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Theory for particle settling and shear-induced migration in thin-film liquid flow

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Particles suspended in a film flow can either settle out of the flow, remain well mixed, or even advance faster than the fluid, accumulating at the moving contact line. Recent experiments by Zhou *et al.* [Phys. Rev. Lett. **94**, 117803 (2005)] have demonstrated that these three settling behaviors can be achieved by control of the average particle concentration ϕ and inclination angle θ. This work presents a theory for determining the settling behavior in the Stokes regime by calculating the depth profile of ϕ and the depth-averaged velocities of the liquid and particle phases. It is found that shear-induced particle fluxes can lead to an inversely stratified flow, in which the particles move on average faster than the liquid. The theory is directly compared to Zhou *et al.*'s experimental data, and the implications of stratification for lubrication-type models are also discussed.

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I. INTRODUCTION

Film flow of particle-laden liquid occurs in many important contexts, from geophysical flows such as erosion and turbidity currents [1] to industrial processes including paper making, food characterization, and the application of fertilizers. While sophisticated constitutive models have been developed for general suspension flow, their complexity often makes them incompatible with fluid techniques such as lubrication theory. As a result, the mathematical description of particle-laden films remains a challenging problem.

This paper addresses one of the most fundamental questions about such flows: whether the particles will remain in suspension, or settle out of the flow. More generally, this is the question of the stratification of the concentration ϕ and is intimately linked with the relative velocities of the two phases. For although the aggregation of particles at the bottom raises the difficult question of a stopping condition, it must be preceded by an overall settling motion, which can be identified in a continuum model by a decreasing profile of ϕ. Also, since the velocity of a film increases strongly with the vertical coordinate z, a decreasing depth profile implies that the particle phase moves on average more slowly than the liquid, again consistent with settling out of the flow. Alternatively, if an increasing depth profile were somehow achieved, not only would it be rarer for particles to contact the bottom, but, far from stopping, the particles would be carried downstream on average faster than the liquid.

Although it may seem that gravity will always enforce a decreasing concentration profile, or stable stratification, increasing depth profiles were seen in the theoretical work of Carpen and Brady for the similar problem of inclined Poiseuille flow [2]. This is due to shear-induced particle fluxes, consisting of a diffusive effect in shearing flow and a migration toward regions of lower shear rate $\dot{\gamma}$ in situations of inhomogeneous shear [3]. These fluxes have been described by diffusive flux models based on that of Leighton and Acrivos [3] and by the suspension balance model due to Nott and Brady [4], and the two models have been seen to behave similarly [5]. Carpen and Brady, employing the suspension balance model, found that increasing concentration profiles in the Poiseuille problem are indeed unstable to finite wavelength disturbances in the spanwise direction.

Recent experiments by Zhou *et al.* demonstrate the complexity of settling in inclined film flows [6]. They observed three distinct settling behaviors: at low inclination angles α and average concentrations ϕ the particles settle to the bottom substrate and are removed from the flow, at intermediate α and ϕ the suspension remained well mixed, and at larger α and ϕ the particles advanced faster than the fluid, accumulating in a thick ridge at the advancing contact line. A lubrication model, introduced in that paper and explored further in [7], addresses the growth of the ridge in the last case, assuming an unstratified flow, by attributing the particle motion to settling in the downstream direction. However, no theory currently exists for predicting the settling regime. Also, in light of the above discussion, the effects of stratification on the lubrication flow deserve further study, since stratification can provide a separate mechanism for the accumulation of particles in the ridge. This work presents a model for stratification in a film flow with the goal of addressing these two issues[1].

Similar models for stratified film flow have been studied by Schaflinger *et al.* [8] and Timberlake and Morris [9]. Neither study, however, applies directly to the present problem. Schaflinger *et al.* found stationary profiles of a stratified film in which the downward settling flux is balanced by shear-induced diffusion, without including a migration flux. A direct consequence of this simplification is that the diffusive flux is always directed upward, so the ϕ profile is always decreasing. Thus migration effects are required in order to observe an unstable stratification.

The work of Timberlake and Morris included theory and experiments for a neutrally buoyant suspension [9]. Their work uses the suspension balance model, which includes ef-

[1]Zhou *et al.* did not observe a spanwise instability analogous to that described by Carpen and Brady; however, this does not necessarily indicate that the film is stably stratified since in the film problem any instability would be complicated by the fingering instability of the contact line.

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fects analogous to both diffusion and migration. In this case, the lack of a buoyant force requires that at steady state particle diffusion balances the migration flux, and this migration is always directed up because in gravity-driven film flow the shear stress decreases toward the free surface. Therefore, in neutrally buoyant suspensions ϕ must increase with z.

II. MODEL

This work uses a Newtonian rheology for the suspension, using an empirical viscosity $\mu(\phi)$ that increases with the concentration. Particle motion is described by the diffusive flux model, and the derivation is similar to that of Schaflinger *et al.* but differs in the inclusion of a migration flux. This effect was introduced by Leighton and Acrivos [3] and quantified by Phillips *et al.* [10] in the expression

$$\frac{D\phi}{Dt} = a^2 \nabla \cdot \left(K_c \phi \nabla (\dot{\gamma}\phi) + K_\eta \dot{\gamma} \frac{\phi^2}{\mu(\phi)} \nabla \mu(\phi) \right) \qquad (1)$$

representing the advection due to all shear-induced flux. The terms involving $\nabla\phi$ and $\nabla\mu(\phi)$, representing diffusion, and $\nabla\dot{\gamma}$, representing migration, are inferred from the scaling of the mechanisms Leighton and Acrivos proposed to explain shear-induced flux, which include anisotropy in the rate of two-particle encounters (proportional to $\phi\dot{\gamma}$) and in the displacement resulting from those encounters [proportional to $1/\mu(\phi)$]. Although the reasoning of Leighton and Acrivos suggests that K_c and K_η could in general be functions of ϕ, the forms of these functions are unknown. This work uses the values $K_c=0.43$ and $K_\eta=0.65$ obtained by Phillips *et al.* from an experiment conducted at high concentrations (0.45, 0.50, and 0.55) and under the assumption that the parameters are approximately constant. Equation (1) corresponds to a particle flux

$$\begin{aligned} F_m &= -a^2 K_c \phi \nabla \left(\frac{\sigma}{\mu(\phi)} \phi \right) - a^2 (K_\eta - K_c) \frac{\sigma \phi^2}{\mu(\phi)^2} \nabla \mu(\phi) \\ &= -\frac{a^2 \phi}{\mu(\phi)} \left(K_c \nabla (\sigma\phi) + (K_\eta - K_c) \frac{\sigma\phi}{\mu(\phi)} \nabla \mu(\phi) \right), \end{aligned} \qquad (2)$$

where the shear rate $\dot{\gamma}$ has been eliminated in favor of the shear stress $\sigma=\mu(\phi)\dot{\gamma}$.

For a flat film on an incline, equilibrium is reached when this flux balances that of gravitational settling in the z direction. Settling rates are commonly expressed as a product of the velocity of a single sphere $v_s=-(2/9)\Delta\rho g/\mu_f$ by a hindered settling function $f(\phi)$ for which many empirical formulas exist. Here ρ and μ_f are the density and viscosity of the fluid, g is the gravitational constant, and $\Delta=(\rho_p-\rho)/\rho$ is the density difference for particles of density ρ_p. The following calculations use $\Delta=1.7$, the value in Zhou *et al.*'s experiments. At this point it is convenient to follow Schaflinger *et al.* and use the hindered settling function $f(\phi)=(1-\phi)/\mu(\phi)$, leading to the settling flux

$$F_s = -\frac{2}{9} \frac{a^2 \Delta\rho g \cos\alpha}{\mu_f} \frac{\phi(1-\phi)}{\mu(\phi)}, \qquad (3)$$

where α is the angle of inclination.

The balance of flux $F_m+F_s=0$ then takes the form

$$K_c(\sigma\phi)' + (K_\eta - K_c)\frac{\sigma\phi}{\mu(\phi)}\mu(\phi)' = -\frac{2}{9}\frac{\Delta\rho g \cos\alpha}{\mu_f}(1-\phi) \qquad (4)$$

where the gradients have been replaced by primes denoting differentiation by z. Substituting the standard formula $\mu(\phi)=\mu_f(1-\phi/\phi_m)^{-2}$ [11] with the maximum packing fraction $\phi_m \approx 0.67$ and differentiating yields

$$\begin{aligned} \left(1 + \frac{2(K_\eta - K_c)}{K_c}\frac{\phi}{\phi_m - \phi}\right)\sigma\phi' &= \phi(1+\Delta\phi) \\ &\quad - \frac{2\Delta}{9K_c}(\cot\alpha)(1-\phi), \end{aligned} \qquad (5)$$

where z and σ have now been nondimensionalized using the depth of the film h and the unit of stress, $(\rho g/h)\sin\alpha$.

For a flat film there is no capillary force, so the pressure can be set to zero at the free surface $z=1$, and is assumed to be hydrostatic in the suspension. The nondimensional shear stress then satisfies the equation

$$\sigma' = -(1+\Delta\phi). \qquad (6)$$

Equations (5) and (6) constitute the system to be studied here, with the understanding that (5) is replaced by $\phi'=0$ when $\phi=0$ or ϕ_m to ensure that pure fluid and packed particles are admissible solutions and to keep the concentration within its meaningful range. The physical boundary conditions both involve the stress: $\sigma(0)=(1+\Delta\phi_0)$ and $\sigma(1)=0$, where ϕ_0 is the imposed average concentration. Thus for these two equations there is only a one-parameter family of physically meaningful solutions, parametrized by ϕ_0. In practice this system was easiest to solve by shooting with a Runge-Kutta method from $z=0$ while adjusting the value of $\phi(0)$. Once σ and ϕ are determined, the mixture velocity can be calculated using $dv/dz=\dot{\gamma}=\sigma(z)/\mu(\phi(z))$ and $v(0)=0$.

Note that while modeling of mixtures at or very near the maximum packing concentration is difficult, as non-Newtonian behavior has been observed, these regions can be expected to contribute little to the overall flow structure. Regardless of the rheological model used, most of the shearing occurs in areas of lower concentration, so the near plug flow resulting from a diverging viscosity is generally a good approximation.

III. SOLUTIONS

Unlike the previous studies, this model incorporates the opposing effects of particle migration and buoyancy which are necessary to allow both stable and unstable stratifications. Still the structure of the equations permits some generalizations about the solutions. Since $\sigma \geqslant 0$, it is apparent from Eq. (5) that a single solution $\phi(z)$ is monotone, because $\sigma\phi'$ is determined by a function of ϕ only with a single unstable root $\phi^*=\phi^*(\alpha)$ in its allowable domain (between 0 and ϕ_m). There are then two possibilities: $\phi_0 > \phi(0) > \phi^*$

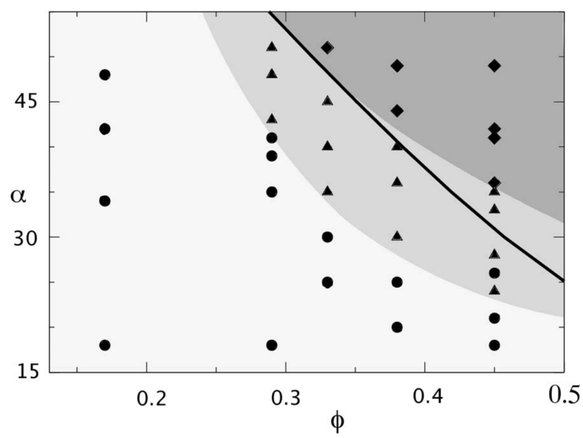

FIG. 1. Function $\phi^*(\alpha)$ determining whether particles tend toward the top or bottom of the film. Overlaid are Zhou *et al.*'s experimental parameters for which particles settle to the substrate (●, white), remain well mixed (▲, light), or accumulate in a ridge (◆, dark). Experimental data are from Fig. 2 of [6].

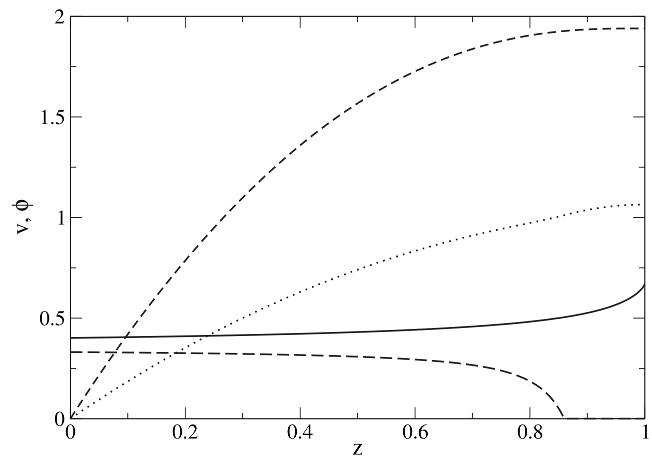

FIG. 2. Depth profiles of ϕ and v for two average concentrations at $\alpha=45°$. Bulk concentration $\phi_0=0.25$: velocity (dotted line) and concentration (long-dashed line). Bulk concentration $\phi_0=0.45$: velocity (short-dashed line) and concentration (solid line). Velocities are scaled by the average velocity of a homogeneous film at the same concentration. With this rescaling the average velocities at $\phi_0=0.25$ of the particle and liquid phases are 0.57 and 0.70, and at $\phi_0=0.45$ the velocities are 1.41 and 1.33, respectively.

with $\phi(1)=\phi_m$, or $\phi_0<\phi(0)<\phi_m$ with $\phi(1)=0$.

It is also possible to deduce that the concentration at the free surface is either 0 or ϕ_m, the conditions under which Eq. (5) is replaced by $\phi'=0$, since otherwise the equation becomes degenerate due to the vanishing shear stress. Incidentally, this same conclusion applies to Schaflinger *et al.*'s model, because the diffusion flux is simply proportional to the also vanishing shear rate. However, in that case $\phi=\phi_m$ is not possible because the profile is always decreasing, whereas here both extreme values can be achieved.

Thus by continuity the concentration either decreases monotonically from $\phi(0)<\phi^*(\alpha)$ to zero or increases monotonically from $\phi(0)>\phi^*(\alpha)$ to ϕ_m with increasing z. As discussed in the Introduction we associate the former case with the regime in Zhou *et al.*'s experimental work, characterized by particles settling out of the flow, and the latter case with the particle-rich ridge regime. While there is no obvious reason why there should be a third regime (other than the single solution $\phi\equiv\phi^*$) where the fluid and particles move at the same velocity, it may be that experiments in which the suspension stayed well mixed had $\phi_0\approx\phi^*$ and the relatively small difference between the two velocities did not have time to produce noticeable segregation on the experimental time scale.

Plotted in Fig. 1 is the calculated transition point $\phi^*(\alpha)$ and the experimental data from [6]. As expected, the transition lies within the well-mixed regime. This calculation involves no fitting parameters, and the agreement is remarkable considering the simplifying assumptions of one-dimensional, time-independent flow. The position of the curve $\phi^*(\alpha)$ also suggests that the experimentally observed well-mixed films mostly lie in the $\phi_0<\phi^*(\alpha)$ range, and therefore would likely result in particles settling out of the flow were the experiments continued longer.

Examples of the two cases ($\phi_0>\phi^*$ and $\phi_0<\phi^*$) are shown in Fig. 2 for $\alpha=45°$, $\phi^*(\alpha)\approx 0.35$. The effect of the increasing concentration profile for $\phi_0=0.45$ is to flatten the velocity near the top from the parabolic shape of an unstratified film, while for $\phi_0=0.25$ the absence of particles near the top increases the shear in this area. Also of interest is the fact that when $d\phi/dz>0$ both phases move faster than the velocity of an unstratified film, because of the high-shear, low-ϕ region at the bottom and the low shear at the top where v is at its greatest. Both phases are slower when $d\phi/dz<0$.

Figure 3 investigates the importance of the differential velocities due to stratification as a mechanism for phase separation and the eventual formation of a ridge in the experiment of Zhou *et al.* In order to facilitate a comparison between this mechanism and that of gravitational settling in the downstream direction, proposed by Zhou *et al.*, the ratio

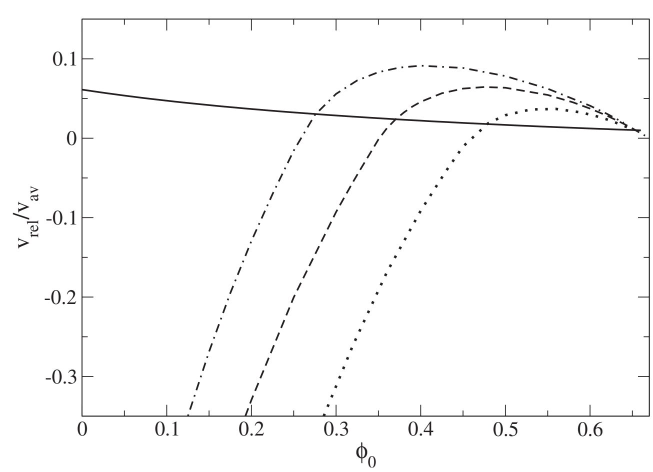

FIG. 3. Ratio $v_{\mathrm{rel}}/v_{\mathrm{av}}=(v_p-v_f)/[\phi v_p+(1-\phi)v_f]$ of velocities relevant for formation of the particle-rich ridge in Zhou *et al.*'s experiment. Calculated velocity difference due to the stratified flow at $\alpha=30°$ (dotted line), 45° (dashed line), and 60° (dot-dashed line). The solid line represents the velocity difference due to direct gravitational settling in the flow direction as described by Zhou *et al.*, which is independent of α.

$v_{\mathrm{rel}}/v_{\mathrm{av}}=(v_p-v_f)/[\phi v_p+(1-\phi)v_f]$ was chosen. This ratio determines the accumulation of particles in an experiment limited by the length of the channel. At large values of ϕ and α, the regime in which the ridge was observed, stratification has a larger effect than in-plane settling. Thus the stratified flow appears to be the more important mechanism in forming the ridge.

A description of the ridge evolution including stratification is possible within the lubrication context if the film is assumed to be always in equilibrium between settling and migration, by using the calculations of Fig. 3 to determine the relative velocity from ϕ. This would result in a system similar to that in [7], which for length scales greater than a modified capillary length describes a ridge that grows linearly with time. If this route is followed, care must be taken to ensure the length scale is also large enough to justify the equilibrium assumption. The experiments and two-dimensional calculations of Timberlake and Morris [9] indicate that the distance traveled before reaching equilibrium [which is proportional to $(h/a)^2$] can be as large as tens of centimeters, even for an experiment with fairly large particles such as [6]. At shorter length scales, such a two-dimensional model may therefore be necessary, which would generalize the above results by allowing nonequilibrium concentration profiles. The most likely effect of nonequilibrium physics would be to lengthen the time scale of phase separation, making the well-mixed regime more likely for length-limited experiments.

This vertical equilibrium theory demonstrates the importance of particle migration in determining the flow, and if extended to two or three dimensions could provide a model for effects such as ridge formation, particle deposition in the clear fluid regime, the contact-line instability, and spanwise particle banding [2].

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