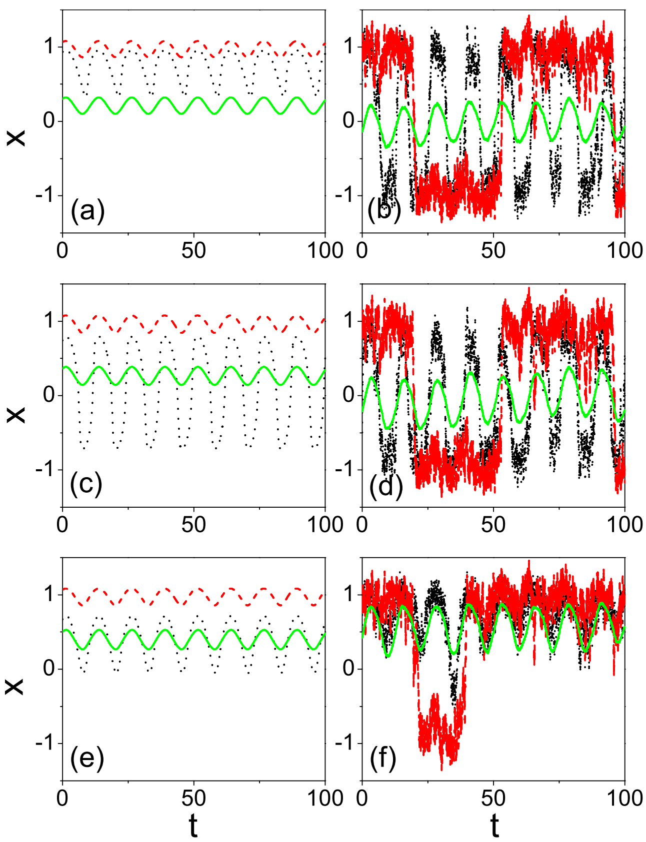

FIG. 1. (Color online) How noise influences the evolution of double-well systems on Barabasi-Albert (BA) network with $N=500$ where the dotted, dashed, and solid lines represent the evolutions on the hub with links $k_{\text{hub}}=68$, a general node with links $k_{ge}=5$, and the average on all the nodes, respectively. The left-hand panels (a), (c), and (e) are for the case of no noise and the right-hand panels (b), (d), and (f) are for the case of noise strength $T=0.2$. The coupling strength is $\lambda=0.02$ in (a) and (b), 0.05 in (c) and (d), and 0.1 in (e) and (f).

$$\dot{x}_i = -dV(x_i)/dx_i + \lambda \sum_{j=1}^{k_i} (x_j - x_i) + A \cos \omega t + \xi_i(t), \quad (1)$$

where the potential $V(x)=(x-1)^2(x+1)^2$, k_i denotes the links of node i, $A \cos \omega t$ is the weak signal, and the noise $\xi_i(t)$ are independent Gaussian white noise with average zero and satisfy

$$\langle \xi_i(t)\xi_j(t')\rangle = 2T\delta_{ij}\delta(t-t'). \quad (2)$$

The potential $V(x)$ has an unstable maximum at $x=0$ and two stable minima at $x=\pm 1$. Without external force, a particle will finally stay at one of the two minima. We choose the signal strength $A=0.8$ so that there is no firing or jumping between the two minima in Eq. (1) when $T=0$, see Fig. 2. In this work, we fix $N=500$, $\langle k\rangle=6$, and $\omega=0.5$.

Reference [15] shows that without noise, the coupling can amplify the amplitude of oscillation at the hub, see Fig. 1(a) for $T=0$ and $\lambda=0.02$ where the dotted, dashed, and solid lines represent the evolutions on the hub with links $k_{\text{hub}}=68$, a general node with links $k_{ge}=5$, and the average on all the nodes, respectively. Obviously, the hub's oscillation amplitude is much larger than that of the general node. For an optimal coupling strength $\lambda=0.05$, the hub's amplitude reaches its maximum, see Fig. 1(c). When λ is over the optimal value, the hub's amplitude will be reduced again, see Fig. 1(e). Thus, the amplitude amplification of the hub shows a resonance on the coupling strength [15]. Here, an important feature is that there is no jumping or firing between the two minima for both the hub and general nodes for all of the λ.

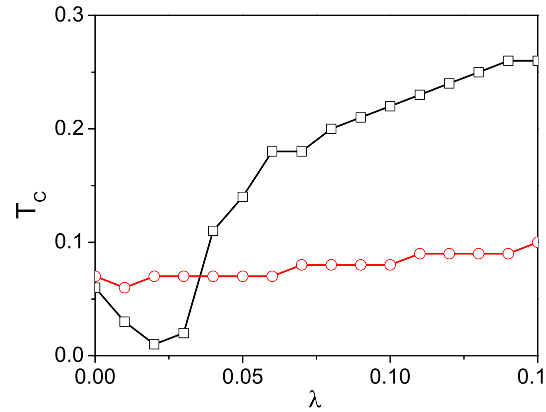

FIG. 2. (Color online) Critical noise strength for inducing jumping in the BA model with $N=500$ where the "squares" and "circles" represent the thresholds for the hub with links $k_{\text{hub}}=68$ and a general node with links $k_{ge}=5$, respectively.

With introduction of noise the situation will be completely changed as noise may induce the jumping or firing between the two equilibria which are located at $x=\pm 1$ if there were no interactions among nodes. Our numerical simulations show that the amplitude of the hub will not change anymore with the increase of coupling strength, i.e., the effect of amplitude amplification on λ disappear because of the jumping induced by noise. The right-hand panels of Fig. 1 show the results for $T=0.2$ and $\lambda=0.02, 0.05$, and 0.1, respectively. It is easy to see that the averages in Figs. 1(b) and 1(d) the solid line waves are oscillations around $x=0$, which is different from the corresponding (a) and (c) for $T=0$ where the waves are away from $x=0$. The reason is that the case without noise has no jumping and thus can keep its memory on initial condition. The asymmetric initial conditions around ± 1 on different nodes result in the average away from $x=0$. In contrast, the case with noise has jumping and thus loses its memory on initial condition, resulting in an approximate symmetric distribution of nodes at the two equilibria and hence the average turns out to perform oscillation around $x=0$.

Very strangely, for the case of larger λ, we find that the jumping rate is not increased further but reduced very much, see Fig. 1(f). For understanding the mechanism we have checked the relationship between the jumping rate and noise strength T and found that there is a threshold T_c for each λ where there is jumping for $T>T_c$ and no jumping for $T<T_c$ during a certain time interval Δt. We specify T_c by checking the jumping in the time interval $\Delta t=100$. Figure 2 shows T_c as a function of λ, where the "squares" and "circles" represent the results for the hub with links $k_{\text{hub}}=68$ and a general node with links $k_{ge}=5$, respectively. It is easy to see that T_c does not change much for the general

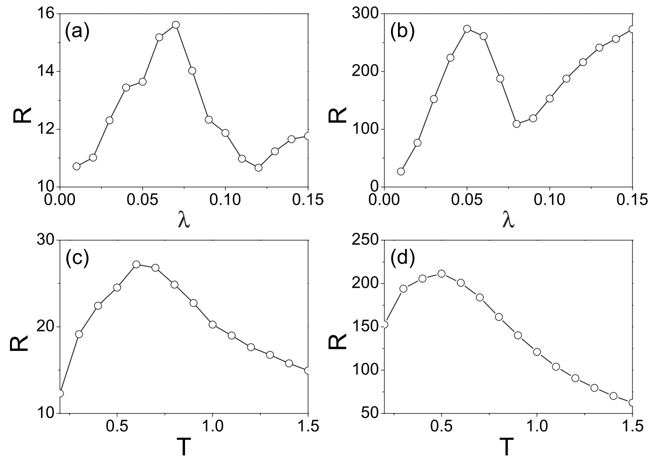

FIG. 3. How SNR changes with the coupling strength and noise strength in the BA model where R represents the SNR, (a) and (c) denote the case of a general node with links $k_{ge}=5$ and (b) and (d) the case of the hub with links $k_{\rm hub}=68$. (a) and (b) SNR versus coupling strength with $T=0.2$; (c) and (d) SNR versus noise strength with $\lambda=0.03$.

node but it changes significantly for the hub. For the hub at $T=0.2$, we have $T>T_c$ for $\lambda \leqslant 0.08$, Figs. 1(b) and 1(d), and $T<T_c$ for $\lambda>0.08$, Fig. 1(f). Figure 2 also shows that there is a specific λ_c for each given T, i.e., λ_c depends on T. For example, λ_c will be over 0.15 for $T=0.3$.

We now turn to study how the coupling strength and noise strength influence the SNR. We first fix T and let the coupling strength λ change. By taking time series for each λ and making the Fourier transformation to get its power spectrum, we can obtain the SNR at the reference frequency ω. Figures 3(a) and 3(b) show how the SNR changes with the coupling strength at $T=0.2$ for the general node and the hub, respectively. It is easy to see that both the general node and the hub show resonance on coupling strength for $0<\lambda<\lambda_c\simeq 0.08$, and SNR for the hub is about 20 times larger than that for the general node, indicating the hub can amplify the weak signal detected by a large number of surrounding general nodes. After $\lambda>\lambda_c$, there is no jumping at the hub and thus the hub just oscillates around one of the two equilibria [see Fig. 1(f)]. Our numerical simulations show that in this region of the coupling strength local clusters of the general nodes are formed and more than one-half of the nodes are staying around in the same equilibrium with the hub. With further increase of λ, this number of clustering nodes will increase until the appearance of synchronization. (The detailed synchronization results will be reported elsewhere.) This synchronization makes the hub oscillate around one equilibrium (no firing) and at the same time increases the SNR with λ [10], which is just what we have observed in Fig. 3(b) for $\lambda>\lambda_c=0.08$.

Next we fix $\lambda=0.03$ and let T change. We find that the SNR also shows resonance on the noise strength T which is the traditional SR. Figures 3(c) and 3(d) show the results on the general node and the hub, respectively. In sum, the SF networked two-state systems show an enhanced double resonance at the hub, i.e., resonance on both coupling strength and noise strength.

The characteristic feature of the SF network is the existence of a few hubs whose link number is much larger than the general nodes. As a hub is connected with a large number of surrounding nodes, the hub and its surrounding nodes can be roughly modeled by a starlike subnetwork where every general node is connected to the hub and there is no connection among the general nodes [15]. That is, the starlike network captures the main trait of SF networks. Therefore, we here use the starlike network to illustrate heuristically the mechanism of the double resonance.

The dynamics on general nodes and the hub of the starlike network are as follows:

$$\dot{x}_i = -\frac{dV(x_i)}{dx_i} + \lambda(x_H - x_i) + A\cos\omega t + \xi_i(t),$$

$$\dot{x}_H = -\frac{dV(x_H)}{dx_H} + \sum_{i=1}^{m-1}\lambda(x_i - x_H) + A\cos\omega t + \xi_H(t), \quad (3)$$

where x_i and x_H denote the general node and the hub, respectively, $i=1,\ldots,m-1$, and ξ_i and ξ_H are independent Gaussian white noise with average zero and satisfy $\langle\xi_i(t)\xi_j(t')\rangle = 2T\delta_{ij}\delta(t-t')$, $\langle\xi_i(t)\xi_H(t')\rangle=0$, and $\langle\xi_H(t)\xi_H(t')\rangle=2T\delta(t-t')$.

For $\lambda \ll 1$, the dynamics on the general nodes can be written as $\dot{x}_i=4x_i-4x_i^3+A\cos\omega t+\xi_i(t)$, which is the standard double-well system with a weak periodic signal and an external noise. It is well known that this system can show SR on noise strength [2], implying that the case of general nodes is trivial. Thus, we here only focus on x_H of the hub. The sum $\sum_{i=1}^{m-1}x_i \equiv (m-1)\langle x(t)\rangle$ is in fact an average over $m-1$ realizations. In the stationary state, each x_i has lost its memory on initial condition because of the jumping and $\langle x(t)\rangle$ becomes a periodic function of time, i.e., $\langle x(t)\rangle = \langle x(t+\tau_\omega)\rangle$ with $\tau_\omega=2\pi/\omega$ [4]. For small λ, the average $\langle x(t)\rangle$ can be written as [4]

$$\langle x(t)\rangle = \bar{x}\cos(\omega t - \bar{\phi}), \quad (4)$$

with amplitude $\bar{x}=\frac{A\langle x^2\rangle_0}{T}\frac{2r_k}{\sqrt{4r_k^2+\omega^2}}$ and a phase lag $\bar{\phi}=\arctan(\frac{\omega}{2r_k})$ where $\langle x^2\rangle_0$ is the T-dependent variance of the stationary unperturbed system ($A=0$) and $r_k=\frac{1}{\sqrt{2}\pi}\exp(-\frac{1}{T})$ is the Kramers rate.

Substituting Eq. (4) into Eq. (3) we obtain

$$\dot{x}_H = [4-\lambda(m-1)]x_H - 4x_H^3 + A'\cos(\omega t - \theta) + \xi_H(t), \quad (5)$$

where $A'=\sqrt{[A+\lambda(m-1)\bar{x}\cos\bar{\phi}]^2+[\lambda(m-1)\bar{x}\sin\bar{\phi}]^2}$ and $\tan\theta=\frac{\lambda(m-1)\bar{x}\sin\bar{\phi}}{A+\lambda(m-1)\bar{x}\cos\bar{\phi}}$. Thus, Eq. (5) becomes a standard double-well system with two minima at $x=\pm\sqrt{4-\lambda(m-1)}/2$ and the barrier height $U_0=1-\lambda^2(m-1)^2/16$. For guaranteeing $U_0>0$, the resonance range of λ is limited to $\lambda<4/(m-1)$.

Considering the fact that a general node has two minima at $x=\pm 1$ and the barrier height $U_0=1$, we see that the barrier height of the hub is reduced and the distance between the minima become shorter. Thus, comparing with the general nodes, the jumping at the hub is much easy to occur. This is

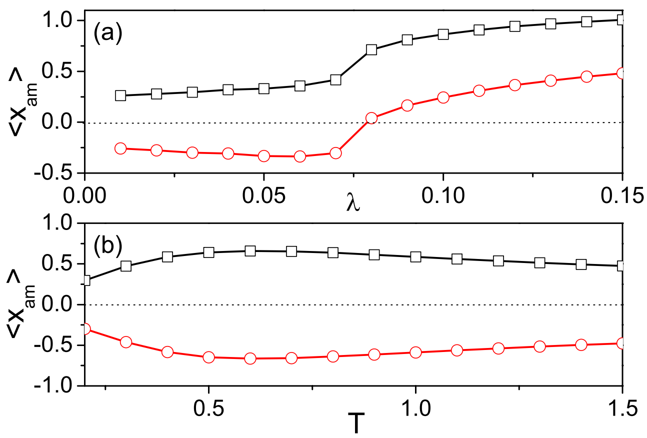

FIG. 4. (Color online) Amplitudes of the average in SF network as a function of λ and T where the "squares" and "circles" represent the up and down amplitudes, respectively. The parameter is $T=0.2$ in (a) and $\lambda=0.03$ in (b).

the reason why the coupling strength in SF network can enhance the SNR at the hub. On the other hand, Eq. (5) suggests a new mechanism for the high sensitivity observed in the visual and auditory systems of animals [17–20]. As we know, most of the technical and biology networks are SF networks [24,25] and there are evidences that the functional connections among different areas of the human brain and the cat brain are also SF networks [26–28]. Suppose each node in the SF network represents a bunch of neuron sensors. The collected signals at nodes are transferred to the hub and then implement the amplification by an optimal λ. Thus, the self-tuning of coupling strength λ is equivalent to the self-tuning of parameter a in potential.

Equation (4) is not always correct and its effective range of validity can be estimated by checking the solid line waves in the right-hand panels of Fig. 1. Take Fig. 1(b) as an example: We may measure the maximum and minimum of the solid line wave for each period τ_ω and then take their averages. Figure 4(a) shows how the two averages change with λ for the SF network. From Fig. 4(a) we see that the two averages are symmetric only when $\lambda<\lambda_c$. When $\lambda>\lambda_c$, they will be both on the same side ($x>0$ in this example), confirming that most of the nodes become a cluster in their single wells, with Fig. 1(f) showing the precursor. Moreover, we find that for fixed λ, the two averages also show resonance for the noise strength T [see Fig. 4(b)], which corresponds to the SR shown in Fig. 3(c) for the SF network. We note that this resonance in Fig. 4(b) can be explained by the expression of $\bar{x}$ in Eq. (4).

The SNR of the double-well system, Eq. (5), can be easily figured out. Following the derivation of Eq. (5.9) in Ref. [2] we obtain the SNR of Eq. (5) as

$$R \approx \frac{\sqrt{2}A'^2[4-\lambda(m-1)]^2}{16T^2}e^{-16-[\lambda^2(m-1)^2]/16T}. \qquad (6)$$

By $dR/dT=0$ we find that the maximum SNR for a fixed λ occurs at $T_{op}=1/2-\lambda^2(m-1)^2/32$; and by $dR/d\lambda=0$ we find that the maximum SNR for a fixed T occurs at $\lambda_{op}=(2\pm\sqrt{4-16T})/(m-1)$, where the A' is approximately treated as a constant. As $k_{\rm hub}=68$ in SF network, we take $m=68$ in Eq. (6), which gives $T_{op}\approx 3/8$ for $\lambda=0.03$ and $\lambda_{op}\approx 0.043$ for $T=0.2$. These two values are very close to the optimal values in Figs. 3(b) and 3(d) and thus explains the double resonance observed in the SF network. On the other hand, we know from Eq. (5) that the barrier height U_0 depends on the coupling strength λ. The existence of the optimal λ_{op} implies that the coupling strength may be self-tuned to its optimal value, confirming the mechanism of the self-tuning of coupling strength.

In sum, we have found a mechanism, i.e., SF topology assisted tuning for the effect of self-tuning. This finding is significant as it may enrich our understanding on the effect of high sensitivity to external noise in biological systems and also have potential applications in designing the signaling devices.

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