

Random cyclic matrices

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(Received 1 January 2008; revised manuscript received 4 August 2008; published 11 September 2008)

We present a Gaussian ensemble of random cyclic matrices on the real field and study their spectral fluctuations. These cyclic matrices are shown to be pseudosymmetric with respect to generalized parity. We calculate the joint probability distribution function of eigenvalues and the spacing distributions analytically and numerically. For small spacings, the level spacing distribution exhibits either a Gaussian or a linear form. Furthermore, for the general case of two arbitrary complex eigenvalues, leaving out the spacings among real eigenvalues, and, among complex conjugate pairs, we find that the spacing distribution agrees completely with the Wigner distribution for a Poisson process on a plane. The cyclic matrices occur in a wide variety of physical situations, including disordered linear atomic chains and the Ising model in two dimensions. These exact results are also relevant to two-dimensional statistical mechanics and ν -parametrized quantum chromodynamics.

DOI: 10.1103/PhysRevE.78.036213

PACS number(s): 05.45.Mt, 03.65.Ge

With the pseudo-Hermitian extension of quantum mechanics [1–3], it has become possible to develop a number of new ideas, opening thereby interesting and important directions of investigation. One of these advances has been in random matrix theory where pseudounitarily invariant ensembles were presented [4] that exhibit completely different kinds of level repulsion as compared to the ensembles known [5–7]. Thus, physical systems that violate parity and time-reversal invariance (\mathcal{PT} symmetric) exhibit level repulsion that could be linear or $\sim -S \ln S$ where S is the nearest-neighbor spacing of levels. However, an explicit analysis has been done only for an ensemble of 2×2 matrices.

In this paper, we present random matrix theory (RMT) of $N \times N$ cyclic matrices with real elements. As we shall show, these matrices are pseudosymmetric with respect to “generalized parity.” Such matrices arise in very significant contexts, the celebrated example being that of the Onsager solution of the two-dimensional Ising model [8,9]. They are encountered in the treatment of linear atomic chains with the Born–von Kármán boundary condition [10] and in understanding overlap matrices for molecules such as benzene. These matrices also occur as transfer matrices in the theory of disordered chains [11] and in the general context of wave propagation in one-dimensional structures [12]. In the latter example, generally, matrices of second order occur—thus, our earlier results [13] throw light on the fluctuation properties of the eigenvalues. Cyclic matrices also appear in the context of phase transitions in the spherical model [14]. In all these varied instances, as soon as there is a random parameter (e.g., external field or a random coupling in the example of the Ising model), the level correlations dictate the long-time tails of the time correlation functions which, in turn, relate to the relaxation of these systems when they are perturbed from thermodynamic equilibrium [15].

RMT appears in seemingly unrelated problems in physics and mathematics ranging from growth models, directed poly-

mers, and random sequences, to the Riemann hypothesis [16–18]. Also, the study of random matrices has been related to quantum chaos and exactly solvable models in a remarkable way [18–20]. Generically, the statistics of spectral fluctuations of classically integrable, pseudointegrable, and chaotic systems follow, respectively, the general features of Poisson, short-range Dyson [21,22] or semi-Poisson [23] model, and Wigner-Dyson ensembles [24]. However, for the physical situations occurring in two-dimensional statistical mechanics where time reversal and parity are violated [25,27–29], there is no general understanding of the statistical nature of spectral fluctuations [30,31]. Perhaps the first example of a billiard system with a \mathcal{PT} -symmetric (violating \mathcal{P} and \mathcal{T}) Hamiltonian was a particle enclosed in a rectangular cavity in the presence of an Aharonov-Bohm flux line [32]. For this classically pseudointegrable system, the spectral statistics of quantum energy levels was found to exhibit level repulsion that is distinctly different from the standard RMT [5]. For these classes of systems, an important step was taken in Ref. [4], and the present work takes us to show the nature of these fluctuations in $N \times N$ cyclic matrices. The general case of $N \times N$ random pseudo-Hermitian matrices remains open, however.

Let us consider an $N \times N$ cyclic matrix with real elements $\{a_i\}$:

$$\mathbf{M} = \begin{bmatrix} a_1 & a_2 & \cdots & a_N \\ a_N & a_1 & \cdots & a_{N-1} \\ \vdots & & & \\ a_2 & a_3 & \cdots & a_1 \end{bmatrix}. \quad (1)$$

It is important to note that this matrix is, in fact, pseudo-Hermitian (pseudo-orthogonal) with respect to $\boldsymbol{\eta}$

$$\boldsymbol{\eta} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \vdots & & & & & \\ 0 & 1 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad (2)$$

that is,

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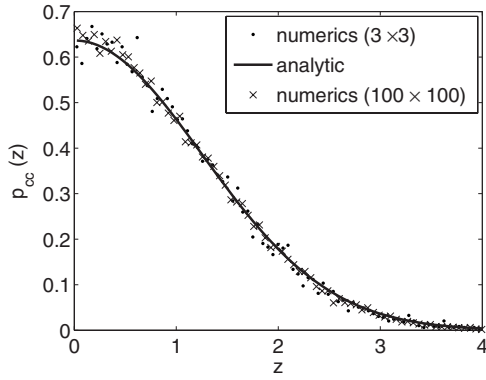


FIG. 1. Probability distribution of the absolute spacing between the complex conjugate pair of eigenvalues of a Gaussian ensemble of 3×3 cyclic matrices. The numerical result obtained by considering 10 000 realizations agrees with the analytic result (9). The Gaussian spacing distribution may be interpreted to give an accumulation of eigenvalues resulting in a maximum at zero spacing, but no tendency to cluster as the first derivative is zero. This is different from a Poisson distribution.

$$\mathbf{M}^\dagger = \mathbf{M}^T = \boldsymbol{\eta} \mathbf{M} \boldsymbol{\eta}^{-1}. \quad (3)$$

Since $\boldsymbol{\eta}^2 = \text{identity } \mathbf{I}$, $\boldsymbol{\eta}$ is introduced here as “generalized parity.” Thus, we have an ensemble of random cyclic matrices (RCMs) that is pseudo-orthogonally invariant in the sense of Eq. (3). There are two distinct scenarios with respect to time reversal \mathcal{T} and parity \mathcal{P} : (a) the standard case where \mathcal{T} and \mathcal{P} are preserved; this case is trivially \mathcal{PT} symmetric and (b) the case of \mathcal{PT} symmetry where both \mathcal{T} and \mathcal{P} both are broken. In case (a), one may study the fluctuation properties of energy levels after classifying the eigenfunctions according to definite parity (odd or even); however, case (b) belongs to a different class altogether. Whereas case (a) corresponds to the invariant ensembles of random matrix theory [5], case (b) has not been fully studied, only some partial results exist [4,13] and RCM belong to this case. The discrete symmetries for operators represented by cyclic matrices are clearly spelled out here. Due to this generality, our final results are expected to be relevant for a wide variety of physical situations occurring in anyon physics [25], ν -parametrized quantum chromodynamics, fractional quantum hall systems [26], etc.

The eigenvalues of \mathbf{M} are given by [33]

$$E_l = \sum_{p=1}^N a_p \exp \frac{2\pi i}{N} (p-1)(l-1) \quad (4)$$

($l=1, 2, \dots, N$), the maximum real eigenvalue being $\sum_i a_i$. The diagonalizing matrix is given by [14]

$$U_{jl} = \frac{1}{\sqrt{N}} \exp \frac{2\pi i}{N} (j-1)(l-1). \quad (5)$$

We consider a Gaussian ensemble of cyclic matrices with a distribution,

$$P(\mathbf{M}) \sim \exp -A \text{Tr}(\mathbf{M}^\dagger \cdot \mathbf{M}), \quad (6)$$

where A sets the scale (of energy, for instance).

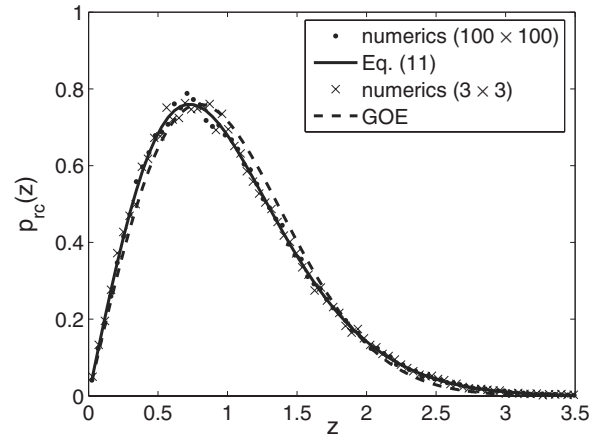


FIG. 2. Probability distribution of the absolute spacing between a real and a complex eigenvalue of a Gaussian ensemble of 3×3 cyclic matrices. The numerical result obtained by considering 10 000 realizations for 3×3 matrices and 1000 realizations of 100×100 matrices agrees with the analytic result (11). We observe a linear level repulsion near zero spacing; however, the result is distinctly different from the Wigner surmise for GOE.

For the sake of simplicity, we present the analysis for an ensemble of 3×3 matrices. We would like to obtain the joint probability distribution function (JPDF) of eigenvalues because all the correlations are related to it. Also, we would like to show results on the spacing distribution as they enjoy a central place in discussions in quantum chaos, universality arguments, and rule the dominant long-time tail in correlation functions. We immediately see that $\text{tr } \mathbf{M}^\dagger \cdot \mathbf{M} = 3(a_1^2 + a_2^2 + a_3^2)$. In effect, we have $P(\{a_i\}) = (\frac{3A}{\pi})^{3/2} e^{-3A \sum_i a_i^2}$. There are three eigenvalues—one real $E_1 = \sum_i a_i$ and one complex conjugate pair (E_2, E_2^*) . We may define spacing as $S_{23} := |E_2 - E_3| = \sqrt{3}(a_3 - a_2)$ as well as $S_{12} := |E_1 - E_2| = |\frac{3}{2}(a_2 + a_3) + \frac{i\sqrt{3}}{2}(a_2 - a_3)|$. Obviously, $S_{12} = S_{13}$. The JPDF of eigenvalues $P(\{E_i\})$ can be written as

$$P(E_1, E_2, E_2^*) = \left(\frac{A}{\pi}\right)^{3/2} e^{-A(E_1^2 + 2|E_2|^2)}. \quad (7)$$

With this JPDF, spacing distributions can be found [34]. The spacing distribution for the complex conjugate pair $P_{CC}(S_{23})$ is given by

$$\begin{aligned} P_{CC}(S_{23}) &= \int \prod_{i=1}^3 da_i P(\{a_i\}) \delta(S_{23} - \sqrt{3}|a_3 - a_2|) \\ &= \sqrt{\frac{2A}{\pi}} e^{-(A/2)S_{23}^2}. \end{aligned} \quad (8)$$

Using this, we may define an average spacing $\overline{S_{23}}$ through the first moment and obtain finally a normalized spacing distribution in terms of the variable $z = S_{23}/\overline{S_{23}}$:

$$p_{CC}(z) = \frac{2}{\pi} e^{-z^2/\pi}. \quad (9)$$

Similarly, the spacing distribution $P_{RC}(S_{12})$ is obtained:

$$P_{RC}(S_{12}) = \frac{4A}{\sqrt{3}} S_{12} e^{-(4/3)S_{12}^2} I_0\left(\frac{2}{3}AS_{12}^2\right). \quad (10)$$

The mean spacing turns out to be $\overline{S_{12}} = \frac{3}{8}\sqrt{\frac{\pi}{A}}c$, where $c = {}_2F_1\left[\frac{3}{4}, \frac{5}{4}, 1, \frac{1}{4}\right] = 1.31112\dots$. Defining $z = S_{12}/S_{12}$,

$$P_{RC}(z) = \frac{3\sqrt{3}\pi}{16} c^2 z \exp\left(-\frac{3\pi}{16}c^2z^2\right) I_0\left(\frac{3\pi}{32}c^2z^2\right). \quad (11)$$

We can now make following observations: (i) the Gaussianity of $p_{CC}(z)$ implies that there is no level repulsion among the complex conjugate pairs; and at the same time there is no attraction and there is no tendency of clustering as in Poissonian spacing distribution and (ii) real and complex eigenvalues display linear level repulsion. These results are also borne out by the numerical simulations in Figs. 1 and 2.

For the general case of $N \times N$ matrices, we need to invert Eq. (4). This inversion leads us to the following relation:

$$a_i = \sum_l \mathbf{S}_{il} E_l, \quad (12)$$

where $\mathbf{S}_{il} = \omega^{(i-1)[N-(l-1)]}$ and $\omega = e^{2\pi i/N}$ is a root of unity. \mathbf{S} is a symmetric matrix and $\mathbf{S}^2 = N\eta$. Employing these relations, we can find $\sum_i a_i^2$, and hence the following result for the JPDF for even N :

$$P(\{E_i\}) = \left(\frac{A}{\pi}\right)^{N/2} \exp\left[-A\left(E_1^2 + E_{N/2+1}^2 + \sum_{i \neq 1, N/2+1}^N E_i E_{N+2-i}\right)\right], \quad (13)$$

where E_1 and $E_{N/2+1}$ are real and the rest of the eigenvalues may be complex. For odd N , the above result will hold except that there will be only one real eigenvalue E_1 and the summation in the second term will extend over all i except 1. Employing this general result on JPDF, we can now calculate the spacing distributions for the general case. There are three cases: (i) spacing among the complex conjugate pair of eigenvalues is found to be distributed again as a Gaussian, (ii) spacing between a real and a complex eigenvalue is distributed according to Eq. (11), and (iii) two complex eigenvalues $E_j = x_j + iy_j$ and $E_k = x_k + iy_k$ are spaced according to

$$p(s) = \frac{\int \prod_i d \operatorname{Re} E_i d \operatorname{Im} E_i P(\{E_i\}) \delta(|E_j - E_k| - s)}{\int \prod_i d \operatorname{Re} E_i d \operatorname{Im} E_i P(\{E_i\})}, \quad (14)$$

which reduces to the following integral on change of variables $\xi(\eta)_{\pm} = x(y)_k \pm x(y)_j$

$$p(s) = \frac{A}{\pi} \int d\xi_- d\eta_- e^{-A(\xi_-^2 + \eta_-^2)} \delta(\sqrt{\xi_-^2 + \eta_-^2} - s) = \frac{\pi s}{2} \exp\left(-\frac{\pi s^2}{4}\right) \quad (15)$$

which is exactly the Wigner distribution (Fig. 3). Let us

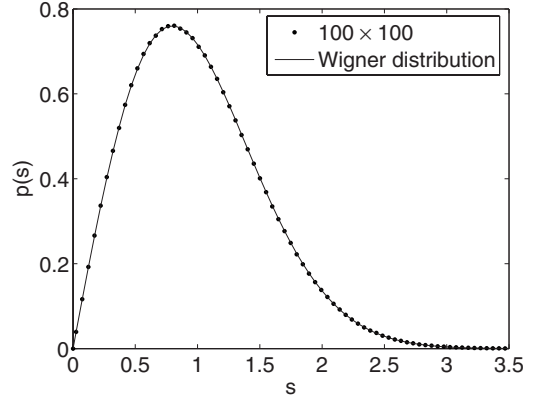


FIG. 3. We observe a linear level repulsion between two eigenvalues which are neither real nor complex conjugate pairs for an ensemble of 100×100 matrices with 5000 realizations. The agreement with GOE is deceptive; in fact, this suggests that the eigenvalues describe a Poisson process on a plane.

recall that Wigner’s result holds exactly for 2×2 real symmetric matrices; it serves as an excellent approximation for $N \times N$, matrices, though. We also know that the spacing distribution for a Poissonian random process in a plane is exactly the same as the Wigner surmise. Thus our result proves that the complex eigenvalues of random cyclic matrices describe such a process. This is a very beautiful, nonintuitive result which brings out yet another characteristic of RCM.

The eigenfunctions of \mathbf{M} corresponding to the real eigenvalues (E_1 and $E_{N/2+1}$) are also simultaneously eigenfunctions of “generalized parity” η . However, the eigenfunctions of \mathbf{M} corresponding to the complex conjugate pair of eigenvalues are not simultaneously eigenfunctions of η . Thus, when these complex eigenvalues occur, “generalized parity” is said to be spontaneously broken. Also, the eigenfunctions corresponding to the complex conjugate pair of eigenvalues have zero \mathcal{PT} norm. This is expected from the recent works [1,17] on \mathcal{PT} -symmetric quantum mechanics. This observation then fully embeds our findings into the new random matrix theory developed recently for pseudo-Hermitian Hamiltonians. However, we also note that the eigenvectors ψ_1 (ψ_2) corresponding to complex conjugate eigenvalues λ (λ^*) satisfy orthogonality defined with respect to η . Since these results are found for $N \times N$ matrices, we believe that this work extends the random matrix theory in a significant way. The findings on the spacing distributions have led us to a linear level repulsion among distinct complex eigenvalues, whereas the spacing between the complex conjugate pair is Gaussian distributed.

The Ginibre orthogonal ensemble with Gaussian distributed real elements has been completely solved only recently [35,36]. The ensemble of asymmetric random cyclic matrices is a simple nontrivial instance for which all the interesting quantities are analytically obtained in an explicit manner. Such examples play an important role in developing a deeper insight, even when formal results exist.

Also, we would like to point out the role played by level repulsion when a system with spectral properties described by RMT approaches equilibrium. In its approach to equilib-

rium, the central quantity of interest is the two-time correlation function; the long-time behavior is decided by the degree of level repulsion as the levels get closer. We can immediately see [15] that linear level repulsion is related to the exponent 2 in the t^{-2} tail at long times.

It is a great pleasure to thank Bob Dorfman, University of Maryland, College Park, for bringing to our attention the role played by cyclic matrices in certain models in statistical mechanics, the work of Ted Berlin and Mark Kac [14] in particular.

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