# Scaling analysis of the site-diluted Ising model in two dimensions

R. Kenna
*Applied Mathematics Research Centre, Coventry University, Coventry, CV1 5FB, England*

J. J. Ruiz-Lorenzo
*Departamento de Fisica, Universidad de Extremadura, Avda Elvas s/n, Badajoz, 06071 Spain*
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A combination of recent numerical and theoretical advances are applied to analyze the scaling behavior of the site-diluted Ising model in two dimensions, paying special attention to the implications for multiplicative logarithmic corrections. The analysis focuses primarily on the odd sector of the model (i.e., that associated with magnetic exponents), and in particular on its Lee-Yang zeros, which are determined to high accuracy. Scaling relations are used to connect to the even (thermal) sector, and a first analysis of the density of zeros yields information on the specific heat and its corrections. The analysis is fully supportive of the strong scaling hypothesis and of the scaling relations for logarithmic corrections.

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## I. INTRODUCTION

The Ising model in two dimensions with uncorrelated, quenched random-site or random-bond disorder is a paradigm for the study of the statistical mechanics of disordered systems. With moderate dilution, the random system exhibits a phase transition different to that of the pure system, and the nature of this transition has been investigated for over three decades.

For the disordered Ising models in two dimensions, two scenarios have arisen. The strong universality hypothesis maintains that the leading critical exponents remain the same as in the pure case and that the disorder induces multiplicative logarithmic corrections to scaling, while the weak universality hypothesis favours dilution-dependent leading critical exponents. While the former is now mostly favored, especially in the random-bond Ising model (RBIM), the debate regarding the scaling behavior of the specific heat has persisted, especially in the site-diluted Ising model (RSIM), focusing on whether this is characterized by an extremely weak double-logarithmic divergence or a finite cusp. Indeed, according to the Harris criterion [1], the vanishing of the specific-heat critical exponent $\alpha$ in the pure Ising model, marks the borderline between the $\alpha > 0$ case, where disorder is relevant and where the critical exponents may change as random quenched disorder is added, and the $\alpha < 0$ scenario, where this type of disorder does not alter critical behavior and the critical exponents are unchanged [1]. In this borderline circumstance, logarithmic corrections to scaling (distinct from the logarithmic divergence in the specific heat in the pure Ising system) may arise.

The issue of scaling in the RSIM is addressed here, as that is the more contentious version of the model. All of the thermodynamic information concerning a statistical mechanical system is contained in the locus and density of its partition function zeros [2]. In Ref. [3], the first numerical analysis of the Lee-Yang zeros in a disordered system (the RSIM) was performed and the leading critical behavior of the lowest lying Lee-Yang zeros was extracted through a finite-size scaling (FSS) analysis. A robust numerical technique to construct the density of zeros from simulational data was given in Ref. [4]. Finally, a self-consistent scaling theory which links the exponents characterising the logarithmic corrections to scaling was recently presented in Refs. [5,6].

Here, these numerical [3,4] and analytical [5,6] advances are combined to unambiguously determine the leading scaling behavior and potential multiplicative logarithmic corrections in the specific heat through the density of Lee-Yang zeros and the scaling relations. This and all multiplicative logarithmic correction-to-scaling exponents for FSS of odd functions are zero. That is, there are no multiplicative logarithmic corrections for the FSS of the magnetic susceptibility, the individual Lee-Yang zeros, and the density of zeros. This comes about through the delicate manner in which the exponents of the logarithms, which are nonzero in thermal scaling, balance each other out. In this way, it is established that the Lee-Yang zeros of disordered systems can be precisely determined numerically, a density-of-zeros analysis is applicable to such a system, also at the level of logarithms, new scaling relations for the logarithmic corrections [5,6] in this model are confirmed and a negative exponent for the contentious specific heat or its multiplicative logarithmic correction is made unlikely.

## II. LOGARITHMIC CORRECTIONS AND SCALING SCENARIOS

Recently a self-consistent scaling theory for logarithmic-correction exponents has been presented [5,6]. Denoting the reduced temperature by $t$ and the reduced external field by $h$, this theory deals with the circumstances where, in the absence of field, the specific heat, magnetization, susceptibility, and correlation length scale, respectively, as

$$C_\infty(t) \sim |t|^{-\alpha}|\ln|t||^{\hat{\alpha}}, \tag{2.1}$$

$$m_\infty(t) \sim |t|^{\beta}|\ln|t||^{\hat{\beta}}, \tag{2.2}$$

$$\chi_\infty(t) \sim |t|^{-\gamma}|\ln|t||^{\hat{\gamma}}, \tag{2.3}$$

 

$$\xi_\infty(t) \sim |t|^{-\nu}|\ln|t||^{\hat{\nu}}. \tag{2.4}$$

At $t=0$ the magnetization scales with reduced field as

$$m_\infty(h) \sim h^{1/\delta}|\ln|t||^{\hat{\delta}}. \tag{2.5}$$

The locus of Lee-Yang zeros along the imaginary $h$-axis is parameterized by $r$ and the Yang-Lee edge, which marks the end of their distribution, is denoted by $r_{\rm YL}(t)$. In the symmetric phase, this scales as

$$r_{\rm YL}(t) \sim |t|^{\Delta}|\ln|t||^{\hat{\Delta}}, \tag{2.6}$$

while the density of these zeros along their locus ($r > r_{\rm YL}$) at criticality ($t=0$) behaves as [4]

$$g_\infty(r) \sim r^{a_2-1}|\ln r|^{\hat{a}_2}. \tag{2.7}$$

The leading critical exponents are related by the standard scaling relations for continuous phase transitions [7]. The theory presented in Refs. [5,6] relates the exponents of the logarithmic corrections in an analogous manner.

With the strong universality hypothesis, the leading critical exponents for the dilute Ising models are identical to their pure counterparts

$$\alpha=0, \quad \beta=\frac{1}{8}, \quad \gamma=\frac{7}{4}, \quad \delta=15, \quad \nu=1, \tag{2.8}$$

with gap exponent

$$\Delta=\beta+\gamma=\frac{15}{8}. \tag{2.9}$$

The latter equation has been verified in Ref. [3]. The exponent $a_2$ characterizing the leading behavior of the density of Lee-Yang zeros in Eq. (2.7) is [5]

$$a_2=\frac{2-\alpha}{\Delta}, \tag{2.10}$$

which gives $a_2=16/15$ if the strong hypothesis (2.8) and (2.9) holds.

Shalaev [8] and later Shankar and Ludwig [9], and then Jug and Shalaev [10], used field theory, bosonization techniques, and conformal invariance to derive theoretical predictions for the logarithmic-correction exponents in the random-bond case with sufficiently small quenched dilution. These corrections, which we term the SSLJ exponents, are [8–10]

$$\hat{\alpha}=0, \quad \hat{\beta}=-\frac{1}{16}, \quad \hat{\gamma}=\frac{7}{8}, \quad \hat{\delta}=0, \quad \hat{\nu}=\frac{1}{2} \tag{2.11}$$

with specific heat diverging as a double logarithm [6,8,9,11]

$$C_\infty(t) \sim |\ln|\ln|t|||. \tag{2.12}$$

The questions addressed in the literature over the past three decades concerned (i) the validity of the strong scaling hypothesis and the veracity of the leading exponents (2.8) in the diluted cases, (ii) the validity of the theoretical derivation of the SSLJ correction exponents (2.11), which relies on certain assumptions regarding the nature of the dilution in the random-bond case (see, also, Ref. [12] for the random-site case), (iii) whether these exponents are dilution independent, (iv) the vanishing of $\hat{\alpha}$ and the validity of the double logarithm in Eq. (2.12), and (v) whether or not these sets of exponents also apply to the random-site version of the model.

For the two-dimensional random Ising models, an alternative scenario to (i) has persisted in the literature. This is the weak universality hypothesis and maintains that certain leading critical exponents change continuously as the concentration of impurity defects is increased [13]. In particular, the exponents $\alpha$, $\beta$, $\gamma$, and $\nu$ are maintained to be dilution dependent while $\delta$, $\eta$ and the ratios $\beta/\nu$ and $\gamma/\nu$ remain independent of the dilution. Furthermore, while agreeing with the SSLJ double-logarithmic form (2.12) for the specific heat, Dotsenko and Dotsenko (DD) used the renormaliztion group to predict [11]

$$\chi_\infty(t) \sim t^{-2}\exp\{-c[\ln(-\ln t)]^2\}, \tag{2.13}$$

where $c$ is a constant related to the concentration of disorder. While the SSLJ and DD predictions for the susceptibility differ substantially, it is fair to say that after much work by various authors the strong hypothesis is now mostly favored (especially in the random-bond model).

The validity of the logarithmic-correction exponents (2.11) proved harder to establish quantitatively. The first direct, clear, quantitative validation of the SSLJ prediction for the magnetic susceptibility (that $\hat{\gamma}=7/8$) came in Ref. [14] through series expansions. While the detailed scaling behavior of Eq. (2.13) has long been ruled out, the weak versus strong controversy persisted. In contrast to the susceptibility, DD and SSLJ agree on the double logarithmic behavior of the specific heat. However, this has been notoriously difficult to confirm numerically, and has been the source of much controversy.

Simulational works, generally supportive of the vanishing of $\alpha$ and $\hat{\alpha}$ and the specific heat diverging as a double logarithm viz. Eq. (2.12), are found in Refs. [15–18] and Refs. [18–20] for the bond-disordered and random-site Ising models, respectively (see also Refs. [12,14,21–27]). Indeed, plots of the measured specific heat as a function of the double logarithm of the lattice extent are contained in Refs. [15,18] and Refs. [18–20] for the RBIM and the RSIM, respectively. However, in Ref. [28] it was claimed that such apparent double-logarithmic FSS behavior does not necessarily imply divergence of the specific heat and numerically based counterclaims that the specific heat remains finite (so that $\alpha<0$ or $\alpha=0$ and $\hat{\alpha}<0$) in the random-bond [28,34] and random-site models [35–38] also exist (see also Refs. [29–33]). The situation is summarized in Table I.

The difficulties in unambiguously discriminating between the weak and strong scenarios on the basis of finite-size data were highlighted in Refs. [28,37,39]. Indeed, in Ref. [14] a fit to

$$c_\infty(t) \propto |\ln t|^{\tilde{\alpha}}, \tag{2.14}$$

was attempted, and it was observed that $\tilde{\alpha}$ decreases from the value of 1 (which corresponds to the pure model) as the strength of disorder is increased. This could be interpreted as supporting almost any reasonable value $\tilde{\alpha}\leq 1$. In Ref. [37] it

TABLE I. Selection of recent works supportive of the weak or strong scaling hypothesis.

| | RBIM | RSIM |
|---|---|---|
| Support strong universality hypothesis | [14,21–24] | [25–27] |
| and theoretical support for $\alpha=\hat{\alpha}=0$ | [8–11] | [12] |
| or numerical support for $\alpha=\hat{\alpha}=0$ | [15–18] | [18–20] |
| Support for weak universality hypothesis | [29] | [30] |
| and theoretical support for finite $C_\infty(t)$ | [31,32] | [33] |
| or numerical support for finite $C_\infty(t)$ | [28,34] | [35–38] |

was pointed out that specific heat data in the literature, which were stated to be supportive of Eq. (2.12), can often equally be fitted to Eq. (2.1) with negative $\alpha$, still consistent with the Rushbrooke relation.

In this paper we address the problem from a fresh perspective, namely that of partition function zeros (see also Ref. [3]). In Ref. [5] the scaling relation

$$\hat{\Delta} = \hat{\beta} - \hat{\gamma} \tag{2.15}$$

for the logarithmic correction to the FSS of the Lee-Yang zeros was derived. For the diluted Ising models in two dimensions, this leads to the prediction

$$\hat{\Delta} = -\frac{15}{16} = -0.9375, \tag{2.16}$$

and verification of this value for the logarithmic correction exponent characterizing the scaling of the Yang-Lee edge in Eq. (2.6) is one of the aims of this work.

In addition to testing the applicability and efficacy of the Lee-Yang-zero technique in random models, also at the level of logarithmic corrections, a central aim of this paper is to measure the density of zeros. Indeed, the correction exponent for the density of zeros is given in Ref. [5] as

$$\hat{a}_2 = \frac{\gamma\hat{\Delta} + \Delta\hat{\gamma}}{\Delta}. \tag{2.17}$$

From the scaling relations for logarithmic corrections, in the special circumstances which prevail the diluted Ising models in two dimensions given in Ref. [6], $\hat{a}_2$ is related to the specific-heat correction exponent $\hat{\alpha}$ appearing in Eq. (2.1) via

$$\hat{\alpha} = 1 + 2\frac{\hat{\Delta}}{\Delta} + \hat{a}_2 = 1 + \frac{16}{15}\hat{\Delta} + \hat{a}_2, \tag{2.18}$$

having used the established value (2.9) for the leading gap exponent [3]. The elusive specific heat scaling exponents $\alpha$, $\hat{\alpha}$ can thus be measured from the density (2.7) together with Eqs. (2.10) and (2.18) and, in this way, one can distinguish between the competing $\alpha=0$, $\hat{\alpha}=0$ and $\alpha<0$ or $\hat{\alpha}<0$ scenarios.

Since it is more contentious, we address the site-diluted version. Unlike the self-dual random-bond version, the location of the critical temperature in the site-diluted Ising model is not exactly known and has to be estimated numerically. In this work, the highly accurate measurements for the critical temperatures reported in Ref. [19] for different values of the site dilution are used.

## III. SIMULATION OF THE RSIM

The partition function for a given realization of the RSIM in a reduced magnetic field $h$ is

$$Z_L(\beta,h) = \sum_{\{\sigma_i\}} \exp\left(\beta \sum_{\langle ij\rangle} \epsilon_i\epsilon_j\sigma_i\sigma_j + h\sum_i \epsilon_i\sigma_i\right), \tag{3.1}$$

where $L$ denotes the linear extent of the lattice and the sum over configurations $\{\sigma_i\}$ is taken over Ising spins $\sigma_i \in \{\pm 1\}$ and where $\epsilon_i$ are independent quenched random variables which take the values unity with probability $p$ and zero with probability $1-p$. For simulational purposes, a regular (square) lattice with periodic boundary conditions is used. The percolation threshold for such a lattice in the thermodynamic limit occurs at $p=p_c=0.592746\cdots$, so that for $p<p_c$ the lattice fragments into finite-size systems on which no true transition can occur [40]. Writing

$$S = \sum_{\langle ij\rangle} \epsilon_i\epsilon_j\sigma_i\sigma_j, \quad M = \sum_i \epsilon_i\sigma_i, \tag{3.2}$$

and

$$\rho_L(\beta;M) = \sum_S \rho_L(S,M)\exp \beta S, \tag{3.3}$$

where the spectral density $\rho_L(S,M)$ gives the relative weight of configurations with given values of $S$ and $M$, the partition function in imaginary field $ih$ is

$$\begin{aligned} Z_L(\beta,h) &= \sum_M \rho_L(\beta;M)\exp(ihM) \\ &= Z_L(\beta,0)\langle\cos(hM) + i\sin(hM)\rangle, \end{aligned} \tag{3.4}$$

where the expectation value has real measure. Assuming the Lee-Yang theorem holds [2,3], since odd moments of the magnetization vanish for $t \geq 0$ ($\beta \leq \beta_c$), the zeros for a given realization of disorder are given by the values of $h$ for which

$$\langle\cos hM\rangle = 0. \tag{3.5}$$

Finally, for each value of $L$ and $p$, these zeros are averaged over realizations of disorder and the resulting $j$th Lee-Yang zero is denoted by $r_j(L)$. Errors associated with the zeros are computed as sample-to-sample fluctuations.

We have simulated, using the Wolff single-cluster algorithm [41], three different values of the dilution, namely, $p=0.88889$, 0.75, and 0.66661. From Ref. [19], the values of the critical temperatures for these three dilutions are $\beta_c=0.53781(2)$ for $p=0.88889$, $\beta_c=0.77125(8)$ for $p=0.75$, while $\beta_c=1.10$ corresponds to $p=0.66661(3)$. In each of these three cases, we have run lattices of extent $L=32$, 48, 64, 96, 128, 196, and 256. The number of samples simulated for each lattice size and each value of the site-occupation probability (and corresponding $\beta_c$ value) is given in Table II.

We have monitored the behavior of the nonlocal observables (such as the susceptibility) with the Monte Carlo time

TABLE II. The number of samples simulated for each lattice size $L$ at each value of the site-occupation probability and corresponding $\beta_c$ value (from Ref. [19]).

| $L$ | | 32 | 48 | 64 | 96 | 128 | 196 | 256 |
|---|---|---|---|---|---|---|---|---|
| $p=0.88889$ | $\beta_c=0.53781$ | 1000 | 1000 | 1000 | 600 | 600 | 250 | 250 |
| $p=0.75$ | $\beta_c=0.77125$ | 1000 | 1000 | 1000 | 1000 | 1000 | 550 | 270 |
| $p=0.66661$ | $\beta_c=1.10$ | 1000 | 1000 | 1000 | 920 | 600 | 270 | 160 |

by using a standard logarithmic binning of the dynamical data. We have checked that, in all the cases, the nonlocal observables have reached a plateau (as a function of the Monte Carlo time).

## IV. SCALING AND DENSITY ANALYSES

The analysis focuses on the odd sector of the RSIM, which is connected to the even sector through the standard scaling relations and their logarithmic counterparts [5,6]. This connection is used to determine the specific-heat exponents through the density of zeros, in addition to a detailed analysis of the susceptibility and the Lee-Yang zeros.

The FSS analyses of the magnetic susceptibility and Yang-Lee edge focuses on the correction-to-scaling exponents. From Eq. (2.4), the reduced temperature is expressed in terms of the correlation length near criticality as

$$t \sim \xi_\infty^{-1/\nu}(\ln \xi_\infty)^{\hat{\nu}/\nu}. \qquad (4.1)$$

Substituting Eq. (4.1) into Eqs. (2.3) and (2.6) gives the scaling behavior for susceptibility and the lowest lying Lee-Yang zeros in terms of the correlation length. Because there are no logarithmic corrections to the FSS behavior of the correlation length [6], for sufficiently large lattices $\xi_\infty$ may be replaced by $L$. This substitution then gives for the FSS of the susceptibility and the $j$th Lee-Yang zeros

$$\chi_L \sim L^{\gamma/\nu}(\ln L)^{(\nu\hat{\gamma}-\gamma\hat{\nu})/\nu}, \qquad (4.2)$$

$$r_j \sim L^{-\Delta/\nu}(\ln L)^{(\nu\hat{\Delta}+\Delta\hat{\nu})/\nu}, \qquad (4.3)$$

respectively. The values for the correction exponents $\hat{\gamma}$ and $\hat{\nu}$ given in Eqs. (2.8) and (2.11) have been numerically established for the RBIM in Refs. [14,22] and that for and $\hat{\nu}$ in the RSIM has been verified in Ref. [27]. Therefore, their confirmation in this setting as $\nu\hat{\gamma}-\gamma\hat{\nu}=0$ serves as a useful check on the accuracy of our method at the logarithmic level.

TABLE III. The estimates for the leading-exponent ratios $\gamma/\nu$ and $\Delta/\nu$ from fits to the scaling behavior of the susceptibility and first Lee-Yang zeros as well as estimates for $(\nu\hat{\gamma}-\gamma\hat{\nu})/\nu$. These estimates agree with the theoretical values, which are 7/4, 15/8, and 0, respectively.

| | $p=0.88889$ | $p=0.75$ | $p=0.66661$ |
|---|---|---|---|
| $\gamma/\nu$ | $1.747\pm0.007$ | $1.755\pm0.005$ | $1.752\pm0.007$ |
| $(\nu\hat{\gamma}-\gamma\hat{\nu})/\nu$ | $-0.01\pm0.03$ | $0.02\pm0.03$ | $0.01\pm0.03$ |
| $\Delta/\nu$ | $1.879\pm0.004$ | $1.878\pm0.006$ | $1.878\pm0.006$ |

The scaling analysis begins with $p=0.88889$ (and $\beta=0.53781$). A double logarithmic plot of $\chi_L$ against $L$ is presented in Fig. 1(a), and a fit to all data points yields $\gamma/\nu=1.747(7)$ in agreement with Eq. (2.8). The goodness of fit corresponds to a $\chi^2$ per degree of freedom ($\chi^2/N_{\rm DF}$) of 0.8. The FSS correction exponent $(\nu\hat{\gamma}-\gamma\hat{\nu})/\nu$ is extracted by a fit, corresponding to Fig. 1(b), of $\ln\chi_L - 7/4\ln L$ against $\ln(\ln L)$, giving a slope $-0.01(3)$ (with $\chi^2/N_{\rm DF}\approx0.8$) compatible with the expected value of zero from Eqs. (2.8) and (2.11). These results are summarized in Table III alongside the results of similar analyses at $p=0.75$ ($\beta=0.77125$) and $p=0.66661$ ($\beta=1.10$). In each case, the $\chi^2/N_{\rm DF}$ indicates a good fit and compatibility with the SSLJ theory is firmly established.

Subleading corrections to Eq. (4.3) are expected to take the form [18]

$$\chi_L = AL^{\gamma/\nu}(\ln L)^{(\nu\hat{\gamma}-\gamma\hat{\nu})/\nu}\left[1+O\left(\frac{1}{\ln L}\right)\right]. \qquad (4.4)$$

The amplitude $A$ of the leading term may firstly be estimated by fitting to $\chi_L=AL^{\gamma/\nu}$. A subsequent fit to the parameters governing corrections to scaling in Eq. (4.4) yields the estimate $(\nu\hat{\gamma}-\gamma\hat{\nu})/\nu=-0.03(2)$ (with $\chi^2/N_{\rm DF}\approx0.8$) for at $p=0.88889$. That is, the inclusion of additive corrections does not lead to an improved estimate for the multiplicative logarithmic exponents. [We have also tested additive corrections of the form $\ln\ln L/\ln L$ in place of $1/\ln L$ in Eq. (4.4). While this leads to small improvement over Eq. (4.4), it also does not significantly affect the estimates for the exponents of the multiplicative logarithms.] This observation holds for all

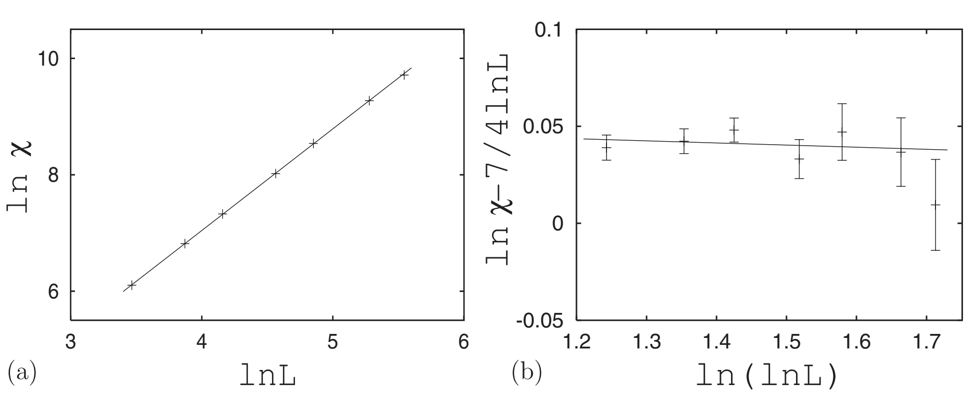


FIG. 1. (a) FSS plot for $\chi_L$ at $p_c=0.88889$. The slope gives an estimate for $\gamma/\nu$ of 1.747(7). (b) Plot of $\ln\chi_L - 7/4\ln L$ against $\ln(\ln L)$ giving slope $-0.01(3)$, indicating no multiplicative logarithmic corrections to the FSS of the susceptibility.

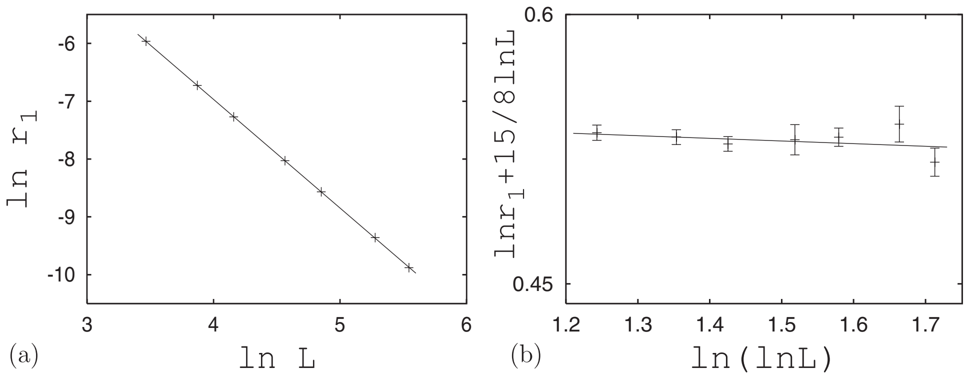


FIG. 2. (a) FSS plot for the lowest Lee-Yang zero at $p_c = 0.88889$. The slope gives an estimate for $\Delta/\nu$ of 1.879(4). (b) Plot of $\ln r_1 + 15/8 \ln L$ against $\ln(\ln L)$ giving slope $-0.01(2)$, indicating the absence of multiplicative logarithmic corrections to the FSS of $r_1$.

quantities analyzed below and for all values of $p$, and we henceforth refrain from reporting on additive corrections.

The analysis of the leading FSS behavior at $p=0.88889$ of the first Lee-Yang zero of Ref. [3] is reconfirmed (to higher precision) in Fig. 2(a), where, ignoring corrections, the slope of the log-log plot yields $\Delta/\nu = 1.879(4)$, compatible with the expected value of 15/8. The $\chi^2/N_{\mathrm{DF}}$ here is 0.7. While the corresponding fit at $p=0.66661$ yields $\Delta/\nu=1.878(6)$, the equivalent fit (using all lattice sizes) for $p=0.75$ gives the estimate 1.883(4), which is two standard deviations from the theoretical value. To avoid contamination due to scaling corrections, the smallest lattices may be removed from the analysis in standard fashion, and compatibility with theory is indeed restored when the smallest pair are removed from the fit; the $L=64$ to $L=256$ data again yields $\Delta/\nu=1.878(6)$ (with $\chi^2/N_{\mathrm{DF}}=1.8$). These reconfirmed results are also summarized in Table III.

To test the scaling relation (2.15) and the prediction (2.16), $\ln r_1 + 15/8 \ln L$ is plotted against $\ln(\ln L)$ in Fig. 2(b) for $p=0.88889$. A fit to all points yields slope

$$\frac{\nu\hat{\Delta} + \Delta\hat{\nu}}{\nu} = -0.01(2), \tag{4.5}$$

compatible with zero. The $\chi^2/N_{\mathrm{DF}}$ here is 0.7. With the by now established theoretical values for $\Delta$, $\nu$ and $\hat{\nu}$ from Eqs. (2.8), (2.9), and (2.11), this yields

$$\hat{\Delta} = -0.95(2), \tag{4.6}$$

a value compatible with (2.16) and confirming Eq. (2.15) in the RSIM. The analysis at $p=0.66661$ gives $(\nu\hat{\Delta}+\Delta\hat{\nu})/\nu = -0.01(3)$, comparable to Eq. (4.5) and compatible with theory [the corresponding $\hat{\Delta}$ value is $-0.95(3)$]. At $p=0.75$ one finds the estimate $-0.04(2)$, using all lattice sizes. Again, compatibility with theory is restored by dropping the two smallest $L$ values, where one again finds $-0.01(3)$ [$\hat{\Delta} = -0.95(3)$]. These estimates for $\hat{\Delta}$ are summarized in Table IV.

Accepting, now, that the values $\Delta = 15/8 = 1.875$ from Eq. (2.9) and $\hat{\Delta} = -15/16 = -0.9375$ from Eq. (2.16) are supported by the data for each dilution level, Eqs. (2.10) and (2.18) give the scaling exponents for the specific heat. In particular, the logarithmic correction exponent is identical to that of the density of zeros

$$\hat{\alpha} = \hat{a}_2. \tag{4.7}$$

Therefore $\alpha = 2 - 16a_2/15$ and $\hat{\alpha} = \hat{a}_2$ can be extracted from the scaling form (2.7) for the density of zeros, a form which is expected to hold for small enough $r$ (i.e., close to the critical region).

A robust method to determine the density of zeros from simulational data was developed in Ref. [4]. Define the density of zeros along the singular line $r > r_{\mathrm{YL}}(t)$ as

$$g_L(r) = L^{-d} \sum_j \delta[r - r_j(L)], \tag{4.8}$$

where $r_j(L)$ is the position of the $j$th zero for a lattice of extent $L$. Here, $j$ is called the index of the zero. Integrating along the locus of zeros gives the cumulative density of zeros (or index density) to be

$$G_L(r) = \int_0^r g_L(s) ds = \frac{j}{L^d} \quad \text{for } r_j(L) < r < r_{j+1}(L), \tag{4.9}$$

so that at a zero, it is given by the average

$$G_L[r_j(L)] = \frac{2j-1}{2L^d}. \tag{4.10}$$

From Eq. (2.7), this may be fitted to the form

$$G(r) = a_1 r^{a_2} (\ln r)^{\hat{a}_2} + a_3, \tag{4.11}$$

allowing for an additional parameter $a_3$ which determines the phase. A value of $a_3$ greater or less than zero indicates that the system is in the broken or symmetric phase, respectively, so that $a_3=0$ only at the transition point $t=0$. The second criterion for a good fit is good data collapse. Note that this

TABLE IV. The estimates for the specific heat exponent $\alpha$ and logarithmic-correction exponents $\hat{\Delta}$ and $\hat{\alpha}$. These are in agreement with the theoretical values from the strong universality hypothesis, namely, $\alpha=0$, $\hat{\Delta}=-15/16$, and $\hat{\alpha}=0$.

| | $p=0.88889$ | $p=0.75$ | $p=0.66661$ |
|---|---|---|---|
| $\hat{\Delta}$ | $-0.95 \pm 0.02$ | $-0.95 \pm 0.03$ | $-0.95 \pm 0.03$ |
| $\alpha$ | $-0.02 \pm 0.03$ | $0.01 \pm 0.02$ | $0.00 \pm 0.03$ |
| $\hat{\alpha}$ | $-0.02 \pm 0.05$ | $-0.01 \pm 0.03$ | $-0.04 \pm 0.05$ |

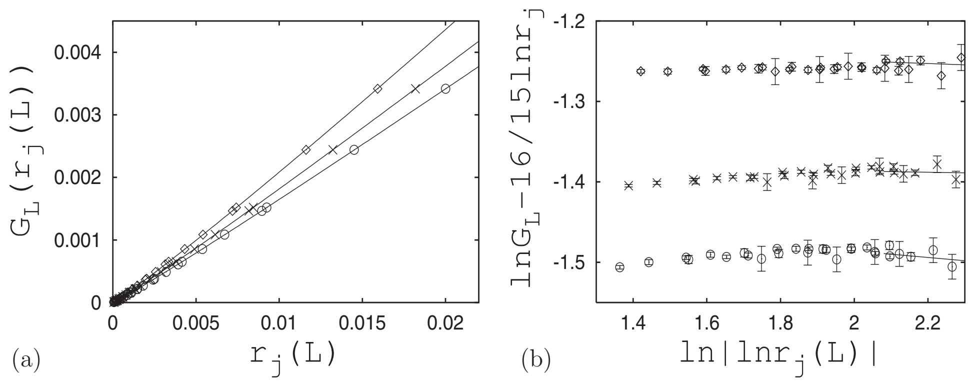


FIG. 3. (a) The integrated density of zeros at $p=0.88889$ (diamonds), $p=0.75$ (crosses) and $p=0.66661$ (circles). The excellent data collapse (four data points for each of seven lattice sizes are involved at each value of $p$) and zero intercept indicate the correct transition point and the fitted curves give $a_2$ compatible with the expected value 16/15 ($\alpha=0$). (b) The absence of multiplicative logarithmic corrections in the integrated density of zeros indicates that the specific heat logarithm exponent $\hat{\alpha}=0$ independent of the degree of dilution.

method does not allow an independent goodness-of-fit test [4].

The integrated density of zeros is plotted in Fig. 3(a) using the first four Lee-Yang zeros for lattices from size $L=32$ to 256 (28 points in all) for each value of the dilution, demonstrating excellent data collapse in each case. For $p=0.88889$, three-parameter fits to Eq. (4.11), for small $r$, with $\hat{a}_2=0$ are supportive of the theoretical values of $a_2=16/15$ and $a_3=0$; using the eight lowest data points, one obtains $a_3=0.0000002(4)$ and $a_2=1.076(16)$. The latter results corresponds to the estimate $\alpha=-0.02(3)$, from (2.10). The corresponding results in the $p=0.75$ and $p=0.66661$ cases are $a_3=-0.0000002(3)$, $a_2=1.062(10)$ [$\alpha=0.01(2)$] and $a_3=-0.0000001(4)$, $a_2=1.066(15)$ [$\alpha=0.00(3)$], respectively. These results for $\alpha$ are gathered in Table IV.

We now accept that $a_3$ is indeed zero and the theoretical value $a_2=16/15$ ($\alpha=0$) from (2.10) holds for each dilution. Potential multiplicative logarithmic corrections are detected by plotting $\ln G-16/15 \ln r$ against $\ln(\ln r)$ in Fig. 3(b). A fit to all data points for $p=0.88889$ gives $\hat{\alpha}=\hat{a}_2=0.012(3)$, which is four standard deviations from the theoretical value of zero. However, focusing on the scaling region closer to the origin establishes compatibility with the theory. For example, fitting to the lowest eight data points yields $\hat{\alpha}=\hat{a}_2=-0.02(5)$. The equivalent results for $p=0.75$ and $p=0.66661$ are $\hat{\alpha}=\hat{a}_2=-0.01(3)$ and $-0.04(5)$, respectively. The corresponding fits are depicted in Fig. 3(b) and the estimates for $\hat{\alpha}$ are summarized in Table IV. These values constitute numerical evidence that $\alpha=\hat{\alpha}=0$, independent of dilution and in favour of strong universality.

## V. CONCLUSIONS

The debate regarding the critical behavior of the disordered Ising model in two dimensions has persisted for over thirty years (most recently in Refs. [18,38]). Here the SSLJ prediction [8–10] for the multiplicative logarithmic corrections for the scaling behavior of the susceptibility has been reconfirmed through a careful FSS analysis. In addition to this, the Lee-Yang zeros have been determined to high accuracy and their logarithmic corrections verified for the first time.

The scaling behavior of the specific heat in the site-diluted version of the model has been particularly difficult to pin down directly, and fits to the measured specific heat as a function of the double logarithm of the lattice extent [15,18–20] have been claimed not to be unambiguous [28]. Here an alternative approach has been taken, involving the density of Lee-Yang zeros. Using scaling relations [5,6] to connect to the even sector of the model, the specific-heat scaling and correction exponents are clearly determined. Since the simulations are performed at three different values of the dilution (some quite large), the analyses presented herein for the susceptibility, the individual Lee-Yang, zeros, and for their densities, are unambiguously supportive of the strong scaling hypothesis.

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