Effective field theory for models defined over small-world networks: First- and second-order phase transitions

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(Received 24 January 2008; revised manuscript received 17 June 2008; published 2 September 2008)

We present an effective field theory to analyze, in a very general way, models defined over small-world networks. Even if the exactness of the method is limited to the paramagnetic regions and to some special limits, it provides, yielding a clear and immediate (also in terms of calculation) physical insight, the exact critical behavior and the exact critical surfaces and percolation thresholds. The underlying structure of the nonrandom part of the model—i.e., the set of spins filling up a given lattice $\mathcal{L}_0$ of dimension d_0 and interacting through a fixed coupling J_0—is exactly taken into account. When $J_0 \geqslant 0$, the small-world effect gives rise, as is known, to a second-order phase transition that takes place independently of the dimension d_0 and of the added random connectivity c. When $J_0 < 0$, a different and novel scenario emerges in which, besides a spin-glass transition, multiple first- and second-order phase transitions may take place. As immediate analytical applications we analyze the Viana-Bray model ($d_0=0$), the one-dimensional chain ($d_0=1$), and the spherical model for arbitrary d_0.

DOI: 10.1103/PhysRevE.78.031102 PACS number(s): 05.50.+q, 64.60.aq, 64.70.P−

I. INTRODUCTION

Since the very beginning of the pioneering work of Watts and Strogatz [1], interest in small-world networks—an interplay between random and regular networks—has been growing "exponentially." Mainly, there are two reasons that have caused such a "diffusion."

The first reason is due to the topological properties of the small-world network. In synthesis, if N is the size of the system, for any finite probability p of rewiring or for any finite added random connectivity c (the two cases correspond to slightly different procedures for building a small-world network) one has a "short-distance behavior," implying that the average shortest distance between two arbitrarily chosen sites grows as $l(N) \sim \ln(N)$, as in random networks, and a large clustering coefficient, $C(N) \sim O(1)$, as in regular lattices. The interplay between these two features makes small-world networks representative of many realistic situations including social networks, communications networks, chemical reactions networks, protein networks, neuronal networks, etc.

The second reason is due the fact that, in models defined over small-world networks, despite the presence of an underlying finite-dimensional structure—a lattice $\mathcal{L}_0$ of dimension $d_0 < \infty$—the existence of shortcut bonds makes such models mean-field like and, hopefully, exactly solvable. However, even if such a claim sounds intuitively correct, the complexity of these models turns out to be in general quite high and, compared to numerical works, there are still few exact results for small-world networks [2–9] (for the percolation and synchronization problem, see [10,11]).

In particular, for $d_0 > 1$ a mean-field critical behavior is expected and has been also supported by Monte Carlo (MC) simulations [12]. Some natural questions then arise. Are we able to prove analytically such a behavior? If, for example, $d_0=2$, does the mean-field critical behavior hold for any situation? Yet does the correlation length diverge at the critical temperature?

Furthermore, even if for $d_0=1$ an exact analytical treatment has been developed at the level of replica symmetry breaking (RSB) [5] and one-step replica symmetry breaking (1RSB) [6], the calculations are quite involved and the solutions of the coupled equations to evaluate the order parameters require a certain numerical work, which becomes rapidly hard in the 1RSB case. In any case, even if these methods are able to give in principle exact results at any temperature, they are not in general suitable to provide a clear simple and immediate physical picture of the model, even within some approximations. The main problem in fact resides in the presence of short loops: as soon as $d_0 > 1$ these loops cannot be neglected and the "traditional" cavity and replica methods seem hardly applicable. In particular, we are not able to predict what happens, for example, if we set J_0 negative. Should we still expect a second-order phase transition? And what about the phase diagram?

In this paper, we present a general method to study random Ising models defined on small-world graphs built up by adding a random connectivity c over an underlying arbitrary lattice $\mathcal{L}_0$ having dimension d_0. We will show that this method, in a very simple and physically sound way, provides an answer to the above questions as well as to many others.

Roughly speaking, as an effective field theory the method generalizes the Curie-Weiss mean-field equation $m = \tanh(\beta J m)$ to take into account the presence of the short-range couplings J_0 besides the long-range ones J. As we will show, the magnetization m of the model defined over the small-world network, the *random model* for short, behaves as the magnetization m_0 of the model defined over $\mathcal{L}_0$, the *unperturbed model* for short, but immersed in an effective external field to be determined self-consistently. Even if the exactness of this method is limited to the paramagnetic (P) region, it provides the exact critical behavior and the exact critical surfaces, as well as simple qualitative good estimates of the correlation functions in the ferromagnetic (F) and spin-glass (SG) regions. Furthermore, in unfrustrated systems, the method becomes exact at any temperature in the two limits $c \to 0^+$ and $c \to \infty$.

The consequences of such a general result are remarkable from both the theoretical and the practical point of view. Once the explicit form of the magnetization of the unperturbed model, $m_0 = m_0(\beta J_0, \beta h)$, is known, analytically or numerically, as a function of the couplings J_0 and of the external field h, we get an approximation to the full solution of the random model, which is analytical or numerical, respectively, and becomes exact in the P region. If we do not have $m_0 = m_0(\beta J_0, \beta h)$, but we know at least some of its properties, we can still use these properties to derive certain exact relations and the critical behavior of the random model.

In the first part of the paper, after presenting the self-consistent equations, we focus on their application for a general study of the critical surfaces and of the critical behavior. In the second part, we apply the method to study models of interest which can be solved analytically (and very easily) as for them we know $m_0(\beta J_0, \beta h)$: the Viana-Bray model (which can be seen as a $d_0 = 0$ dimensional small-world model), the one-dimensional chain small-world model, and the spherical small-world model in arbitrary d_0 dimension.

We stress that the critical surfaces as well as the correlation functions in the P region provided by the present method are exact and not based on any special ansatz as the replica-symmetry and the treelike ansatz. We prove in particular that independently of the added random connectivity c, of the underlying dimension d_0, of the structure of the underlying lattice $\mathcal{L}_0$, and of the nature of the phase transition present in the unperturbed model (if any), for $J_0 \geqslant 0$ we always have a second-order phase transition with the classical mean-field critical indices, but with a finite correlation length if calculated along the "Euclidean distance" defined in $\mathcal{L}_0$; on the other hand, for $J_0 < 0$ we show that, as soon as c is sufficiently large, there exist at least two critical temperatures which, depending on the behavior of $\chi_0(\beta J_0, \beta h)$—the susceptibility of the unperturbed system—correspond to first-or second-order phase transitions. This phenomenon will be explicitly shown in the example of the one-dimensional small-world model. Note that, as will result from the detailed analysis of the self-consistent equation (Sec. III B), in any case, the critical behavior of the unperturbed model, if any, can never influence the behavior of the random model.

The paper is organized as follows. In Sec. II we introduce the class of small-world networks over which we define the random Ising models, stressing some important differences concerning the definition of the correlation functions with respect to those usually considered in "ordinary" random models. In Sec. III we present our method: in Sec. III A we provide the self-consistent equations and their relations with physical correlation functions, in Sec. III B we analyze the stability of the solutions of the self-consistent equations and the critical surface and behavior of the system. We separate Sec. III B into the subcases $J_0 \geqslant 0$ and $J_0 < 0$. In Sec. III C we discuss the limits of the method. In Sec. III D we study the stability between the F and SG phases and the phase diagram. Finally, in Sec. III E we mention how to generalize the method to cases with more different short-range couplings J_0 and to analyze possible disordered antiferromagnetic systems. In Secs. IV–VI the theory is applied to the three above-mentioned example cases. The successive Secs. VII–IX are devoted to the derivation of the method. The starting point of the proof is given in Sec. VII and is based on a general mapping between a random model and a nonrandom one [13–15] suitably adapted to the present case. The self-consistent equations are then easily derived in Sec. VIII. Note that, apart from the equations concerning the stability between the P-F and the P-SG transitions, which are derived in Sec. IX, the derivations of the equations presented in Sec. III B are mostly left to the reader, since they can be easily obtained by standard arguments of statistical mechanics using the Landau free energy $\psi(m)$ that we provide and that is derived in Sec. VIII too. Finally, in Sec. X we draw some conclusions. In the Appendix we generalize the method to inhomogeneous external fields to make clear the subtle behavior of the correlation functions in small-world models.

II. RANDOM ISING MODELS ON SMALL-WORLD NETWORKS

The family of models we shall consider are random Ising models constructed by superimposing random graphs with finite average connectivity c onto some given lattice $\mathcal{L}_0$ whose set of bonds (i,j) and dimension will be indicated by Γ_0 and d_0, respectively. Given an Ising model (the unperturbed model) of N spins coupled over $\mathcal{L}_0$ through a coupling J_0 with Hamiltonian

$$H_0 \stackrel{def}{=} -J_0 \sum_{(i,j)\in\Gamma_0} \sigma_i\sigma_j - h\sum_i \sigma_i \tag{1}$$

and given an ensemble $\mathcal{C}$ of unconstrained random graphs $\boldsymbol{c}$, $\boldsymbol{c} \in \mathcal{C}$, whose bonds are determined by the adjacency matrix elements $c_{i,j} = 0, 1$, we define the corresponding small-world model as described by the Hamiltonian

$$H_{\boldsymbol{c},\boldsymbol{J}} \stackrel{def}{=} H_0 - \sum_{i<j} c_{ij} J_{ij} \sigma_i \sigma_j, \tag{2}$$

the free energy F and the averages $\overline{\langle \mathcal{O}\rangle^l}$, with $l = 1, 2$, being defined in the usual (quenched) way as ($\beta = 1/T$)

$$-\beta F \stackrel{def}{=} \sum_{\boldsymbol{c}\in\mathcal{C}} P(\boldsymbol{c}) \int d\mathcal{P}(\boldsymbol{J}) \ln(Z_{\boldsymbol{c},\boldsymbol{J}}) \tag{3}$$

and

$$\overline{\langle \mathcal{O}\rangle^l} \stackrel{def}{=} \sum_{\boldsymbol{c}\in\mathcal{C}} P(\boldsymbol{c}) \int d\mathcal{P}(\boldsymbol{J}) \langle \mathcal{O}\rangle^l_{\boldsymbol{c},\boldsymbol{J}}, \quad l = 1, 2, \tag{4}$$

where $Z_{\boldsymbol{c},\boldsymbol{J}}$ is the partition function of the quenched system,

$$Z_{\boldsymbol{c},\boldsymbol{J}} = \sum_{\{\sigma_i\}} e^{-\beta H_{\boldsymbol{c},\boldsymbol{J}}(\{\sigma_i\})}, \tag{5}$$

$\langle \mathcal{O}\rangle_{\boldsymbol{c},\boldsymbol{J}}$ the Boltzmann average of the quenched system ($\langle \mathcal{O}\rangle$ depends on the given realization of the J's and of $\boldsymbol{c}$: $\langle \mathcal{O}\rangle = \langle \mathcal{O}\rangle_{\boldsymbol{c};\boldsymbol{J}}$; for shortness, we later will omit to write these dependences),

$$\langle \mathcal{O} \rangle_{\boldsymbol{c},\boldsymbol{J}} \overset{def}{=} \frac{\sum_{\{\sigma_i\}} \mathcal{O} e^{-\beta H_{\boldsymbol{c},\boldsymbol{J}}(\{\sigma_i\})}}{Z_{\boldsymbol{c},\boldsymbol{J}}}, \qquad (6)$$

and $d\mathcal{P}(\boldsymbol{J})$ and $P(\boldsymbol{c})$ are two product measures given in terms of two normalized measures $d\mu(J_{i,j}) \geq 0$ and $p(c_{i,j}) \geq 0$, respectively:

$$d\mathcal{P}(\boldsymbol{J}) \overset{def}{=} \prod_{(i,j),i<j} d\mu(J_{i,j}), \qquad \int d\mu(J_{i,j}) = 1, \qquad (7)$$

$$P(\boldsymbol{c}) \overset{def}{=} \prod_{(i,j),i<j} p(c_{i,j}), \qquad \sum_{c_{i,j}=0,1} p(c_{i,j}) = 1. \qquad (8)$$

The variables $c_{i,j} \in \{0,1\}$ specify whether a "long-range" bond between the sites i and j is present ($c_{i,j}=1$) or absent ($c_{i,j}=0$), whereas the $J_{i,j}$'s are the random variables of the given bond (i,j). For the $J_{i,j}$'s we will not assume any particular distribution, while, to be specific, for the $c_{i,j}$'s we shall consider the distribution

$$p(c_{ij}) = \frac{c}{N}\delta_{c_{ij},1} + \left(1 - \frac{c}{N}\right)\delta_{c_{ij},0}. \qquad (9)$$

This choice leads in the thermodynamic limit $N \to \infty$ to a number of long-range connections per site distributed according to a Poisson law with mean $c > 0$ (so that in average there are in total $cN/2$ bonds). Note, however, that the main results we report in the next section are easily generalizable to any case in which Eq. (8) holds, or holds only in the thermodynamic limit due a sufficiently small number of constrains among the matrix elements $c_{i,j}$.

When we will need to be specific, for the $J_{i,j}$'s we will assume either the distribution

$$\frac{d\mu(J_{i,j})}{dJ_{i,j}} = \delta(J_{i,j} - J) \qquad (10)$$

or

$$\frac{d\mu(J_{i,j})}{dJ_{i,j}} = p\,\delta(J_{i,j} - J)dJ_{i,j} + (1-p)\,\delta(J_{i,j} + J) \qquad (11)$$

to consider ferromagnetism or glassy phases, respectively. In Eq. (11), $p \in [0,1]$.

The quantities of major interest are the averages and the quadratic averages of the correlation functions which for shortness will be indicated by $C^{(1)}$ and $C^{(2)}$. For example, the following are nonconnected correlation functions of order k:

$$C^{(1)} = \overline{\langle \sigma_{i_1} \cdots \sigma_{i_k} \rangle}, \qquad (12)$$

$$C^{(2)} = \overline{\langle \sigma_{i_1} \cdots \sigma_{i_k} \rangle^2}, \qquad (13)$$

where $k \geq 1$ and the indices $i_1, \ldots, i_k$ are supposed all different. For shortness we will continue to use the symbols $C^{(1)}$ and $C^{(2)}$ also for the connected correlation function since, as we shall see in the next section, they obey to the same rules of transformations. We point out that the set of indices $i_1, \ldots, i_k$ is fixed along the process of the two averages. This implies in particular that, if we consider the spin with index i and the spin with index j, their distance remains undefined, or more precisely, the only meaningful distance between i and j is the distance defined over $\mathcal{L}_0$—i.e., the Euclidean distance between i and j—which we will indicate as $\|i-j\|_0$.

Therefore, throughout this paper, it must be kept in mind that, for example, $C^{(1)}(\|i-j\|_0) = \overline{\langle \sigma_i \sigma_j \rangle}$ is very different from the correlation function $G^{(1)}(l)$ of two points at a fixed distance l, l being here the distance defined over both $\mathcal{L}_0$ and the random graph $\boldsymbol{c}$—i.e., the minimum number of bonds to join two points among both the bonds of Γ_0 and the bonds of the random graph $\boldsymbol{c}$. In fact, if, for $J_0=0$, one considers all possible realizations of the Poisson graph and then all possible distances l between two given points i and j, one has

$$C^{(1)}(\|i-j\|_0) = \overline{\langle \sigma_i \sigma_j \rangle} - \overline{\langle \sigma_i \rangle \langle \sigma_j \rangle} = \sum_{l=1}^{N} P_N(l) G^{(1)}(l), \qquad (14)$$

where here $P_N(l)$ is the probability that, in the system with N spins, the shortest path between the vertices i and j has length l. If we now use $G^{(1)}(l) \sim [\tanh(\beta J)]^l$ [16] (in the P region holds the equality) and the fact that the average of l with respect to $P_N(l)$ is of the order $\ln(N)$, we see that the two-point connected correlation function (14) goes to 0 in the thermodynamic limit. Similarly, all the connected correlation functions defined in this way are zero in this limit. Note, however, that this independence of the variables holds only if $J_0=0$. This discussion will be more deeply analyzed along the proof by using another point of view, based on mapping the random model to a suitable fully connected model.

III. EFFECTIVE FIELD THEORY

A. Self-consistent equations

Depending on the temperature T and on the parameters of the probability distributions, $d\mu(\cdot)$ and $p(\cdot)$, the random model may stably stay in either the P, F, or SG phase. In our approach for the F and SG phases there are two natural order parameters that will be indicated by $m^{(\mathrm{F})}$ and $m^{(\mathrm{SG})}$. Similarly, for any correlation function, quadratic or not, there are two natural quantities indicated by $C^{(\mathrm{F})}$ and $C^{(\mathrm{SG})}$, and which in turn will be calculated in terms of $m^{(\mathrm{F})}$ and $m^{(\mathrm{SG})}$, respectively. To avoid confusion, it should be kept in mind that in our approach, for any observable $\mathcal{O}$, there are, in principle, always two solutions that we label as F and SG, but as we shall discuss in Sec. III D, for any temperature, only one of the two solutions is stable and useful in the thermodynamic limit.

In the following, we will use the label "0" to specify that we are referring to the unperturbed model with Hamiltonian (1). Note that all the equations presented in this paper have meaning and usefulness also for sufficiently large but finite size N. For shortness we shall omit to write the dependence on N.

Let $m_0(\beta J_0, \beta h)$ be the stable magnetization of the unperturbed model with coupling J_0 and in the presence of a uniform external field h at inverse temperature β. Then, the order parameters $m^{(\Sigma)}$, $\Sigma = \mathrm{F}, \mathrm{SG}$, satisfy the self-consistent decoupled equations

$$m^{(\Sigma)} = m_0(\beta J_0^{(\Sigma)}, \beta J^{(\Sigma)} m^{(\Sigma)} + \beta h), \tag{15}$$

where the effective couplings $J^{(\mathrm{F})}$, $J^{(\mathrm{SG})}$, $J_0^{(\mathrm{F})}$, and $J_0^{(\mathrm{SG})}$ are given by

$$\beta J^{(\mathrm{F})} = c \int d\mu(J_{i,j}) \tanh(\beta J_{i,j}), \tag{16}$$

$$\beta J^{(\mathrm{SG})} = c \int d\mu(J_{i,j}) \tanh^2(\beta J_{i,j}), \tag{17}$$

$$J_0^{(\mathrm{F})} = J_0, \tag{18}$$

and

$$\beta J_0^{(\mathrm{SG})} = \tanh^{-1}[\tanh^2(\beta J_0)]. \tag{19}$$

Note that $|J_0^{(\mathrm{F})}| > J_0^{(\mathrm{SG})}$.

For the correlation functions $C^{(\Sigma)}$, $\Sigma=\mathrm{F},\mathrm{SG}$, for sufficiently large N we have

$$C^{(\Sigma)} = C_0(\beta J_0^{(\Sigma)}, \beta J^{(\Sigma)} m^{(\Sigma)} + \beta h) + O\left(\frac{1}{N}\right), \tag{20}$$

where $C_0(\beta J_0, \beta h)$ is the correlation function of the unperturbed (nonrandom) model.

Concerning the free energy density f we have

$$\begin{aligned}\beta f^{(\Sigma)} = &-\frac{c}{2}\int d\mu(J_{i,j})\ln[\cosh(\beta J_{i,j})] \\ &-\lim_{N\to\infty}\frac{1}{N}\sum_{(i,j)\in\Gamma_0}\ln[\cosh(\beta J_0)] - \ln[2\cosh(\beta h)] \\ &+\left\{\lim_{N\to\infty}\frac{1}{N}\sum_{(i,j)\in\Gamma_0}\ln[\cosh(\beta J_0^{(\Sigma)})] + \ln[2\cosh(\beta h)]\right\} \\ &\times\frac{1}{l} + \frac{1}{l}L^{(\Sigma)}(m^{(\Sigma)}),\end{aligned} \tag{21}$$

where $l=1,2$ for $\Sigma=\mathrm{F},\mathrm{SG}$, respectively, and the nontrivial free energy term $L^{(\Sigma)}$ is given by

$$L^{(\Sigma)}(m) \overset{def}{=} \frac{\beta J^{(\Sigma)}(m)^2}{2} + \beta f_0(\beta J_0^{(\Sigma)}, \beta J^{(\Sigma)} m + \beta h), \tag{22}$$

$f_0(\beta J_0, \beta h)$ being the free energy density in the thermodynamic limit of the unperturbed model with coupling J_0 and in the presence of an external field h, at inverse temperature β.

For given β, among all the possible solutions of Eqs. (15), in the thermodynamic limit, for both $\Sigma=\mathrm{F}$ and $\Sigma=\mathrm{SG}$, the true solution $\bar{m}^{(\Sigma)}$, or leading solution, is the one that minimizes $L^{(\Sigma)}$:

$$L^{(\Sigma)}(\bar{m}^{(\Sigma)}) = \min_{m\in[-1,1]} L^{(\Sigma)}(m). \tag{23}$$

Finally, let k be the order of a given correlation function $C^{(1)}$ or $C^{(2)}$. The averages and the quadratic averages over the disorder, $C^{(1)}$ and $C^{(2)}$, are related to $C^{(\mathrm{F})}$ and $C^{(\mathrm{SG})}$, as follows:

$$C^{(1)} = C^{(\mathrm{F})}, \quad \text{in F}, \tag{24}$$

$$C^{(1)} = 0, \quad k \text{ odd}, \quad \text{in SG}, \tag{25}$$

$$C^{(1)} = C^{(\mathrm{SG})}, \quad k \text{ even}, \quad \text{in SG}, \tag{26}$$

and

$$C^{(2)} = (C^{(\mathrm{F})})^2, \quad \text{in F}, \tag{27}$$

$$C^{(2)} = (C^{(\mathrm{SG})})^2, \quad \text{in SG}. \tag{28}$$

From Eqs. (27) and (28) for $k=1$, we note that the Edward-Anderson order parameter [17] $C^{(2)}=\overline{\langle\sigma\rangle^2}=q_{EA}$ is equal to $(C^{(\mathrm{SG})})^2=(m^{(\mathrm{SG})})^2$ only in the SG phase, whereas in the F phase we have $q_{EA}=(m^{(\mathrm{F})})^2$. Therefore, since $m^{(\mathrm{SG})} \neq m^{(\mathrm{F})}$, $m^{(\mathrm{SG})}$ is not equal to $\sqrt{q_{EA}}$; in our approach, $m^{(\mathrm{SG})}$ represents a sort of a spin-glass order parameter [18].

The localization and the reciprocal stability between the F and SG phases will be discussed in Sec. III D. Note, however, that, at least for lattices $\mathcal{L}_0$ having only loops of even length, the stable P region is always that corresponding to a P-F phase diagram, so that in the P region the correlation functions must be calculated only through Eqs. (24) and (27).

As an immediate consequence of Eq. (15) we get the susceptibility $\tilde{\chi}^{(\Sigma)}$ of the random model:

$$\tilde{\chi}^{(\Sigma)} = \frac{\tilde{\chi}_0(\beta J_0^{(\Sigma)}, \beta J^{(\Sigma)} m^{(\Sigma)} + \beta h)}{1 - \beta J^{(\Sigma)} \tilde{\chi}_0(\beta J_0^{(\Sigma)}, \beta J^{(\Sigma)} m^{(\Sigma)} + \beta h)}, \tag{29}$$

where $\tilde{\chi}_0$ stands for the susceptibility χ_0 of the unperturbed model divided by β (we will adopt throughout this dimensionless definition of the susceptibility),

$$\tilde{\chi}_0(\beta J_0, \beta h) \overset{def}{=} \frac{\partial m_0(\beta J_0, \beta h)}{\partial(\beta h)} = \frac{1}{\beta}\frac{\partial m_0(\beta J_0, \beta h)}{\partial h}, \tag{30}$$

and similarly for the random model.

For the case $\Sigma=\mathrm{F}$ without disorder $[d\mu(J') = \delta(J'-J)dJ']$, Eq. (29) was already derived in [7] by series expansion techniques at zero field $(h=0)$ in the P region (where $m=0$).

Another remarkable consequence of our theory comes from Eq. (20). We see in fact that in the thermodynamic limit any correlation function of the random model fits with the correlation function of the unperturbed model but immersed in an effective field that is exactly zero in the P region and zero external field $(h=0)$. In other words, in terms of correlation functions, in the P region, the random model and the unperturbed model are indistinguishable (modulo the transformation $J_0 \to J_0^{(\mathrm{SG})}$ for $\Sigma=\mathrm{SG}$). Note, however, that this assertion holds only for a given correlation function calculated in the thermodynamic limit. In fact, the corrective $O(1/N)$ term appearing on the right-hand side (rhs) of Eq. (20) cannot be neglected when we sum the correlation functions over all the sites $i \in \mathcal{L}_0$, as to calculate the susceptibility; yet it is just this corrective $O(1/N)$ term that gives rise to the singularities in the random model.

More precisely, for the two-point connected correlation function

$$\tilde{\chi}_{i,j}^{(\Sigma)} \overset{def}{=} \overline{\langle\sigma_i\sigma_j\rangle^l - \langle\sigma_i\rangle^l\langle\sigma_j\rangle^l}, \tag{31}$$

where $l=1,2$ for $\Sigma=\mathrm{F}$, SG, respectively, if

$$\tilde{\chi}_{0;i,j} \overset{def}{=} \langle\sigma_i\sigma_j\rangle_0 - \langle\sigma_i\rangle_0\langle\sigma_j\rangle_0, \tag{32}$$

we have

$$\tilde{\chi}^{(\Sigma)}_{i,j} = \tilde{\chi}_{0;i,j}(\beta J^{(\Sigma)}_0, \beta J^{(\Sigma)} m^{(\Sigma)} + \beta h) + \frac{\beta J^{(\Sigma)}}{N} \times \frac{[\tilde{\chi}_0(\beta J^{(\Sigma)}_0, \beta J^{(\Sigma)} m^{(\Sigma)} + \beta h)]^2}{1 - \beta J^{(\Sigma)} \tilde{\chi}_0(\beta J^{(\Sigma)}_0, \beta J^{(\Sigma)} m^{(\Sigma)} + \beta h)}, \tag{33}$$

where the dependence on N in $\tilde{\chi}^{(\Sigma)}_{i,j}$ and $\tilde{\chi}_0$ is understood. Equation (33) clarifies the structure of the correlation functions on small-world models. On the rhs we have two terms: the former is a distance-dependent short-range term whose finite correlation length, for $T \neq T^{(\Sigma)}_{c0}$ ($T^{(\Sigma)}_{c0}$ being the critical temperature of the unperturbed model with coupling $J^{(\Sigma)}_0$), makes it normalizable; the latter is instead a distance-independent long-range term which turns out to be normalizable thanks to the $1/N$ factor. Once summed, both the terms give a finite contribution to the susceptibility. It is immediate to verify that by summing $\tilde{\chi}^{(\Sigma)}_{i,j}$ over all the indices $i,j \in \mathcal{L}_0$ and dividing by N we get back, as it must be, Eq. (29). Equation (33) will be derived in the Appendix where we generalize the theory to a nonhomogeneous external field.

B. Stability: Critical surfaces and critical behavior

Note that, for β sufficiently small (see later), Eq. (15) has always the solution $m^{(\Sigma)}=0$, and furthermore, if $m^{(\Sigma)}$ is a solution, $-m^{(\Sigma)}$ is a solution as well. From now on, if not explicitly said, we will refer only to the positive (possibly zero) solution, the negative one being understood. A solution $m^{(\Sigma)}$ of Eq. (15) is stable (but in general not unique) if

$$1 - \beta J^{(\Sigma)} \tilde{\chi}_0(\beta J^{(\Sigma)}_0, \beta J^{(\Sigma)} m^{(\Sigma)} + \beta h) > 0. \tag{34}$$

For what follows, we need to rewrite the nontrivial part of the free energy density $L^{(\Sigma)}(m)$ as

$$L^{(\Sigma)}(m) = \beta f_0(\beta J^{(\Sigma)}_0, 0) - m_0(\beta J^{(\Sigma)}_0, 0)\beta h + \psi^{(\Sigma)}(m), \tag{35}$$

where the introduced term $\psi^{(\Sigma)}$ plays the role of a Landau free energy density and is responsible for the critical behavior of the system. Around $m=0$, up to terms $O(h^2)$ and $O(m^3 h)$, $\psi^{(\Sigma)}(m)$ can be expanded as follows:

$$\psi^{(\Sigma)}(m) = \frac{1}{2}a^{(\Sigma)}m^2 + \frac{1}{4}b^{(\Sigma)}m^4 + \frac{1}{6}c^{(\Sigma)}m^6 - m\beta\tilde{h}^{(\Sigma)} + \Delta(\beta f_0)(\beta J^{(\Sigma)}_0, \beta J^{(\Sigma)} m), \tag{36}$$

where

$$a^{(\Sigma)} = [1 - \beta J^{(\Sigma)} \tilde{\chi}_0(\beta J^{(\Sigma)}_0, 0)]\beta J^{(\Sigma)}, \tag{37}$$

$$b^{(\Sigma)} = -\frac{\partial^2}{\partial(\beta h)^2} \tilde{\chi}_0(\beta J^{(\Sigma)}_0, \beta h)|_{\beta h=0} \frac{(\beta J^{(\Sigma)})^4}{3!}, \tag{38}$$

$$c^{(\Sigma)} = -\frac{\partial^4}{\partial(\beta h)^4} \tilde{\chi}_0(\beta J^{(\Sigma)}_0, \beta h)|_{\beta h=0} \frac{(\beta J^{(\Sigma)})^6}{5!}, \tag{39}$$

$$\tilde{h}^{(\Sigma)} = m_0(\beta J^{(\Sigma)}_0, 0) J^{(\Sigma)} + \tilde{\chi}_0(\beta J^{(\Sigma)}_0, 0)\beta J^{(\Sigma)} \beta h. \tag{40}$$

Finally, the last term $\Delta(\beta f_0)(\beta J_0, \beta J^{(\Sigma)} m)$ is defined implicitly to render Eqs. (35) and (36) exact, but terms $O(h^2)$ and $O(m^3 h)$, explicitly:

$$\Delta(\beta f_0)(\beta J^{(\Sigma)}_0, \beta J^{(\Sigma)} m) = -\sum_{k=4}^{\infty} \frac{\partial^{2k-2}}{\partial(\beta h)^{2k-2}} \tilde{\chi}_0(\beta J^{(\Sigma)}_0, \beta h)|_{\beta h=0} \times \frac{(\beta J^{(\Sigma)})^{2k}}{(2k)!}. \tag{41}$$

We recall that the $(k-2)$th derivative of $\tilde{\chi}_0(\beta J^{(\Sigma)}_0, \beta h)$ with respect to the second argument, calculated at $h=0$, gives the total sum of all the kth cumulants normalized to N: $\partial^{k-2}_{\beta h}\tilde{\chi}_0(\beta J^{(\Sigma)}_0, \beta h)|_{h=0} = \Sigma_{i_1,\ldots,i_k}\langle\sigma_{i_1}\cdots\sigma_{i_k}\rangle^{(c)}_0/N$, where $\langle\sigma_{i_1}\cdots\sigma_{i_k}\rangle^{(c)}_0$ stands for the cumulant, or connected correlation function, of order k of the unperturbed model, $\langle\sigma_{i_1}\sigma_{i_2}\rangle^{(c)}_0 = \langle\sigma_{i_1}\sigma_{i_2}\rangle_0 - \langle\sigma_{i_1}\rangle_0\langle\sigma_{i_2}\rangle_0$, etc. Note that, apart from the sign, these terms are proportional to the Binder cumulants [19] (which are all zero above T_{c0} for $k>2$) only for N finite. In the thermodynamic limit the terms $b^{(\Sigma)}$, $c^{(\Sigma)}, \ldots$, in general are nonzero and take into account the large deviations of the block-spin distribution functions from the Gaussian distribution.

Let $T^{(\Sigma)}_c = 1/\beta^{(\Sigma)}_c$ be the critical temperatures, if any, of the random model and let $t^{(\Sigma)}$ be the corresponding reduced temperatures:

$$t^{(\Sigma)} \overset{def}{=} \frac{T - T^{(\Sigma)}_c}{T^{(\Sigma)}_c} = \frac{\beta^{(\Sigma)}_c - \beta}{\beta^{(\Sigma)}_c} + O(t^{(\Sigma)})^2. \tag{42}$$

Here, the term "critical temperature" stands for any temperature where some singularity shows up. However, if we limit ourselves to consider only the critical temperatures crossing which the system passes from a P region to a non P region, from Eq. (34) it is easy to see that, independently of the sign of J_0 and of the nature of the phase transition, we have the important inequalities

$$\beta^{(\Sigma)}_c < \beta^{(\Sigma)}_{c0}, \tag{43}$$

where we have introduced $\beta^{(\Sigma)}_{c0}$, the inverse critical temperature of the unperturbed model with coupling $J^{(\Sigma)}_0$ and zero external field. If more than one critical temperature is present in the unperturbed model, $\beta^{(\Sigma)}_{c0}$ is the value corresponding to the smallest value of these critical temperatures (highest in terms of β). Formally we set $\beta^{(\Sigma)}_{c0}=\infty$ if no phase transition is present in the unperturbed model. A consequence of Eq. (40) is that, in studying the critical behavior of the system for $h=0$, we can put $\tilde{h}^{(\Sigma)}=0$. Throughout this paper, we shall reserve the name critical temperature of the unperturbed model as a P-F critical temperature through which the magnetization $m_0(\beta J_0, 0)$ passes from a zero to a nonzero value, continuously or not. This implies, in particular, that for $J_0<0$ we have, formally, $\beta_{c0}=\infty$.

In this paper we shall study only the order parameters $m^{(\mathrm{F})}$ and $m^{(\mathrm{SG})}$, whereas we will give only few remarks on how to generalize the method for possible antiferromagnetic order parameters. We point out, however, that the existence of possible antiferromagnetic transitions of the unperturbed model does not affect the results we present in this paper.

It is convenient to distinguish the cases $J_0 \geqslant 0$ and $J_0 < 0$, since they give rise to two strictly different scenarios.

1. Case $J_0 \geqslant 0$

In this case $\beta J^{(\Sigma)} \tilde{\chi}_0(\beta J_0, \beta J^{(\Sigma)} m^{(\Sigma)} + \beta h)$ is an increasing function of β for $\beta < \beta_{c0}^{(\Sigma)}$ (and for $h \geqslant 0$). As a consequence, we have that for sufficiently low temperatures, the solution $m^{(\Sigma)}=0$ of Eq. (15) becomes unstable and two, and only two, nonzero solutions $\pm m^{(\Sigma)}$ are instead favored. The inverse critical temperatures $\beta_c^{(\mathrm{F})}$ and $\beta_c^{(\mathrm{SG})}$ can be determined by developing -for $h=0$ -Eqs. (15) for small $m^{(\mathrm{F})}$ and $m^{(\mathrm{SG})}$, respectively, which, in terms of $\tilde{\chi}_0$ gives the following exact equation

$$\tilde{\chi}_0(\beta_c^{(\Sigma)} J_0^{(\Sigma)}, 0) \beta_c^{(\Sigma)} J^{(\Sigma)} = 1, \quad \beta_c^{(\Sigma)} < \beta_{c0}^{(\Sigma)}, \tag{44}$$

where the constraint $\beta_c^{(\Sigma)} < \beta_{c0}^{(\Sigma)}$ excludes other possible spurious solutions that may appear when $d_0 \geqslant 2$ (since in this case $\beta_{c0}^{(\Sigma)}$ may be finite).

The critical behavior of the system can be derived by developing Eqs. (15) for small fields. Alternatively, one can study the critical behavior by analyzing the Landau free energy density $\psi^{(\Sigma)}(m^{(\Sigma)})$ given by Eq. (36).

In the following we will suppose that for $J_0 > 0$, $b^{(\Sigma)}$ is positive. We have checked this hypothesis in all the models we have until now considered and that will be analyzed in Secs. IV–VI. Furthermore, even if the sign of $c^{(\Sigma)}$ cannot be in general *a priori* established, for the convexity of the function f_0 with respect to βh, the sum of the sixth term with $\Delta(\beta f_0)(\beta J_0^{(\Sigma)}, \beta J^{(\Sigma)} m^{(\Sigma)})$, in Eq. (36), must go necessarily to $+\infty$ for $m^{(\Sigma)} \to \infty$. In conclusion, when $J_0 \geqslant 0$, for the critical behavior of the system, the only relevant parameters of $\psi^{(\Sigma)}$ are $a^{(\Sigma)}$, $b^{(\Sigma)}$, and $h^{(\Sigma)} = \tilde{\chi}(\beta J_0^{(\Sigma)}, 0) J^{(\Sigma)} h$, so that the critical behavior can be immediately derived as in the Landau theory for the so-called m^4 *model* [20]. On noting that

$$\begin{aligned} a^{(\Sigma)} &\geqslant 0, \quad \text{for } t^{(\Sigma)} \geqslant 0, \\ a^{(\Sigma)} &< 0, \quad \text{for } t^{(\Sigma)} < 0, \end{aligned} \tag{45}$$

it is convenient to define

$$A^{(\Sigma)} \overset{def}{=} -\beta \frac{\partial}{\partial \beta} a^{(\Sigma)}, \tag{46}$$

so that we have

$$a^{(\Sigma)} = A^{(\Sigma)}|_{\beta=\beta_c^{(\Sigma)}} t^{(\Sigma)} + O(t^{(\Sigma)})^2. \tag{47}$$

Note that, due to the fact that $J_0 \geqslant 0$, $A^{(\Sigma)} > 0$, and, as already mentioned, $b^{(\Sigma)} \geqslant 0$ as well. By using Eq. (47) for $\beta < \beta_{c0}$ and near $\beta_c^{(\Sigma)}$, we see that the minimum $\bar{m}^{(\Sigma)}$ of $\psi^{(\Sigma)}$—i.e., the solution of Eq. (15) near the critical temperature—is given by

$$\bar{m}^{(\Sigma)} = \begin{cases} 0, & t^{(\Sigma)} \geqslant 0, \\ \sqrt{-\left.\dfrac{A^{(\Sigma)}}{b^{(\Sigma)}}\right|_{\beta=\beta_c^{(\Sigma)}} t^{(\Sigma)}} + O(t^{(\Sigma)}), & t^{(\Sigma)} < 0. \end{cases} \tag{48}$$

Similarly, we can write general formulas for the susceptibility and the equation of state. We have

$$\tilde{\chi}^{(\Sigma)} = \begin{cases} \left.\dfrac{\beta J^{(\Sigma)} \tilde{\chi}_0(\beta J_0^{(\Sigma)}, 0)}{A^{(\Sigma)}}\right|_{\beta=\beta_c^{(\Sigma)}} \dfrac{1}{t^{(\Sigma)}} + O(1), & t^{(\Sigma)} \geqslant 0, \\ \left.\dfrac{\beta J^{(\Sigma)} \tilde{\chi}_0(\beta J_0^{(\Sigma)}, 0)}{-2A^{(\Sigma)}}\right|_{\beta=\beta_c^{(\Sigma)}} \dfrac{1}{t^{(\Sigma)}} + O(1), & t^{(\Sigma)} < 0, \end{cases} \tag{49}$$

$$\bar{m}^{(\Sigma)}(h) = \left[\frac{\beta J^{(\Sigma)} \tilde{\chi}_0(\beta J_0^{(\Sigma)}, 0)}{A^{(\Sigma)}} \right]_{\beta=\beta_c^{(\Sigma)}}^{1/3} h^{1/3} + O(h^{2/3}). \tag{50}$$

Finally, on using Eqs. (36) and (48) we get that the specific heat $C^{(\Sigma)}$ has the following finite jump discontinuity at $\beta_c^{(\Sigma)}$:

$$C^{(\Sigma)} = \begin{cases} C_c^{(\Sigma)}, & t^{(\Sigma)} \geqslant 0, \\ C_c^{(\Sigma)} + \left.\dfrac{(A^{(\Sigma)})^2}{2b^{(\Sigma)}}\right|_{\beta=\beta_c^{(\Sigma)}}, & t^{(\Sigma)} < 0, \end{cases} \tag{51}$$

where $C_c^{(\Sigma)}$ is the continuous part of the specific heat corresponding to the part of the free energy density without $\psi^{(\Sigma)}$.

Hence, as a very general result, independently of the structure of the underlying graph $\mathcal{L}_0$ and its dimension d_0, independently of the nature of the phase transition present in unperturbed model (if any), and independently of the added random connectivity c, provided positive, we recover that the random model has always a mean-field critical behavior with a second-order phase transition with the classical exponents $\beta=1/2$, $\gamma=\gamma'=1$, $\delta=3$, and $\alpha=\alpha'=0$ and certain constant coefficients depending on the susceptibility $\tilde{\chi}_0$ and its derivatives calculated at $\beta=\beta_c^{(\Sigma)}$ and external field $h=0$. Note, however, that the correlation length of the system calculated along the distance of $\mathcal{L}_0$, $\|\cdot\|_0$, remains finite also at $\beta_c^{(\Sigma)}$. In fact, from Eq. (20), for the two-point correlation function at distance $r \overset{def}{=} \|i-j\|_0$ in $\mathcal{L}_0$ we have

$$C^{(\Sigma)}(r) = C_0(\beta J_0^{(\Sigma)}, \beta J^{(\Sigma)} m^{(\Sigma)} + \beta h; r). \tag{52}$$

If we now assume for $C_0(\beta J_0, 0; r)$ the general Ornstein-Zernike form

$$C_0(\beta J_0, 0; r) = \frac{e^{-r/\xi_0}}{f_0(r)}, \tag{53}$$

$f_0(r) = f_0(\beta J_0; r)$ being a smooth function of r (which has not to be confused with the free energy density), and $\xi_0 = \xi_0(\beta J_0)$ the correlation length, which is supposed to diverge only at β_{c0} (if any), on comparing Eqs. (52) and (53) for $\beta \geqslant \beta_c^{(\Sigma)}$ we have (notice that, as explained in Sec. III A, at least for lattices $\mathcal{L}_0$ having only loops of even length, the physical correlation function is only that corresponding to $\Sigma=\mathrm{F}$, i.e., $C^{(1)} = C^{(\mathrm{F})}$)

$$C^{(\Sigma)}(r) = \frac{e^{-r/\xi^{(\Sigma)}}}{f^{(\Sigma)}(r)}, \tag{54}$$

where

$$f^{(\Sigma)}(r) = f_0(\beta J_0^{(\Sigma)};r) \tag{55}$$

and

$$\xi^{(\Sigma)} = \xi_0(\beta J_0^{(\Sigma)}). \tag{56}$$

Therefore, due to the inequalities (43), we see that

$$\xi^{(\Sigma)}|_{\beta=\beta_c^{(\Sigma)}} = \xi_0(\beta_c^{(\Sigma)} J_0^{(\Sigma)}) < \infty. \tag{57}$$

Knowledge of $C_0(\beta J_0, \beta h; r)$ also for $h \neq 0$ would allow us to find the general expression for $C^{(\Sigma)}(r)$ through Eq. (52) also for $\beta > \beta_c^{(\Sigma)}$. However, since $C_0(\beta J_0, \beta h; r)$ has no critical behavior for $h \neq 0$, it follows that $C^{(\Sigma)}(r)$ cannot have a critical behavior for $\beta > \beta_c^{(\Sigma)}$ either (and then also for $\beta \to \beta_c^{(\Sigma)}$ from the right). This result is consistent with [2].

2. Case $J_0 < 0$

In this case $J_0^{(F)} < 0$, so that, in general, $\beta J^{(F)} \tilde{\chi}_0(\beta J_0, \beta J^{(F)} m^{(F)})$ is no longer a monotonic function of β. However, it is easy to see that that for $\beta = 0$ and $\beta \to \infty$, this function goes to 0. Therefore, for a sufficiently large connectivity c, from Eq. (34) we see that there may appear at least two regions where the paramagnetic solution $m^{(F)} = 0$ is stable, separated by a third region in which a nonzero solution is instead stable. However, the situation is even more complicated since, unlike the case $J_0 \geq 0$, the nonmonotonicity of $\beta J^{(F)} \tilde{\chi}_0(\beta J_0, \beta J^{(F)} m^{(F)})$ reflects also in the fact that the self-consistent equation (15) for $\Sigma = F$ may have more solutions of the kind $\pm m^{(F)}, \pm m'^{(F)}, \ldots$, which are still stable with respect to the stability condition (34), for $h = 0$. We face in fact here the problem of comparing more stable solutions. According to Eq. (23), in the thermodynamic limit, among all the possible stable solutions, only $\bar{m}^{(F)}$, the solution that minimizes $L^{(F)}$, survives, whereas the ones not leading play the role of metastable states. This kind of scenario, which includes also finite jump discontinuities, has been besides observed in the context of small-world neural networks in [21] where we even observe some analogy in the formalism used, at least for the simplest case of one binary pattern.

From Eqs. (38) and (39) we see that the signs of the Landau coefficients $a^{(\Sigma)}$, $b^{(\Sigma)}$, $c^{(\Sigma)}$, … are functions of β and J_0 only. Given $J_0 < 0$, the most important quantity that features the nonmonotonicity of $\beta J^{(F)} \tilde{\chi}_0(\beta J_0, \beta J^{(F)} m^{(F)})$ is the minimum value of β over which $b^{(F)}$ becomes negative:

$$b^{(F)} \leq 0, \quad \beta \geq \beta_*^{(F)}. \tag{58}$$

The equation for $\beta_*^{(F)}$, as a function of J_0, defines a point where $b^{(F)} = 0$. If $J_0 < 0$, the most general equation for a generic critical temperature is no longer given by Eq. (44). In fact, in general, a critical temperature now is any temperature where the stable and leading solution $\bar{m}^{(F)}$ may have a singular behavior, also with finite jumps between two non zero values.

There are some simplification when for the Landau coefficient $c^{(F)}$ we have $c^{(F)} > 0$, or at least $c^{(F)} > 0$ out of the P region In this situation, in fact, from Eq. (36) we see that $a^{(F)}$, $b^{(F)}$, and $c^{(F)}$ are the only relevant terms for the critical behavior of the system and, for small values of $\bar{m}^{(F)}$, we can again apply the Landau theory, this time for the so-called m^6 *model* [20]. In such a case, for the solution $\bar{m}^{(F)}$ we have

$$\bar{m}^{(F)} = \sqrt{\frac{1}{2c^{(F)}}\left(\sqrt{(b^{(F)})^2 - 4a^{(F)}c^{(F)}} - b^{(F)}\right)}, \quad \text{if } a^{(F)} < 0$$
$$\text{or } a^{(F)} \geq 0 \text{ and } b^{(F)} \leq -4\sqrt{\frac{a^{(F)}c^{(F)}}{3}}, \tag{59}$$

whereas

$$\bar{m}^{(F)} = 0, \quad \text{if } a^{(F)} \geq 0 \text{ and } b^{(F)} > -4\sqrt{\frac{a^{(F)}c^{(F)}}{3}}. \tag{60}$$

From Eqs. (59) and (60) we see that, if $b^{(F)} > 0$, we have a second-order phase transition and Eqs. (44)–(57) are recovered with Eq. (59) becoming the second equation of Eqs. (48) for small and negative values of $a^{(F)}$. However, from Eq. (59) we see that, if $b^{(F)}$ is sufficiently negative, we have a first-order phase transition which, for small values of $a^{(F)}$, gives

$$\bar{m}^{(F)} = \sqrt{-\frac{b^{(F)}}{c^{(F)}}\left(1 - \frac{a^{(F)}c^{(F)}}{2(b^{(F)})^2}\right)}, \quad \text{if } a^{(F)} < 0 \text{ and}$$
$$b^{(F)} < 0 \text{ or } a^{(F)} \geq 0 \text{ and } b^{(F)} \leq -4\sqrt{\frac{a^{(F)}c^{(F)}}{3}}. \tag{61}$$

From Eq. (59) we see that the line $b^{(F)} = -4\sqrt{a^{(F)}c^{(F)}/3}$ with $a^{(F)} \geq 0$ establishes a line of first-order transitions through which $\bar{m}^{(F)}$ changes discontinuously from zero to

$$\Delta \bar{m}^{(F)} = \left(\frac{3a^{(F)}}{c^{(F)}}\right)^{1/4}. \tag{62}$$

The point $a^{(F)} = b^{(F)} = 0$ is a tricritical point where the second- and first-order transition lines meet. If we approach the tricritical point along the line $b^{(F)} = 0$, we get the critical indices $\alpha = 1/2$, $\alpha' = 0$, $\beta = 1/4$, $\gamma = \gamma' = 1$, and $\delta = 5$. However, this critical behavior along the line $b^{(F)} = 0$ has no great practical interest since from Eq. (38) we see that it is not possible to keep $b^{(F)}$ constant and zero as the temperature varies. Finally, we point out that, even if $c^{(F)} > 0$, when the transition is of first order, Eqs. (59) and (61) hold only for $b^{(F)}$, and then $a^{(F)}$, sufficiently small, since only in such a case the finite discontinuity of $\bar{m}^{(F)}$ is small and then the truncation of the Landau free energy term $\psi^{(F)}$ to a finite order meaningful. Note that this question implies also that we cannot establish a simple and general rule to determine the critical temperature of a first-order phase transition (we will return soon on this point).

When $c^{(F)} < 0$, the Landau theory of the m^6 model cannot, of course, be applied. However, as in the case $J_0 > 0$, even if the sign of $c^{(F)}$ cannot be *a priori* established, for the convexity of the function f_0 with respect to βh, the sum of the sixth term with $\Delta(\beta f_0)(\beta J_0^{(F)}, \beta J^{(F)} m^{(F)})$, in Eq. (36), must go necessarily to $+\infty$ for $m^{(F)} \to \infty$ and a qualitative similar behavior of the m^6 model is expected. In general, when $J_0 < 0$, the exact results are limited to the following ones.

From now on, if not otherwise explicitly stated, we shall reserve the name critical temperature, whose inverse value of β we still indicate with $\beta_c^{(\mathrm{F})}$, to any temperature on the boundary of a P region (through which $\bar{m}^{(\mathrm{F})}$ passes from 0 to a nonzero value, continuously or not). For each critical temperature, depending on the value of $\beta_*^{(\mathrm{F})}$, we have three possible scenario of phase transitions:

$$\beta_c^{(\mathrm{F})} > \beta_*^{(\mathrm{F})} \Leftarrow \text{first order}, \tag{63a}$$

$$\beta_c^{(\mathrm{F})} = \beta_*^{(\mathrm{F})} \Leftrightarrow \text{tricritical point}, \tag{63b}$$

$$\beta_c^{(\mathrm{F})} < \beta_*^{(\mathrm{F})} \Rightarrow \text{second order}. \tag{63c}$$

Note that, according to our definition of critical temperature, the critical behavior described by Eqs. (59)–(62) represents a particular case of the general scenario expressed by Eqs. (63). We see also that, in general, when $b^{(\mathrm{F})} \leqslant 0$, approaching the tricritical point, for the critical exponent β we have $\beta \leqslant 1/4$.

In the case in which $\beta_c^{(\mathrm{F})}$ corresponds to a second-order phase transition, or in the case in which $a^{(\mathrm{F})} < 0$ out of the P region (at least immediately near the critical temperature), $\beta_c^{(\mathrm{F})}$ can be exactly calculated by Eq. (44). When we are not in such cases, the only exact way to determine the critical temperature is to find the full solution for $\bar{m}^{(\mathrm{F})}$, which consists in looking numerically for all the possible solutions of Eq. (15) and, among those satisfying the stability condition (34), selecting the one that gives the minimum value of $L^{(\mathrm{F})}$.

C. Level of accuracy of the method

In the P region, Eqs. (15)–(33) are exact, whereas in the other regions provide an effective approximation whose level of accuracy depends on the details of the model. In particular, in the absence of frustration the method becomes exact at any temperature in two important limits: in the limit $c \to 0^+$, in the case of second-order phase transitions, due to a simple continuity argument, and in the limit $c \to \infty$, due to the fact that in this case the system becomes a suitable fully connected model exactly described by the self-consistent equations (15) (of course, when $c \to \infty$, to have a finite critical temperature one has to renormalize the average of the coupling by c).

However, for any $c > 0$, off of the P region and infinitely near the critical temperature, Eqs. (15)–(20) are able to give the exact critical behavior in the sense of the critical indices and, in the limit of low temperatures, Eqs. (15)–(19) provide the exact percolation threshold. In general, as for the Sherrington-Kirkpatrick (SK) model, which can be seen as a particular model with $J_0 = 0$, the level of accuracy is better for the F phase rather than for the SG one and this is particularly true for the free energy density $f^{(\Sigma)}$, Eq. (21). In fact, though the derivatives of $f^{(\Sigma)}$ are expected to give a good qualitative and partly also a quantitative description of the system, $f^{(\mathrm{SG})}$ itself can give wrong results when the SG phase at low temperatures is considered. We warn the reader that in a model with $J_0 = 0$, and a symmetrical distribution $d\mu(J_{i,j})$ with variance $\tilde{J}$, the method gives a ground-state energy per site $u^{(\mathrm{SG})}$, which grows with c as $u^{(\mathrm{SG})} \sim -\tilde{J}c$, whereas the correct result is expected to be $u^{(\mathrm{SG})} \sim -\tilde{J}\sqrt{c}$ [22]. As a consequence, in the SK model, in the limit $\beta \to \infty$, the method gives a completely wrong result with an infinite energy. We stress, however, that the order parameters $m^{(\mathrm{F})}$ and $m^{(\mathrm{SG})}$, and then also the correlation functions, by construction, are exact in the zero-temperature limit.

D. Phase diagram

The physical inverse critical temperature β_c of the random model is in general a non-single-value function of $\boldsymbol{X}$: $\beta_c = \beta_c(\boldsymbol{X})$, where $\boldsymbol{X}$ represents symbolically the parameters of the probability $d\mu$ for the couplings $J_{i,j}$ and the parameter c the average connectivity (which is also a parameter of the probability distribution of the shortcut bonds). The parameters of $d\mu$ can be expressed through the moments of $d\mu$, and as they vary, the probability $d\mu$ changes. For example, if $d\mu$ is a Gaussian distribution, as in the SK model, there are only two parameters given by the first and second moments. A concrete example for the one-dimensional small-world model will be shown in Figs. (17) and (18).

In the thermodynamic limit, only one of the two solutions with label F or SG survives, and it is the solution having minimum free energy. In principle, were our method exact at all temperatures, we were able to derive exactly all the phase diagram. However, in our method, the solution with label F or SG are exact only in their own P region—i.e., the region where $m^{(\mathrm{F})} = 0$ or $m^{(\mathrm{SG})} = 0$, respectively. Unfortunately, according to what we have seen in Sec. III C, whereas the solution with label F is still a good approximation also out of the P region, in the frustrated model (where the variance of $d\mu$ is large if compared to its first moment) the free energy of the solution with label SG becomes completely wrong at low temperatures. Therefore, we are not able to give in general the exact boundary between the solution with label F and the solution with label SG, and in particular we are not able to give the physical frontier F/SG. However, within some limitations which we now prescribe, we are able to give the exact critical surface—i.e., the boundary with the P phase—establishing which one, in the thermodynamic limit, of the two critical boundaries P-F or P-SG is stable (we will use here the more common expression "stable" instead of the expression "leading") and to localize some regions of the phase diagram for which we can say exactly whether the stable solution is P, F, or SG. We will prove the stability of these solutions in Sec. IX. When for a region we are not able to discriminate between the solution with label F and the solution with label SG and they are both out of their own P region, we will indicate such a region with the symbol "SG" and/or "F" (stressing in this way that in this region there may be also mixed phases and reentrance phenomena).

In Sec. IX we prove that there are four possible kind of phase diagrams that may occur according to the cases (i) ($J_0 \geqslant 0; d_0 < 2$, or $d_0 = \infty$), (ii) ($J_0 \geqslant 0; 2 \leqslant d_0 < \infty$), (iii) ($J_0 < 0; d_0 < 2$, or $d_0 = \infty$), and (iv) ($J_0 < 0; 2 \leqslant d_0 < \infty$). The four kinds of possible phase diagrams are schematically depicted in Figs. 1–4 in the plane $(T, \boldsymbol{X})$.

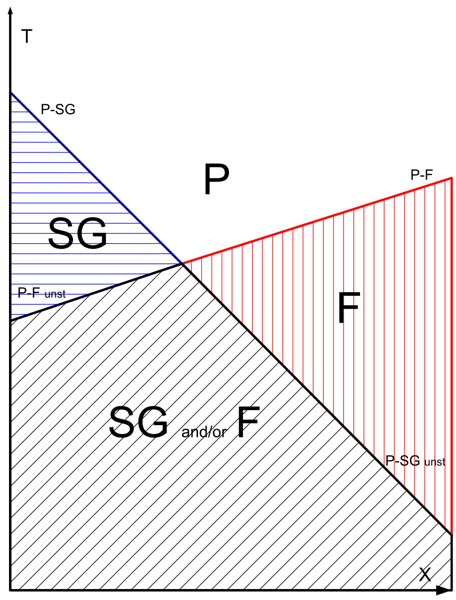

FIG. 1. (Color online) Phase diagram for the case (i): $J_0 \geqslant 0$ and $d_0 < 2$ or $d_0 = \infty$ in a broad sense. Here T is the temperature, while $\boldsymbol{X}$ represents symbolically the connectivity c and the parameters of the probability distribution $d\mu$.

1. $J_0 \geqslant 0$

we have seen in Sec. III B 1, if $J_0 \geqslant 0$, for both the solution with labels F and SG, we have one, and only one, critical temperature. In the following, to avoid confusion, the distinction should be kept in mind between the physical $\beta_c = \beta_c(\boldsymbol{X})$ and $\beta_c^{(\Sigma)}$, with Σ=F or SG. β_c satisfies the following rules.

Case (i). If $d_0 < 2$ and J_0 is a finite range coupling, or else $d_0 = \infty$ at least in a broad sense (see [15]), $\beta_c(\boldsymbol{X})$ is a single-value function of $\boldsymbol{X}$, and we have

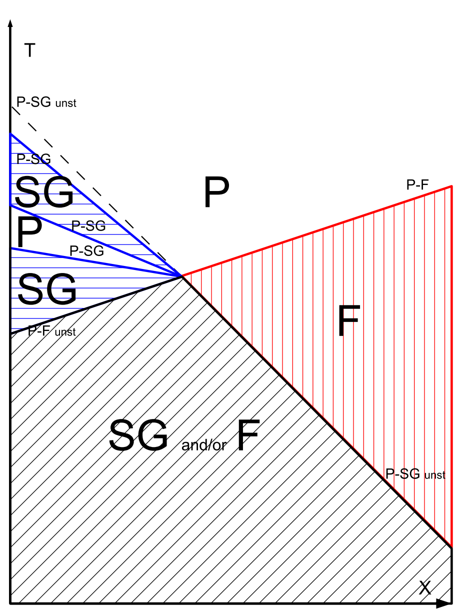

FIG. 2. (Color online) Phase diagram for the case (ii): $J_0 \geqslant 0$ and $2 \leqslant d_0 < \infty$. T and $\boldsymbol{X}$ as in Fig. 1.

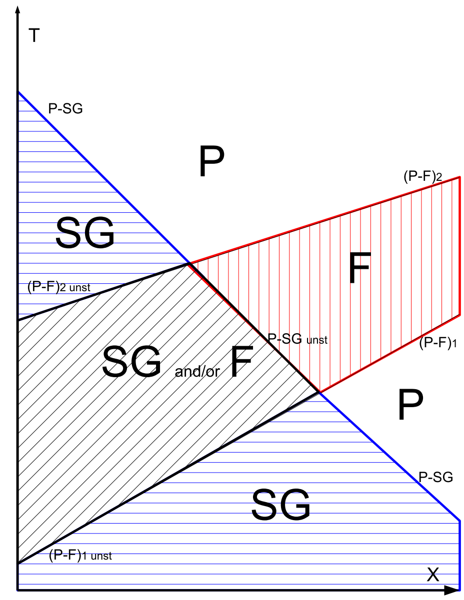

FIG. 3. (Color online) Phase diagram for the case (iii): $J_0 < 0$ and $d_0 < 2$ or $d_0 = \infty$ in a broad sense. T and $\boldsymbol{X}$ as in Fig. 1.

$$\beta_c = \min\{\beta_c^{(\mathrm{F})}, \beta_c^{(\mathrm{SG})}\}. \tag{64}$$

A schematic representation of this case is given in Fig. 1.

Case (ii). If instead $2 \leqslant d_0 < \infty$, we have

$$\begin{cases} \beta_c = \beta_c^{(\mathrm{F})}, & \text{if } \beta_c^{(\mathrm{SG})} \geqslant \beta_c^{(\mathrm{F})}, \\ \beta_c^{(\mathrm{F})} \geqslant \beta_c > \beta_c^{(\mathrm{SG})}, & \text{if } \beta_c^{(\mathrm{SG})} < \beta_c^{(\mathrm{F})}. \end{cases} \tag{65}$$

Notice in particular that the second line of Eq. (65) does not exclude that $\beta_c(\boldsymbol{X})$ might be a non-single-value function of $\boldsymbol{X}$. A schematic representation of this case is given in Fig. 2.

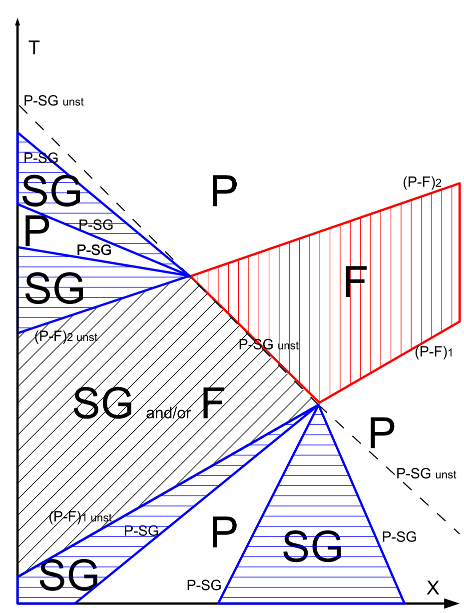

FIG. 4. (Color online) Phase diagram for the case (iv): $J_0 < 0$ and $2 \leqslant d_0 < \infty$. T and $\boldsymbol{X}$ as in Fig. 1.

2. $J_0 < 0$

As we have seen in Sec. III B 2, if $J_0 < 0$, for a sufficiently large connectivity c, the solution with label F has at least two separated P regions corresponding to two critical temperatures. Here we assume that the underlying lattice $\mathcal{L}_0$ has only loops of even length so that, for example, triangular lattices are here excluded. Let us suppose to have for the solution with label F only two critical temperatures (the minimum number, if $J_0 < 0$), and let be

$$\beta_{c1}^{(\mathrm{F})} \geq \beta_{c2}^{(\mathrm{F})}. \tag{66}$$

In general we have the following scenario.

Case (iii). If $d_0 < 2$ and J_0 is a finite range coupling, or $d_0 = \infty$ in a broad sense (see [15]), $\beta_{c2}(\boldsymbol{X})$ is a single-value function of $\boldsymbol{X}$ and satisfies Eq. (64). The other critical inverse temperature $\beta_{c1}(\boldsymbol{X})$ is instead either a two-value function of $\boldsymbol{X}$ and we have

$$\beta_{c1} = \begin{pmatrix} \beta_{c1}^{(\mathrm{F})} \\ \beta_{c}^{(\mathrm{SG})} \end{pmatrix}, \quad \text{if } \beta_{c1}^{(\mathrm{F})} \leq \beta_{c}^{(\mathrm{SG})}, \tag{67}$$

or

$$\nexists\ \beta_{c1}, \quad \text{if } \beta_{c1}^{(\mathrm{F})} > \beta_{c}^{(\mathrm{SG})}, \tag{68}$$

where $\nexists$ in Eq. (68) means that if $\beta_{c1}^{(\mathrm{F})} > \beta_{c}^{(\mathrm{SG})}$, there is no stable boundary with the P region. A schematic representation of this case is given in Fig. 3.

Case (iv). If $2 \leq d_0 < \infty$, β_{c2} satisfies Eq. (65), whereas for β_{c1} we have either

$$\beta_{c1} = \begin{pmatrix} \beta_{c1}^{(\mathrm{F})} \\ \beta_{c}^{(\mathrm{SG}>)} \end{pmatrix}, \quad \text{if } \beta_{c1}^{(\mathrm{F})} \leq \beta_{c}^{(\mathrm{SG})}, \tag{69}$$

or

$$\text{if } \exists\, \beta_{c1} \Rightarrow \beta_{c1} > \beta_{c}^{(\mathrm{SG})}, \quad \text{if } \beta_{c1}^{(\mathrm{F})} > \beta_{c}^{(\mathrm{SG})}, \tag{70}$$

where in Eq. (69) we have introduced the symbol SG> to indicate that in general the stable P-SG surface is above (or below in terms of temperatures) the surface coming from the solution with label SG:$\beta_{c}^{(\mathrm{SG}>)} > \beta_{c}^{(\mathrm{SG})}$. Notice that, similarly to the case (iii), we cannot exclude that β_{c1} in Eq. (70) be a non-single-value function of $\boldsymbol{X}$, as well as $\beta_{c}^{(\mathrm{SG}>)}$ in Eq. (69). A schematic representation of this case is given in Fig. 4.

If more than two critical temperatures are present, the above scheme generalizes straightforwardly.

Keeping our definition for the introduced symbol "SG" and/or "F," we stress that in all the fours cases the phases F and "SG" and/or "F" are exactly localized; in cases (i) and (iii) the phases P and SG are exactly localized; in cases (ii) and (iv) the SG phase is always limited below (in terms of temperatures) by the unstable P-SG surface coming from the solution with label SG (indicated as P-SG unst in Figs. 2 and 4). Finally, we stress that, under the hypothesis that $\mathcal{L}_0$ has only loops of even length, the stable P regions correspond always to the solution with label F.

For $2 \leq d_0 < \infty$, from the second line of Eqs. (65) and (69) and from Eq. (70), we see that the method is not able to give the complete information about the P-SG boundary since we have only inequalities, not equalities. Furthermore, in these regions of the phase diagram the physical critical temperature in general may be a non-single-value function of $\boldsymbol{X}$. On the other hand, we have the important information that in these equations the inequalities between the physical β_c and $\beta_{c}^{(\mathrm{SG})}$ (the solution with label SG) are always strict. As a consequence, we see that, when $2 \leq d_0 < \infty$, in these regions the SG "magnetization" $m^{(\mathrm{SG})}$ will always have a finite jump discontinuity in crossing the surface given by β_c. In other words, along such a branch of the critical surface corresponding to the second line of Eqs. (65) and (69) and Eq. (70), we have a first-order phase transition, independently of the fact that the phase transition corresponding to the $\beta_{c}^{(\mathrm{SG})}$ surface is second order and independently of the sign of J_0.

E. Generalizations

The generalization to the cases in which the unperturbed model has an Hamiltonian H_0 involving couplings depending on the bond $b \in \Gamma_0$ is straightforward. In this case we have just to substitute everywhere in the formulas (15)–(41), $J_0^{(\Sigma)}$ with the set $\{J_{0b}^{(\Sigma)}\}$. However, the critical behavior will be in general different and more complicated than that depicted in the Secs. III B 1 and III B 2. In particular, even in the case in which all the couplings $J_{0b}^{(\Sigma)}$ are positive, we cannot assume that the Landau coefficient $b^{(\Sigma)}$ be positive so that, even in such a case, first-order phase transitions are in principle possible, as has been seen via Monte Carlo simulations in directed small-world models [23].

As anticipated, our method can be generalized also to study possible antiferromagnetic phase transitions in the random model. There can be two kind of sources of antiferromagnetism: one due to a negative coupling J_0 in the unperturbed model, the other due to random shortcuts $J_{i,j}$ having a measure $d\mu$ with a negative average.

In the first case, if for example the sublattice $\mathcal{L}_0$ is bipartite into two sublattices $\mathcal{L}_0^{(a)}$ and $\mathcal{L}_0^{(b)}$, the unperturbed model will have an antiferromagnetism described by two fields $m_0^{(a)}$ and $m_0^{(b)}$. Correspondingly, in the random model we will have to analyze two effective fields $m^{(a)}$ and $m^{(b)}$ which will satisfy a set of two coupled self-consistent equations similar to Eqs. (15) and involving knowledge of $m_0^{(a)}$ and $m_0^{(b)}$. More in general, we can introduce the site-dependent solution m_{0i} to find correspondingly in a set of coupled equations (at most N), the effective fields m_i of the random model.

In the second case, following [24] we consider a lattice $\mathcal{L}_0$ which is composed of, say, p sublattices $\mathcal{L}_0^{(\nu)}$, $\nu = 1, \ldots, p$. Then, we build up the random model with the rule that any shortcut may connect only sites belonging to two different sublattices. Hence, as already done in [13] for the generalized SK model, we introduce p effective fields $m^{(\nu)}$ which satisfy a system of p self-consistent equations involving the p fields $m_0^{(\nu)}$ and calculated in the p external fields $J^{(\mathrm{F})} m^{(\nu)}$ [note that here the symbol F stresses only the fact that the effective coupling must be calculated through Eq. (16)].

IV. SMALL WORLD IN $d_0 = 0$ DIMENSION

A. Viana-Bray model

As an immediate example, let us consider the Viana-Bray model [25]. It can be seen as the simplest small-world model in which N spins with no short-range couplings (here $J_0=0$) are randomly connected by long-range connections J (possibly also random). Note that formally here $\mathcal{L}_0$ has dimension $d_0=0$. Since $J_0=0$, for the unperturbed model we have

$$-\beta f_0(0,\beta h) = \ln[2\cosh(\beta h)],$$

$$m_0(0,\beta h) = \tanh(\beta h),$$

$$\tilde{\chi}_0(\beta J_0,\beta h) = 1 - \tanh^2(\beta h)|_{\beta h=0} = 1. \tag{71}$$

It is interesting to check that the first and second derivatives of $\tilde{\chi}_0$ in $h=0$ are null and negative, respectively. In fact, we have

$$\frac{\partial}{\beta h}\tilde{\chi}_0(0,\beta h) = -2\tanh^2(\beta h)\times[1-\tanh^2(\beta h)]|_{\beta h=0} = 0 \tag{72}$$

and

$$\frac{\partial^2}{(\beta h)^2}\tilde{\chi}_0(0,\beta h) = -2[1-\tanh^2(\beta h)]^2 + 4\tanh^2(\beta h) \times[1-\tanh^2(\beta h)]|_{\beta h=0} = -2. \tag{73}$$

Applying these results to Eqs. (15)–(19) we get immediately the self-consistent equations for the F and SG magnetizations,

$$m^{(\mathrm{F})} = \tanh\left[m^{(\mathrm{F})}c\int d\mu\,\tanh(\beta J)\right], \tag{74}$$

$$m^{(\mathrm{SG})} = \tanh\left[m^{(\mathrm{SG})}c\int d\mu \tanh^2(\beta J)\right], \tag{75}$$

and the Viana-Bray critical surface

$$c\int d\mu\,\tanh(\beta_c^{(\mathrm{F})}J) = 1, \tag{76}$$

$$c\int d\mu\,\tanh^2(\beta_c^{(\mathrm{SG})}J) = 1. \tag{77}$$

On choosing for $d\mu$ a measure having average and variance scaling as $O(1/c)$, for $c\propto N$, we recover the equations for the SK model [26] already derived in this form in [13,14]. In these papers, Eqs. (74)–(77) were derived by mapping the Viana-Bray model and, similarly, the SK model to the nonrandom fully connected Ising model. In this sense it should be also clear that, at least for $\beta\leqslant\beta_c$ and zero external field, in the thermodynamic limit, the connected correlation functions (of order k greater than 1) in the SK and in the Viana-Bray model are exactly zero. In fact, in the thermodynamic limit, the non random fully connected model can be exactly reduced to a model of non interacting spins immersed in an effective medium so that among any two spins there is no correlation. Such a result is due to the fact that, in these models, all the N spins interact through the same coupling J/N, no matter how far apart they are, and the net effect of this is that in the thermodynamic limit the system becomes equivalent to a collection of N noninteracting spins seeing only an effective external field (the medium) like in Eq. (71) with βh replaced by $\beta J m$.

For the measure (10) our approximated equation (74) can be compared with the exact known equation that can be derived by using the Bethe-Peierls or the replica approach and is given by (see, for example [27], and references therein)

$$m^{(\mathrm{F})} = \sum_{q=0}^{\infty}\frac{e^{-c}c^q}{q!}\int \tanh\left(\beta\sum_{m=1}^{q}H_m\right)\prod_{m=1}^{q}\Psi(H_m)dH_m,$$

where the effective field H is determined by the integral equation

$$\Psi(H) = \sum_{q=0}^{\infty}\frac{e^{-c}c^{q-1}}{(q-1)!}\int \delta\Bigg(H - T\tanh^{-1}\left[\tanh\beta J\tanh\left(\beta\sum_{m=1}^{q-1}H_m\right)\right]\Bigg)\prod_{m=1}^{q-1}\Psi(H_m)dH_m.$$

In the limit $\beta\to\infty$, Eqs. (74) and (75) give the following size (normalized to 1) of the giant connected component:

$$m^{(\mathrm{F})} = \tanh(m^{(\mathrm{F})}c), \tag{78}$$

$$m^{(\mathrm{SG})} = \tanh(m^{(\mathrm{SG})}c). \tag{79}$$

These equations are not exact; however, they succeed in giving the exact percolation threshold $c=1$. In fact, concerning Eq. (78) for the F phase, the exact equation for $m^{(\mathrm{F})}$ is (see, for example [27], and references therein)

$$1 - m^{(\mathrm{F})} = e^{-cm^{(\mathrm{F})}}, \tag{80}$$

which, in terms of the tanh function, becomes

$$\frac{2m^{(\mathrm{F})} + (m^{(\mathrm{F})})^2}{2 - m^{(\mathrm{F})} + (m^{(\mathrm{F})})^2} = \tanh(m^{(\mathrm{F})}c),$$

so that Eqs. (78) and (80) are equivalent at the order $O(m^{(\mathrm{F})})$. We see also that, as stated in the Sec. III C, Eqs. (78) and (80) become equal in the limits $c\to 0$ and $c\to\infty$.

B. Gas of dimers

Let us consider for $\mathcal{L}_0$ a set of $2N$ spins coupled through a coupling J_0 two by two. The expression "gas of dimers" stresses the fact that the dimers—i.e., the couples of spins—do not interact each other. As a consequence, the free energy, the magnetization, and the susceptibility of the unperturbed model can be immediately calculated. We have

$$-\beta f_0(\beta J_0,\beta h) = \frac{1}{2}\ln[2e^{\beta J_0}\cosh(2\beta h) + 2e^{-\beta J_0}],$$

$$m_0(\beta J_0,\beta h) = \frac{e^{\beta J_0}\sinh(2\beta h)}{e^{\beta J_0}\cosh(2\beta h) + e^{-\beta J_0}},$$

$$\tilde{\chi}_0(\beta J_0,\beta h)= \left.\frac{2e^{\beta J_0}+2\cosh(2\beta h)}{[e^{\beta J_0}\cosh(2\beta h)+e^{-\beta J_0}]^2}\right|_{\beta h=0}=\frac{e^{\beta J_0}}{\cosh(\beta J_0)}, \tag{81}$$

Let us calculate also the second derivative of $\tilde{\chi}_0$. From

$$\frac{\partial}{\beta h}\tilde{\chi}_0(\beta J_0,\beta h)=4\sinh(\beta h) \times\frac{e^{-\beta J_0}-2e^{3\beta J_0}-e^{\beta J_0}\cosh(2\beta h)}{[e^{\beta J_0}\cosh(2\beta h)+e^{-\beta J_0}]^3}, \tag{82}$$

we get

$$\frac{\partial^2}{(\beta h)^2}\tilde{\chi}_0\,(\beta J_0,\beta h)|_{\beta h=0}=-2\frac{\sinh(\beta J_0)+e^{3\beta J_0}}{[\cosh(\beta J_0)]^3}.$$

We note that, as expected, the second derivative of $\tilde{\chi}_0$ in $h=0$, for $J_0\geqslant 0$, is always negative, whereas, for $J_0<0$, it becomes positive as soon as $\beta|J_0|>\ln(\sqrt{2})$.

By using the above equations, from Sec. III we get immediately the following self-consistent equation for the magnetizations,

$$m^{(\Sigma)}=\frac{\tanh(2\beta J^{(\Sigma)}m^{(\Sigma)}+2\beta h)}{1+e^{-2\beta J_0}\,\mathrm{sech}(2\beta J^{(\Sigma)}m^{(\Sigma)}+2\beta h)},$$

and, at least for $J_0\geqslant 0$, the equation for the critical temperature:

$$\frac{e^{\beta_c^{(\Sigma)}J_0^{(\Sigma)}}}{\cosh(\beta_c^{(\Sigma)}J_0^{(\Sigma)})}\beta_c^{(\Sigma)}J^{(\Sigma)}=1.$$

As will be clear soon, this model lies between the Viana-Bray model and the more complex $d_0=1$ dimensional chain small-world model, which will be analyzed in detail in the next section. Our major interest in this simpler gas of dimers small-world model is related to the fact that, in spite of its simplicity and $d_0=0$ dimensionality—since the second derivative of $\tilde{\chi}_0$ may be positive when J_0 is negative—according to the general result of Sec. III B, it is able to give rise to also multiple first- and second-order phase transitions.

V. SMALL WORLDS IN $d_0=1$ DIMENSION

In this section we will analyze the case in which $\mathcal{L}_0$ is the $d_0=1$-dimensional chain with periodic boundary conditions (PBCs). The corresponding small-world model with Hamiltonian (2) in zero field has already been analyzed in [5] by using the replica method. Here we will recover the results found in [5] for β_c and will provide the self-consistent equations for the magnetizations $m^{(\mathrm{F})}$ and $m^{(\mathrm{SG})}$ whose solution, as expected, turns out to be in good agreement with the corresponding solutions found in [5] for c small and large (the latter when the frustration is relatively small). It will be, however, rather evident how much the two methods differ in terms of simplicity and intuitive meaning. Furthermore, we will derive also an explicit expression for the two-point connected correlation function. Finally, we will analyze in the detail the completely novel scenario for the case $J_0<0$, which, as mentioned, produces multiple first- and second-order phase transitions.

In order to apply the method of Sec. III we have to solve the one-dimensional Ising model with PBCs immersed in an external field. The solution of this non random model is easy and well known (see, for example [28],). If we indicate by λ_1 and λ_2 the two eigenvalues coming from the transfer matrix method, one has

$$\lambda_{1,2}=e^{\beta J_0}\cosh(\beta h)\pm[e^{2\beta J_0}\sinh^2(\beta h)+e^{-2\beta J_0}]^{1/2},$$

from which it follows that, for the free energy density, the magnetization, and the two-point connected correlation function, we have

$$-\beta f_0(\beta J_0,\beta h)=\ln(\lambda_1),$$

$$m_0(\beta J_0,\beta h)=\frac{e^{\beta J_0}\sinh(\beta h)}{[e^{2\beta J_0}\sinh^2(\beta h)+e^{-2\beta J_0}]^{1/2}}, \tag{83}$$

$$C_0(\beta J_0,\beta h;\|i-j\|_0)\overset{def}{=}\langle\sigma_i\sigma_j\rangle_0-\langle\sigma\rangle_0^2=\sin^2(2\varphi)\left(\frac{\lambda_2}{\lambda_1}\right)^{\|i-j\|_0}, \tag{84}$$

where the phase φ is defined by

$$\cot(2\varphi)=e^{2\beta J_0}\sinh(\beta h),\quad 0<\varphi<\frac{\pi}{2}, \tag{85}$$

and $\|i-j\|_0$ is the (Euclidean) distance between i and j.

Let us calculate $\tilde{\chi}_0$ and its first and second derivatives. From Eq. (83) we have

$$\tilde{\chi}_0(\beta J_0,\beta h)=\frac{e^{-\beta J_0}\cosh(\beta h)}{[e^{2\beta J_0}\sinh^2(\beta h)+e^{-2\beta J_0}]^{3/2}}, \tag{86}$$

$$\frac{\partial}{\partial\beta h}\tilde{\chi}_0(\beta J_0,\beta h)=\sinh(\beta h) \times\frac{e^{-3\beta J_0}-2e^{\beta J_0}\cosh^2(\beta h)-e^{\beta J_0}}{[e^{2\beta J_0}\sinh^2(\beta h)+e^{-2\beta J_0}]^{5/2}}, \tag{87}$$

$$\frac{\partial^2}{\partial(\beta h)^2}\tilde{\chi}_0(\beta J_0,\beta h)=\cosh(\beta h) \times\frac{e^{-3\beta J_0}-2e^{\beta J_0}\cosh^2(\beta h)-e^{\beta J_0}}{[e^{2\beta J_0}\sinh^2(\beta h)+e^{-2\beta J_0}]^{5/2}} +O(\beta h)^2. \tag{88}$$

From Eq. (88) we see that for $J_0>0$ and any β we have, for sufficiently small h,

$$\frac{\partial^2}{\partial(\beta h)^2}\tilde{\chi}_0(\beta J_0,\beta h) < \cosh(\beta h) \times \frac{1-3e^{\beta J_0}}{[e^{2\beta J_0}\sinh^2(\beta h)+e^{-2\beta J_0}]^{5/2}} + O(\beta h)^2 < 0, \quad (89)$$

whereas for $J_0<0$ we have

$$\frac{\partial^2}{\partial(\beta h)^2}\tilde{\chi}_0(\beta J_0,\beta h) \geqslant 0 \quad \text{for } e^{-4\beta J_0} > 3. \quad (90)$$

We see therefore that, according to Sec. III B, when $J_0<0$ for $\beta|J_0| \geqslant \ln(3)/4=0.1193\ldots$ the Landau coefficient $b^{(\mathrm{F})}$ is negative and we may have a first-order phase transition.

From Eqs. (15) and (83), for the magnetizations $m^{(\mathrm{F})}$ and $m^{(\mathrm{SG})}$ at zero external field we have

$$m^{(\Sigma)} = \frac{e^{\beta J_0^{(\Sigma)}}\sinh(\beta J^{(\Sigma)}m^{(\Sigma)})}{[e^{2\beta J_0^{(\Sigma)}}\sinh^2(\beta J^{(\Sigma)}m^{(\Sigma)})+e^{-2\beta J_0^{(\Sigma)}}]^{1/2}}. \quad (91)$$

From Eqs. (44) and (86) we see that a solution $m^{(\Sigma)}$ becomes unstable at the inverse temperature $\beta_c^{(\Sigma)}$ given by

$$e^{2\beta_c^{(\Sigma)}J_0^{(\Sigma)}}\beta_c^{(\Sigma)}J^{(\Sigma)} = 1. \quad (92)$$

For $J_0 \geqslant 0$ the above equation gives the exact P-F and P-SG critical temperatures in agreement with [5]. When $J_0<0$, unless the transition is of second order, Eq. (92) for $\Sigma=\mathrm{F}$ does not signal a phase transition. In general, as $J_0<0$ the P-F critical temperature must be determined by looking at all the stable solutions $m^{(\mathrm{F})}$ of the self-consistent equation (91) and by choosing the one minimizing the effective free energy $L^{(\mathrm{F})}(m)$ of Eq. (22).

Finally, for the two-point connected correlation function, from Eqs. (20), (84), and (85), we have

$$C^{(\Sigma)}(\|i-j\|_0) = \sin^2(2\varphi^{(\Sigma)})e^{-\|i-j\|_0/\xi^{(\Sigma)}},$$

where

$$2\varphi^{(\Sigma)} = \cot^{-1}[e^{2\beta J_0^{(\Sigma)}}\sinh(\beta J^{(\Sigma)}m^{(\Sigma)})],$$

and the correlation length $\xi^{(\Sigma)}$ is given by performing the effective substitutions $\beta J_0 \to \beta J_0^{(\Sigma)}$ and $\beta h \to \beta J^{(\Sigma)}m^{(\Sigma)}$ in $\ln(\lambda_1/\lambda_2)$.

Note that $C_0(\beta J_0,\beta h)$ is even in βh, so that $C(-m)=C(m)$. Near the critical temperature we have

$$\sin(2\varphi^{(\Sigma)}) = 1 - \frac{(e^{2\beta J_0^{(\Sigma)}}m^{(\Sigma)})^2}{2} + O(m^{(\Sigma)})^4$$

and

$$(\xi^{(\Sigma)})^{-1} = \left| \ln[\tanh(\beta J_0^{(\Sigma)})] - \frac{(\beta J^{(\Sigma)}m^{(\Sigma)})^2}{4\sinh(\beta J_0^{(\Sigma)})}\left[\left(e^{\beta J_0^{(\Sigma)}}+e^{3\beta J_0^{(\Sigma)}}\right)\times\tanh(\beta J_0^{(\Sigma)}) + e^{3\beta J_0^{(\Sigma)}} - e^{\beta J_0^{(\Sigma)}}\right] + O(m^{(\Sigma)})^4 \right|.$$

According to the general result, Eqs. (53)–(57), we see that the correlation length remains finite at all temperatures.

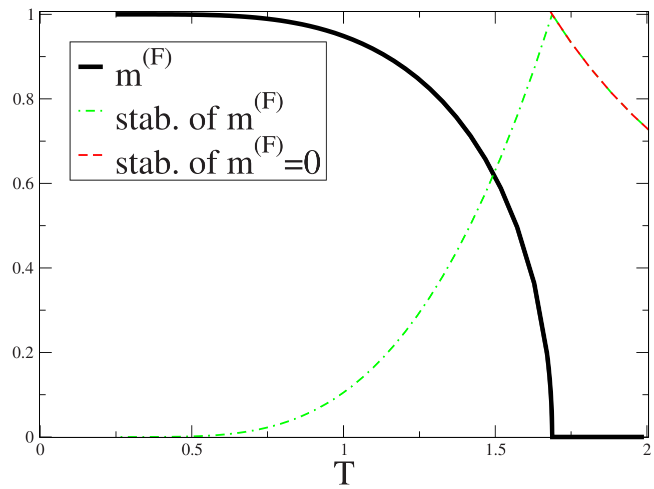

FIG. 5. (Color online) Magnetization (thick solid line) and curves of stability (dashed and dot-dashed lines) for the case $c=0.5$, $J_0=1$, and $J=3/5/$. Here $T_c=1.687$.

In Figs. 5–14 we plot the stable and leading magnetization $m^{(\mathrm{F})}$ (thick solid line), $\tilde{\chi}_0(\beta^{(\mathrm{F})}J_0^{(\mathrm{F})},0)\beta^{(\mathrm{F})}J^{(\mathrm{F})}$ (dashed line), and $\tilde{\chi}_0(\beta^{(\mathrm{F})}J_0^{(\mathrm{F})},\beta J^{(\mathrm{F})}m^{(\mathrm{F})})\beta^{(\mathrm{F})}J^{(\mathrm{F})}$ (dot-dashed line) for several cases obtained by solving Eq. (91) numerically with $\Sigma=\mathrm{F}$. The stable and leading solution (the only one drawn) corresponds to the solution that minimizes $L^{(\mathrm{F})}(m)$ [see Eq. (23)]. In all these examples we have chosen the measure (10). Figures 5 and 6 concern two cases with $J_0>0$ so that one and only one second-order phase transition is present. The input data of these two cases are the same as those analyzed numerically in [5] (note that in the model considered in [5], the long-range coupling J is divided by c). As already stated in Sec. III, the self-consistent equations become exact in the limit $c\to 0$, for second-order phase transitions, and in the limit $c\to\infty$. Therefore, for the magnetiza-

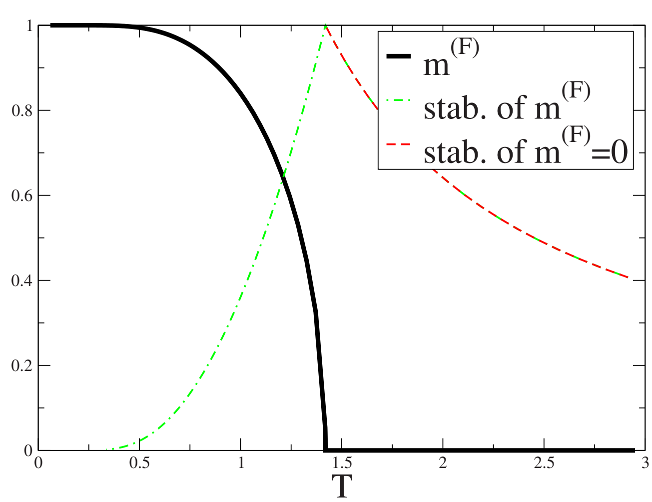

FIG. 6. (Color online) Magnetization (thick solid line) and curves of stability (dashed and dot-dashed lines) for the case $c=10$, $J_0=0.25$, and $J=1/c$. Here $T_c=1.419$.

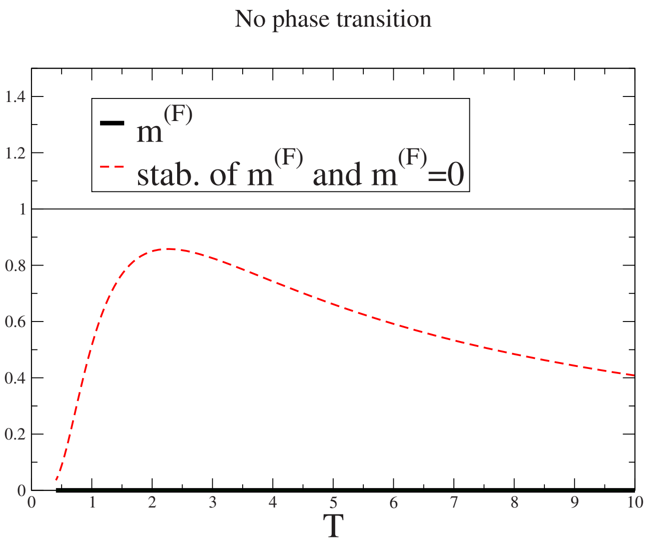

FIG. 7. (Color online) Magnetization (thick solid line) and curves of stability (dashed and dot-dashed lines) for the case $c=5$, $J_0=-1$, and $J=1$. Note that here $m^{(\mathrm{F})}=0$ and the two curves of stability coincide everywhere.

tion, by comparison with [5], in Figs. 5 and 6, where c is relatively small and large, respectively, we see good agreement also below the critical temperature.

Figures 7–14 concern eight cases with $J_0<0$. In these figures we plot also the line $y=1$, to make evident when the stability conditions for the solutions $m^{(\mathrm{F})}=0$ and $m^{(\mathrm{F})}\neq 0$, which are given by $\tilde{\chi}_0(\beta^{(\mathrm{F})}J_0^{(\mathrm{F})},0)\beta^{(\mathrm{F})}J^{(\mathrm{F})}<1$ (dashed line), and $\tilde{\chi}_0(\beta^{(\mathrm{F})}J_0^{(\mathrm{F})},\beta J^{(\mathrm{F})}m^{(\mathrm{F})})\beta^{(\mathrm{F})}J^{(\mathrm{F})}<1$ (dot-dashed line), are satisfied, respectively. As explained above, the critical behavior and the localization of the critical temperatures is more complicated when $J_0<0$. In particular, given $|J_0|$, if c is not sufficiently high, the solution $m^{(\mathrm{F})}=0$ remains stable at all temperatures, and if it is also a leading solution, no phase

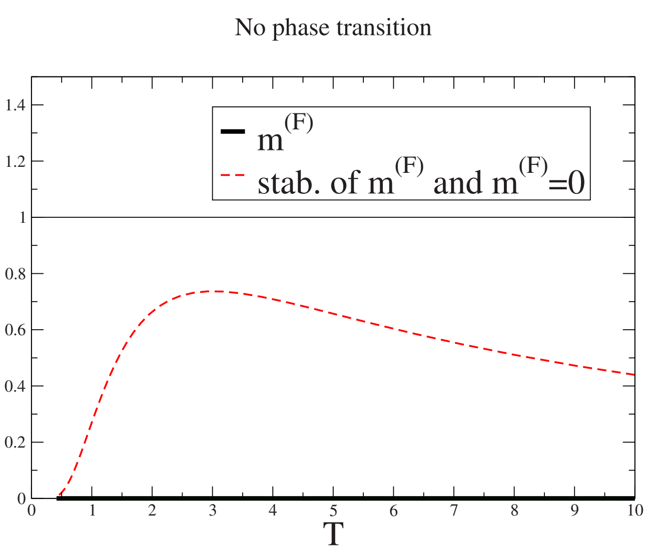

FIG. 8. (Color online) Magnetization (thick solid line) and curves of stability (dashed and dot-dashed lines) for the case $c=5.828$, $J_0=-1.4$, and $J=1$. Note that here $m^{(\mathrm{F})}=0$ and the two curves of stability coincide everywhere.

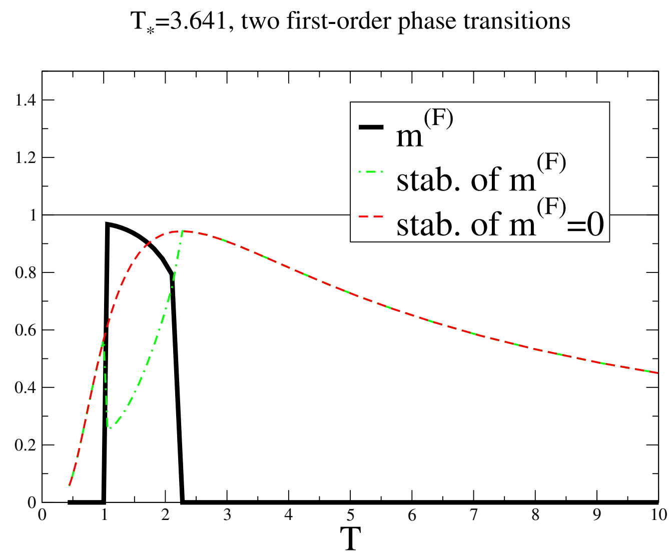

FIG. 9. (Color online) Magnetization (thick solid line) and curves of stability (dashed and dot-dashed lines) for the case $c=5.5$, $J_0=-1$, and $J=1$. Here $T_{c1}=1.02$ and $T_{c2}=2.27$.

transition occurs. Let us consider Eq. (92). For $J_0<0$ the lhs of this equation has some maximum at a finite value $\bar{\beta}$ given by

$$\bar{\beta}J=\frac{1}{2}\ln\left[\frac{1+\delta(r)}{1-\delta(r)}\right],$$

where $r\overset{def}{=}|J_0|/J$ and we have introduced

$$\delta(r)\overset{def}{=}\sqrt{1+r^2}-r.$$

Hence, we see that a sufficient condition for the solution $m^{(\mathrm{F})}=0$ to become unstable is that be

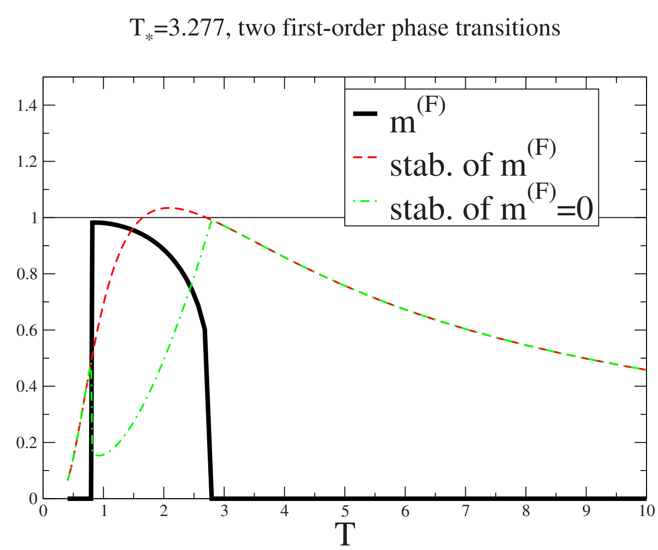

FIG. 10. (Color online) Magnetization (thick solid line) and curves of stability (dashed and dot-dashed lines) for the case $c=5.5$, $J_0=-0.9$, and $J=1$. Here $T_{c1}=0.85$ and $T_{c2}=2.78$.

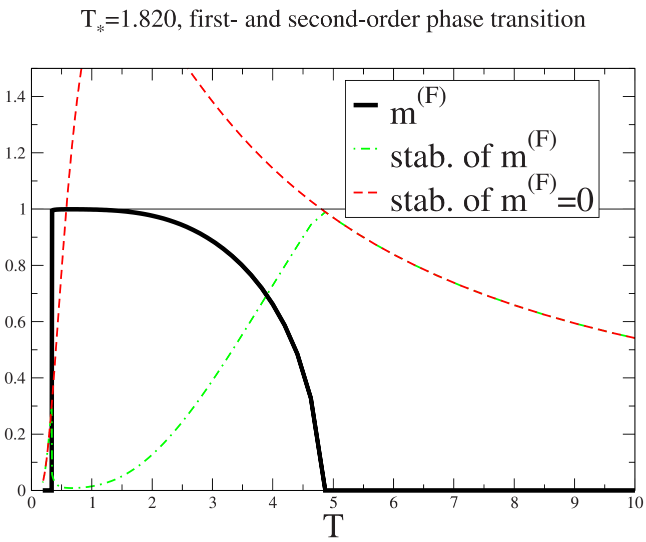

FIG. 11. (Color online) Magnetization (thick solid line) and curves of stability (dashed and dot-dashed lines) for the case $c=6$, $J_0=-0.5$, and $J=1$. Here $T_{c1}=0.35$ and $T_{c2}=4.87$. T_{c2} corresponds to a second-order phase transition.

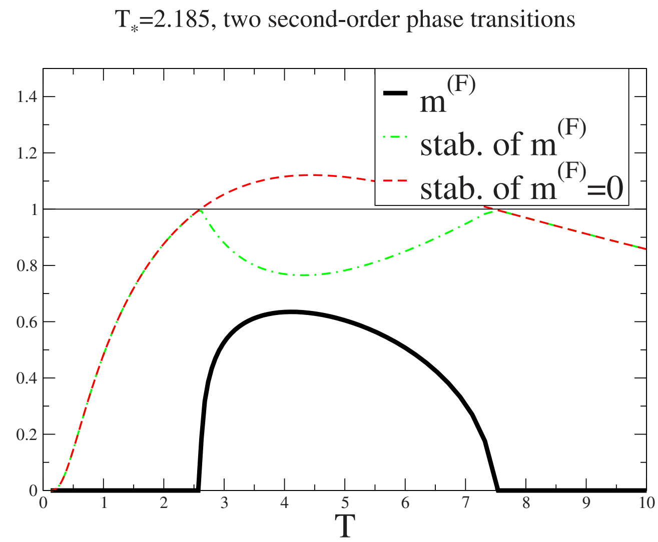

FIG. 13. (Color online) Magnetization (thick solid line) and curves of stability (dashed and dot-dashed lines) for the case $c=1.6$, $J_0=-0.6$, and $J=7$. Here $T_{c1}=2.58$ and $T_{c2}=7.55$.

$$c\left(\frac{1+\delta(r)}{1-\delta(r)}\right)^r \delta(r) \geqslant 1. \tag{93}$$

Note that the above represents only a condition for the instability of the solution $m^{(F)}=0$, but the true solution is the one that is both stable and leading. In fact, when $J_0<0$, a phase transition in general may be present also when Eq. (93) is not satisfied and, correspondingly the possible critical temperatures will be not determined by Eq. (92).

In Fig. 7 we report a case with $J=1$, $J_0=-1$ ($r=1$) and a relatively low value of c, $c=5$, so that no phase transition is present. Similarly, in Fig. 8 we report again a case in which no phase transition is present due to the fact that here r is relatively big, $r=1.1$. It is interesting to observe that for $r=1$ Eq. (93) requires a value of c greater than the limit value $c=3+2\sqrt{2}=5.8284\ldots$. In both Figs. 9 and 10 we report a case in which Eq. (93) is still not satisfied, but nevertheless two first-order phase transitions are present. In both Figs. 11 and 12 we have one first- and one second-order phase transition. In both Figs. 13 and 14 we have two second-order phase transitions. As anticipated in Sec. III B, we note that in Figs. 9–12—i.e., the cases in which there is at least one first-order phase transition—there are always regions where both solutions $m^{(F)}=0$ and $m^{(F)}\neq 0$ are simultaneously stable, but only one solution is leading (the one drawn), whereas in Figs. 13 and 14, as in Figs. 5 and 6, since we have only second-order phase transitions, the stability condition turns out to be a

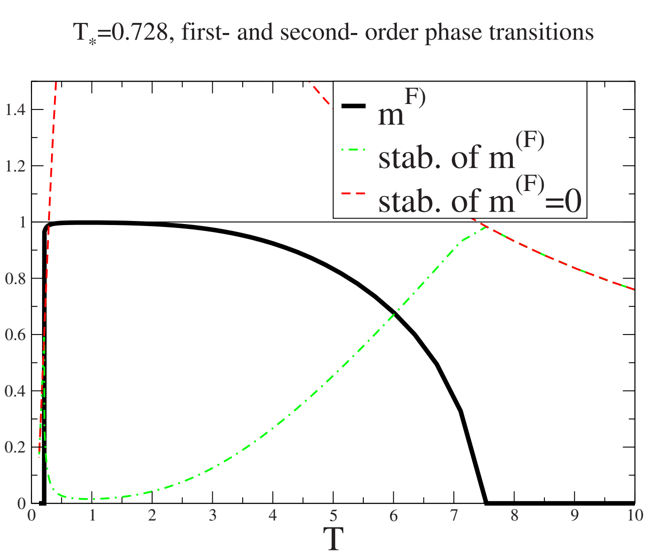

FIG. 12. (Color online) Magnetization (thick solid line) and curves of stability (dashed and dot-dashed lines) for the case $c=4$, $J_0=-0.2$, and $J=2$. Here $T_{c1}=0.22$, and $T_{c2}=7.55$. T_{c2} corresponds to a second-order phase transition.

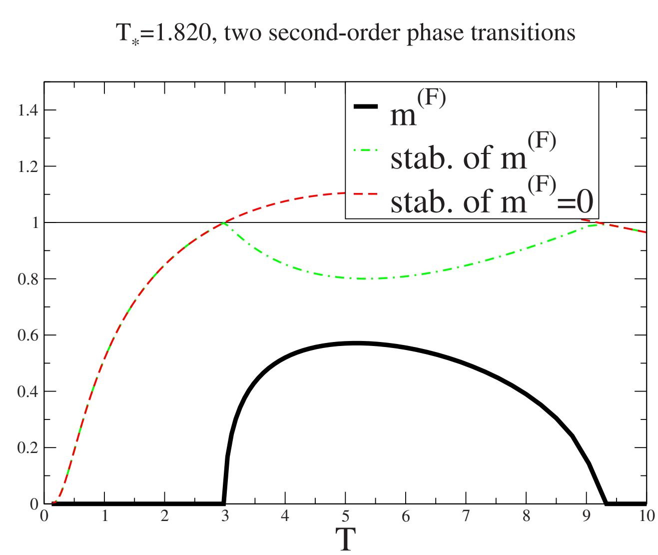

FIG. 14. (Color online) Magnetization (thick solid line) and curves of stability (dashed and dot-dashed lines) for the case $c=1.4$, $J_0=-0.5$, and $J=10$. Here $T_{c1}=3.00$ and $T_{c2}=9.34$.

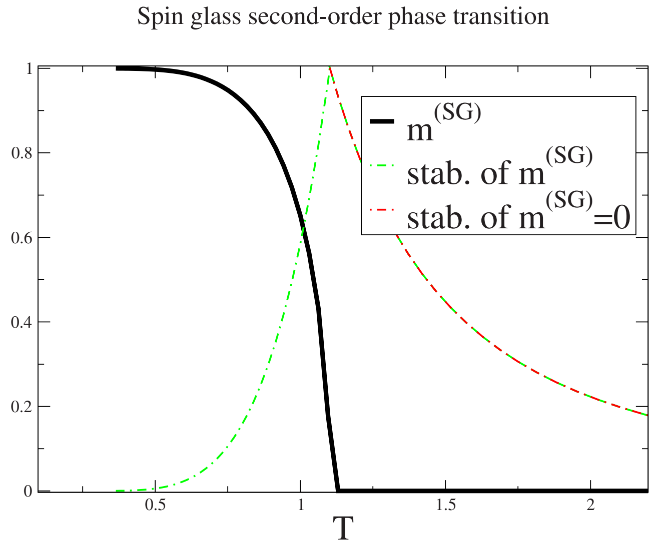

FIG. 15. (Color online) Spin-glass order parameter (solid line) and curves of stability (dashed and dot-dashed lines) for the case $c=0.5$, $J_0=1$, and $J=3/5/c$. Here $T_c=1.130$.

necessary and sufficient condition for determining the leading solution and the critical temperature can be determined also by Eq. (92) with $\Sigma=\mathrm{F}$.

In the top of Figs. 9–14 we write the discriminant temperature $T_*=4|J_0|/\ln(3)$ below which a phase transition (if any) may be first order [see Eqs. (63) and Eqs. (88)–(90)].

Finally in Figs. 15 and 16 we plot the spin-glass order parameter $m^{(\mathrm{SG})}$ (solid line), $\tilde{\chi}_0(\beta^{(\mathrm{SG})}J_0^{(\mathrm{SG})},0)\beta^{(\mathrm{SG})}J^{(\mathrm{SG})}$ (dashed line), and $\tilde{\chi}_0(\beta^{(\mathrm{SG})}J_0^{(\mathrm{SG})},\beta J^{(\mathrm{SG})}m^{(\mathrm{SG})})\beta^{(\mathrm{SG})}J^{(\mathrm{SG})}$ (dot-dashed line) obtained by solving Eq. (91) numerically with $\Sigma=\mathrm{SG}$. In these two examples we have chosen the measure (11) and, for c, $|J|$, and J_0, we have considered the same parameters of Figs. 5 and 6 of the ferromagnetic case. As anticipated in Sec. III B, due to the fact that the effective

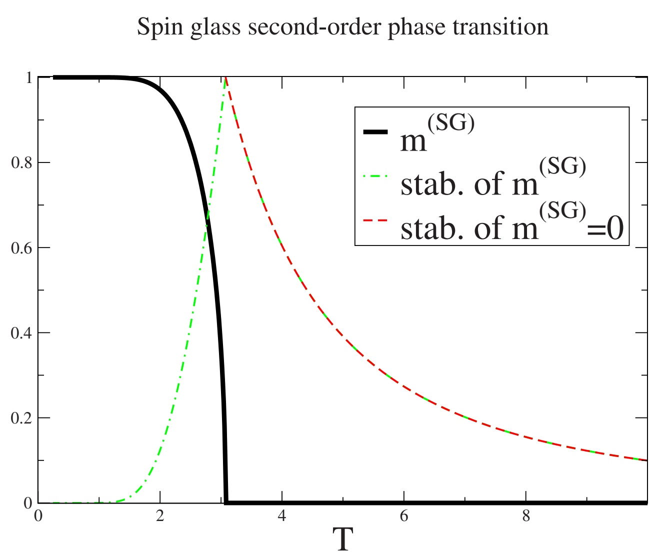

FIG. 16. (Color online) Spin-glass order parameter (solid line) and curves of stability (dashed and dot-dashed lines) for the case $c=10$, $J_0=0.25$, and $J=1/c$. Here $T_c=0.424$.

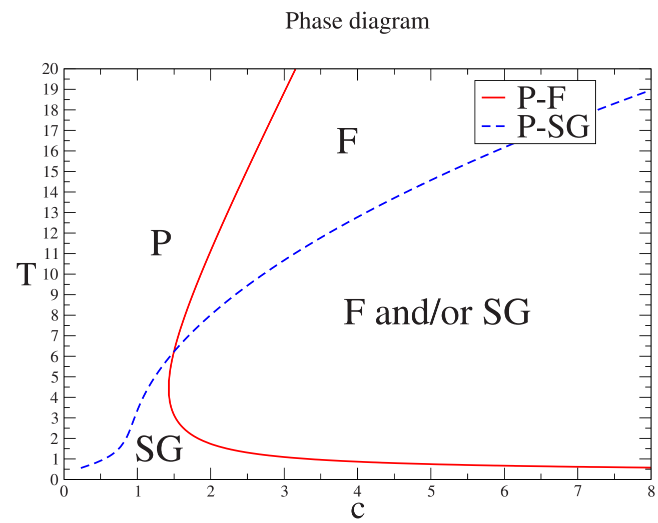

FIG. 17. (Color online) Phase diagram for the case considered in Fig. 13 with the measure of Eq. (10).

coupling $J_0^{(\mathrm{SG})}$ is positive, there is only a second-order phase transition; the stability condition turns out to be a necessary and sufficient condition for determining the leading solution, and the critical temperature can be determined also by Eq. (92) with $\Sigma=\mathrm{SG}$.

Note that, unlike the P-F critical surface, the P-SG critical surface does not depend on the parameter p entering in Eq. (11). For the reciprocal stability between the P-F and P-SG critical surfaces we remind the reader of the general rules of Sec. III D [see cases (i) and (iii)], which, for $J_0 \geqslant 0$, reduce to the results reported in Sec. 6.1 of the Ref. [5]. Here we stress just that, if $J_0 \geqslant 0$, for $p \leqslant 0.5$, only the P-SG transition is possible. However, when $J_0<0$ and c is not sufficiently large, the SG phase may be the only stable phase even when $p=1$. In fact, although when $J_0<0$ the solution $m^{(\mathrm{F})}$ may have two P-F critical temperatures, in general, if the P-SG temperature is between these, we cannot exclude that the solution $m^{(\mathrm{SG})}$ starts to be the leading solution at sufficiently low temperatures. In Figs. 17 and 18, on the plane (T,c), we plot the phase diagrams corresponding to the cases of Figs. 13 and 14, respectively. These phase diagrams are obtained by solving Eq. (44) supposing that here, as in the cases of Figs. 13 and 14, where $c=1.4$ and $c=0.5$, respectively, the P-F transition is always second order. We plan to investigate in more detail the phase diagram in future works.

VI. SMALL-WORLD SPHERICAL MODEL IN ARBITRARY DIMENSION d_0

In this section we will analyze the case in which the unperturbed model is the spherical model built up over a d_0-dimensional lattice $\mathcal{L}_0$ (see [28] and references therein).[1] In this case the σ's are continuous "spin" variables ranging in the interval $(-\infty,\infty)$ subjected to the sole constraint

[1] Note that from the point of view of the statistical mechanics the spherical model is an infinite-dimensional model; however, we will continue to reserve the symbol d_0 for the dimension of $\mathcal{L}_0$.

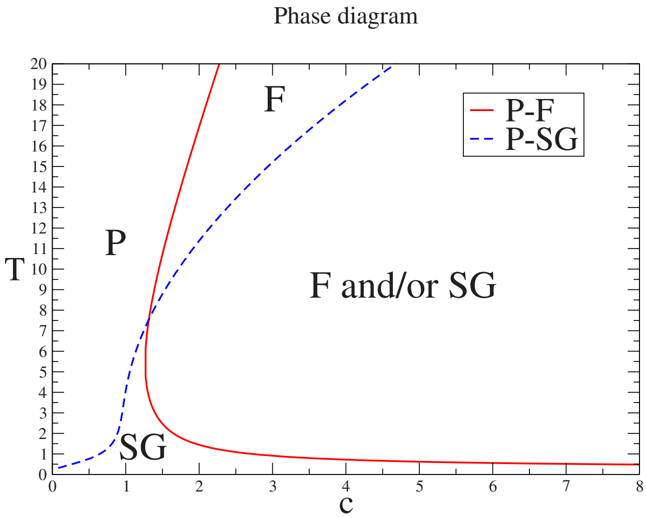

FIG. 18. (Color online) Phase diagram for the case considered in Fig. 14 with the measure of Eq. (10).

$\Sigma_{i\in\mathcal{L}_0}\sigma_i^2=N$; however, our theorems and formalism can be applied as well and give results that, within the same limitations prescribed in Sec. III, are exact.

Following [28], for the unperturbed model we have

$$-\beta f_0(\beta J_0,\beta h)=\frac{1}{2}\ln\left(\frac{\pi}{\beta J_0}\right)+\phi(\beta J_0,\beta h,\bar{z}),$$

$$m_0(\beta J_0,\beta h)=\frac{\beta h}{2\beta J_0\bar{z}}, \tag{94}$$

where

$$\phi(\beta J_0,\beta h,z)=\beta J_0 d_0+\beta J_0 z-\frac{1}{2}g(z)+\frac{(\beta h)^2}{4\beta J_0 z},$$

$$g(z)=\frac{1}{(2\pi)^{d_0}}\int_0^{2\pi}\cdots\int_0^{2\pi}d\omega_1\cdots d\omega_{d_0}\times\ln[d_0+z-\cos(\omega_1)-\cdots-\cos(\omega_{d_0})],$$

and $\bar{z}=\bar{z}(\beta J_0,\beta h)$ is the (unique) solution of the equation $\partial_z\phi(\beta J_0,\beta h,z)=0$,

$$\beta J_0-\frac{(\beta h)^2}{4\beta J_0\bar{z}^2}=\frac{1}{2}g'(\bar{z}), \tag{95}$$

from which follows the equation for m_0:

$$\beta J_0(1-m_0^2)=\frac{1}{2}g'\left(\frac{\beta h}{2\beta J_0 m_0}\right). \tag{96}$$

The derivative g' can in turn be expressed as

$$g'(z)=\int_0^{\infty}e^{-t(z+d_0)}[\mathcal{J}_0(it)]^{d_0}dt, \tag{97}$$

$\mathcal{J}_0(it)$ being the usual Bessel function whose behavior for large t is given by

$$\mathcal{J}_0(it)=\frac{e^t}{(2\pi t)^{1/2}}\left[1+O\left(\frac{1}{t}\right)\right].$$

The critical behavior of the unperturbed system depends on the values of $g'(z)$ and $g''(z)$ near $z=0$. It turns out that for $d_0\leq 2$ one has $g'(0)=\infty$ and there is no spontaneous magnetization, whereas for $d_0>2$ one has $g'(0)<\infty$ and at $h=0$ the unperturbed system undergoes a second-order phase transition with magnetization given by Eq. (96), which, for β above β_{c0}, becomes

$$m_0(\beta J_0,0)=\sqrt{1-\frac{\beta_{c0}}{\beta}},$$

where the inverse critical temperature β_{c0} is given by

$$\beta_{c0}J_0=\frac{1}{2}g'(0).$$

Furthermore, it turns out that for $d_0\leq 4$ one has $g''(0)=\infty$, whereas for $d_0>4$ one has $g''(0)<\infty$. This reflects on the critical exponents α, γ, and δ, which take the classical mean-field values only for $d_0>4$.

According to Sec. III, to solve the random model, for simplicity, at zero external field, we have to perform the effective substitutions $\beta J_0\to\beta J_0^{(\Sigma)}$ and $\beta h\to\beta J^{(\Sigma)}m^{(\Sigma)}$ in the above equations. From Eqs. (94)–(96), we get immediately

$$\bar{z}^{(\Sigma)}=\frac{\beta J^{(\Sigma)}}{2\beta J_0^{(\Sigma)}},$$

the equations for the inverse critical temperature $\beta_c^{(\Sigma)}$,

$$\beta_c^{(\Sigma)}J_0^{(\Sigma)}=\frac{1}{2}g'\left(\frac{\beta_c^{(\Sigma)}J^{(\Sigma)}}{2\beta_c^{(\Sigma)}J_0^{(\Sigma)}}\right), \tag{98}$$

and the magnetizations $m^{(\Sigma)}$,

$$m^{(\Sigma)}=\begin{cases}\sqrt{1-\dfrac{1}{2\beta J_0^{(\Sigma)}}g'\left(\dfrac{\beta J^{(\Sigma)}}{2\beta J_0^{(\Sigma)}}\right)}, & \beta>\beta_c^{(\Sigma)},\\ 0, & \beta<\beta_c^{(\Sigma)}\geq 0.\end{cases} \tag{99}$$

Note that, as must be from the general result of Sec. III B, unlike the unperturbed model, as soon as the connectivity c is not zero, Eq. (98) has always a finite solution $\beta_c^{(\Sigma)}$, independently of the dimension d_0. In fact, one has a finite-temperature second-order phase transition even for $d_0\to 0^+$ where from Eq. (97) we have

$$g'(z)=\frac{1}{z},\quad d_0=0,$$

so that the equations for the critical temperature Eq. (98), become

$$\beta_c^{(\Sigma)}J^{(\Sigma)}=1,\quad d_0=0,$$

which, as expected, coincide with Eqs. (76) and (77) of the Viana-Bray model.

Similarly, unlike the unperturbed model, in the random model all the critical exponents take the classical mean-field values, independently of the dimension d_0. In the specific case of the spherical model, this behavior is due to the fact that $g'(z)$ and $g''(z)$ can be singular only at $z=0$, but as soon as the connectivity c is not zero, there is an effective external field $\beta J^{(\Sigma)}m^{(\Sigma)}$ so that $\overline{z}^{(\Sigma)}$ is not zero. For the critical behavior, the dependence on the dimension d_0 reflects only in the coefficients, not on the critical exponents. In particular, concerning the argument of the square root of the rhs of Eq. (99), by expanding in the reduced temperature $t^{(\Sigma)}$, for $|t^{(\Sigma)}| \ll 1$ we have

$$1-\frac{1}{2\beta_c^{(\Sigma)}J_0^{(\Sigma)}}g'\left(\frac{\beta_c^{(\Sigma)}J^{(\Sigma)}}{2\beta_c^{(\Sigma)}J_0^{(\Sigma)}}\right)=B^{(\Sigma)}t^{(\Sigma)}+O(t^{(\Sigma)})^2, \tag{100}$$

where

$$B^{(\mathrm{F})}=-1+\frac{1}{2\beta_c^{(\mathrm{F})}J_0^{(\mathrm{F})}}g''\left(\frac{\beta_c^{(\mathrm{F})}J^{(\mathrm{F})}}{(2\beta_c^{(\mathrm{F})}J_0^{(\mathrm{F})})^2}\right)\times\left(c\int d\mu(J_{i,j})[1-\tanh^2(\beta_c^{(\mathrm{F})}J_{i,j})]\beta_c^{(\mathrm{F})}J_{i,j}-c\int d\mu(J_{i,j})\tanh(\beta_c^{(\mathrm{F})}J_{i,j})\right),$$

$$B^{(\mathrm{SG})}=\frac{1}{2\beta_c^{(\mathrm{SG})}J_0^{(\mathrm{SG})}}\Bigg[-4\frac{\tanh(\beta_c^{(\mathrm{SG})}J_0)\beta_c^{(\mathrm{SG})}J_0}{1+\tanh^2(\beta_c^{(\mathrm{SG})}J_0)}+g''\left(\frac{\beta_c^{(\mathrm{SG})}J^{(\mathrm{SG})}}{(2\beta_c^{(\mathrm{SG})}J_0^{(\mathrm{SG})})^2}\right)\Bigg(2c\int d\mu(J_{i,j})\times[1-\tanh^2(\beta_c^{(\mathrm{SG})}J_{i,j})]\tanh(\beta_c^{(\mathrm{SG})}J_{i,j})\beta_c^{(\mathrm{SG})}J_{i,j}-4c\int d\mu(J_{i,j})\tanh^2(\beta_c^{(\mathrm{SG})}J_{i,j})\times\frac{\tanh(\beta_c^{(\mathrm{SG})}J_0)\beta_c^{(\mathrm{SG})}J_0}{[1+\tanh^2(\beta_c^{(\mathrm{SG})}J_0)]\beta_c^{(\mathrm{SG})}J^{(\mathrm{SG})}}\Bigg)\Bigg],$$

so that from Eqs. (99) and (100) for the critical behavior of the magnetizations we get explicitly the mean-field behavior

$$m^{(\Sigma)}=\begin{cases}\sqrt{B^{(\Sigma)}t^{(\Sigma)}}+O(t^{(\Sigma)}), & t^{(\Sigma)}<0,\\ 0, & t^{(\Sigma)}\geq 0.\end{cases}$$

VII. MAPPING TO NON RANDOM MODELS

In Sec. VIII we will derive the main result presented in Sec. III. To this aim in the next section VII A we will recall the general mapping between a random model built up over a given graph and a non random one built up over the same graph, whereas in Sec. VII B we will generalize this mapping to random models built up over random graphs. We point out that the mapping does not consist in a sort of annealed approximation.

A. Random models defined on quenched graphs

Let us consider the following random model. Given a graph $\boldsymbol{g}$, which can be determined through the adjacency matrix for shortness also indicate by $\boldsymbol{g}=\{g_b\}$, with $g_b=0,1$, b being a bond, let us indicate with $\Gamma_{\boldsymbol{g}}$ the set of the bonds b of $\boldsymbol{g}$ and let us define over $\Gamma_{\boldsymbol{g}}$ the Hamiltonian

$$H(\{\sigma_i\};\{J_b\})\overset{def}{=}-\sum_{b\in\Gamma_{\boldsymbol{g}}}J_b\sigma_{i_b}\sigma_{j_b}-\sum_i h_i\sigma_i, \tag{101}$$

where J_b is the random coupling at the bond b and σ_{i_b} and σ_{i_b} are the Ising variables at the end points of b. The free energy F and the physics are defined as in Sec. II by Eqs. (3)–(6):

$$-\beta F\overset{def}{=}\int d\mathcal{P}(\{J_b\})\ln[Z(\{J_b\})], \tag{102}$$

$$\overline{\langle\mathcal{O}\rangle^l}\overset{def}{=}\int d\mathcal{P}(\{J_b\})\langle\mathcal{O}\rangle^l, \quad l=1,2, \tag{103}$$

where $d\mathcal{P}(\{J_b\})$ is a product measure over all the possible bonds b given in terms of normalized measures $d\mu_b\geq 0$ (we are considering a general measure $d\mu_b$ allowing also for a possible dependence on the bonds),

$$d\mathcal{P}(\{J_b\})\overset{def}{=}\prod_{b\in\Gamma_{\mathrm{full}}}d\mu_b(J_b), \qquad \int d\mu_b(J_b)=1, \tag{104}$$

where Γ_{full} stands for the set of bonds of the fully connected graph. As in Sec. II, we will indicate a generic correlation function, connected or not, by C with understood indices $i_1,\ldots,i_k$ all different; see Eqs. (12) and (13).

In the following, given an arbitrary vertex i of $\boldsymbol{g}$, we will consider as first neighbors j of i only those vertices for which $\int d\mu_{i,j}(J_{i,j})J_{i,j}$ or $\int d\mu_{i,j}(J_{i,j})J_{i,j}^2$ are at least $O(1/N)$. Note that we can always neglect couplings having lower averages. We will indicate by $D(\Gamma_{\boldsymbol{g}})$ the average number of first neighbors of the graph $\boldsymbol{g}$. For a d-dimensional lattice, $D(\Gamma_{\boldsymbol{g}})=2d-1$; for a Bethe lattice of coordination number q, $D(\Gamma_{\boldsymbol{g}})=q-1$; and for long-range models, $D(\Gamma_{\boldsymbol{g}})\propto N$. We will exploit in particular the fact that $D(\Gamma_{\mathcal{L}_0}\cup\Gamma_{\mathrm{full}})=D(\Gamma_{\mathrm{full}})\propto N$.

Given a random model defined through Eqs. (101)–(104), we define, on the same set of bonds $\Gamma_{\boldsymbol{g}}$, its *related Ising model* through the Ising Hamiltonian

$$H_I(\{\sigma_i\};\{J_b^{(I)}\})\overset{def}{=}-\sum_{b\in\Gamma_{\boldsymbol{g}}}J_b^{(I)}\sigma_{i_b}\sigma_{j_b}-\sum_i h_i\sigma_i, \tag{105}$$

where the Ising couplings $J_b^{(I)}$ have nonrandom values such that $\forall b,b'\in\Gamma_{\boldsymbol{g}}$:

$$J_{b'}^{(I)}=J_b^{(I)} \quad \text{if} \quad d\mu_{b'}\equiv d\mu_b, \tag{106}$$

$$J_b^{(I)}\neq 0 \quad \text{if} \begin{cases}\int d\mu_b(J_b)J_b=O\left(\frac{1}{N}\right),\\ \int d\mu_b(J_b)J_b^2=O\left(\frac{1}{N}\right).\end{cases} \tag{107}$$

In the following a suffix I over quantities such as H_I, F_I, f_I, g_I, etc., or $J_b^{(I)}$, $\beta_c^{(I)}$, etc., will be referred to the related Ising system with Hamiltonian (105).

We can always split the free energy of the random system with N spins as follows:

$$-\beta F = \sum_b \int d\mu_b \ln[\cosh(\beta J_b)] + \sum_i \ln[2\cosh(\beta h_i)] + \phi, \tag{108}$$

ϕ being the high-temperature part of the free energy. Let φ be the density of ϕ in the thermodynamic limit:

$$\varphi \overset{def}{=} \lim_{N\to\infty} \phi/N. \tag{109}$$

Let us indicate by φ_I the high-temperature part of the free energy density of the related Ising model defined through Eqs. (105)–(107). As is known, φ_I can be expressed in terms of the quantities $z_b = \tanh(\beta J_b^{(I)})$ and $z_i = \tanh(\beta h_i)$—i.e., the parameters of the high-temperature expansion:

$$\varphi_I = \varphi_I(\{\tanh(\beta J_b^{(I)})\};\{\tanh(\beta h_i)\}). \tag{110}$$

The related Ising model is defined by a set of, typically few, independent couplings $\{J_b^{(I)}\}$, through Eqs. (106) and (107) and, for $h_i=0$, $i=1,\ldots,N$, its critical surface will be determined by the solutions of an equation, possibly vectorial, $G_I(\{\tanh(\beta J_b^{(I)})\})=0$.

In [13] we have proved the following mapping.

Let $\beta_c^{(SG)}$ and $\beta_c^{(F/AF)}$ be, respectively, solutions of the two equations

$$G_I\left(\left\{\int d\mu_b \tanh^2(\beta_c^{(SG)} J_b)\right\}\right) = 0, \tag{111}$$

$$G_I\left(\left\{\int d\mu_b \tanh(\beta_c^{(F/AF)} J_b)\right\}\right) = 0. \tag{112}$$

Asymptotically, at sufficiently high dimensions $D(\Gamma_g)$, the critical inverse temperature of the spin-glass model β_c is given by

$$\beta_c = \min\{\beta_c^{(SG)}, \beta_c^{(F/AF)}\}, \tag{113}$$

and in the paramagnetic phase for $D(\Gamma_g)>2$ the following mapping holds:

$$\left|\frac{\varphi-\varphi_{eff}}{\varphi}\right| = \left|\frac{C-C_{eff}}{C}\right| = O\left(\frac{1}{D(\Gamma_g)}\right), \tag{114}$$

$$\varphi_{eff} = \frac{1}{l}\varphi^{(\Sigma)} \overset{def}{=} \frac{1}{l}\varphi_I\left(\left\{\int d\mu_b \tanh^l(\beta J_b)\right\}\right), \tag{115}$$

and

$$C_{eff} = \frac{1}{l}C^{(\Sigma)} \overset{def}{=} \frac{1}{l}C_I\left(\left\{\int d\mu_b \tanh^l(\beta J_b)\right\}\right), \tag{116}$$

where

$$l = \begin{cases} 2, & \text{if } \varphi_I\left(\left\{\int d\mu_b \tanh^2(\beta J_b)\right\}\right) \geqslant 2\left|\varphi_I\left(\left\{\int d\mu_b \tanh(\beta J_b)\right\}\right)\right|, \\ 1, & \text{if } \varphi_I\left(\left\{\int d\mu_b \tanh^2(\beta J_b)\right\}\right) < 2\left|\varphi_I\left(\left\{\int d\mu_b \tanh(\beta J_b)\right\}\right)\right|, \end{cases} \tag{117}$$

and Σ=F/AF or SG for $l=1$ or 2, respectively.

In the limit $D(\Gamma_g)\to\infty$ and $h_i=0$, $i=1,\ldots,N$, Eqs. (111)–(117) give the exact free energy and correlation functions in the paramagnetic phase (P); the exact critical paramagnetic-spin glass (P-SG), $\beta_c^{(SG)}$, and paramagnetic-F/AF (P-F/AF), $\beta_c^{(F/AF)}$, surfaces, whose reciprocal stability depends on which of the two ones has higher temperature. In the case of a measure $d\mu$ not depending on the bond b, the suffices F and AF stand for ferromagnetic and antiferromagnetic, respectively. In the general case, such a distinction is possible only in the positive and negative sectors in the space of the parameters of the probability distribution, $\{\int d\mu_b J_b \geqslant 0\}$ and $\{\int d\mu_b J_b < 0\}$, respectively, whereas, for the other sectors, we use the symbol F/AF only to stress that the transition is not P-SG.

It is not difficult to see that, when the measure $d\mu$ does not depend on the specific bond b—i.e., if $d\mu_b \equiv d\mu_{b'}, \forall\ b,b' \in \Gamma_g$—in the P region, Eqs. (111)–(117) lead to the following exact limit for φ and C [15]:

$$\lim_{D(\Gamma_g)\to\infty} \varphi = \lim_{D(\Gamma_g)\to\infty} C = 0, \quad \text{for } \beta \leqslant \beta_c. \tag{118}$$

Therefore, the basic role of Eqs. (114)–(117) is to show how, in the limit $D(\Gamma_g)\to\infty$, φ and C approach zero and which are their singularities. In particular, this proves that for all (random) infinite-dimensional models and any disorder non-bond-dependent, the critical exponent α' for the specific heat has the mean-field classical value $\alpha'=0$ and that the correlation functions (with different indices) above the critical temperature are exactly zero. We point out, however, that, when the measure $d\mu_b$ depends explicitly on the bond b, Eq. (118) in general does not hold.[2] In fact, when the measure $d\mu_b$ is bond dependent, the symmetry expressed by Eq. (118) is broken since the bonds are no longer equivalent. As we will see in the next section, in small-world models with an underlying lattice $\mathcal{L}_0$ having $d_0<2$, even if Eq. (118) still holds

[2]This was not strongly emphasized in (15).

for φ, the symmetry is broken for C since the direction(s) of the axis(es) of $\mathcal{L}_0$ is (are) now favored direction(s). Yet, if $2 \leq d_0 < \infty$, the symmetry (118) for φ is broken as well.

The analytic continuation of Eqs. (114)–(117) to $\beta > \beta_c$ and/or for $h \neq 0$ provides certain estimations which are expected to be qualitatively good. In general, such estimations are not exact, and this is particularly evident for the free energy density of the SG phase. However, the analytic continuation for the other quantities gives a good qualitative result and provides the exact critical behavior (in the sense of the critical indices) and the exact percolation threshold.

For models defined over graphs satisfying a weak definition of infinite dimensionality, as happens on a Bethe lattice with coordination number $q>2$, a more general mapping has been established [15], In this case, all the above equations, along the critical surface (at least), still hold exactly in the thermodynamic limit, where we can set effectively $D(\Gamma_{\mathbf{g}}) = \infty$. However, for the aims of this paper we do not need here to consider this generalization of the mapping.

We have yet to make an important comment about Eqs. (25), (26), and (28), concerning the evaluation of a correlation function in the SG phase here for a random system with $J_0=0$ (for the moment being). In fact, Eq. (116), for both a normal and a quadratic correlation function $C^{(1)}$ or $C^{(2)}$, has a factor $1/2$ not entering in the physical equations (25), (26), and (28). The difference is just due to an artifact of the mapping that separates the Gibbs state into two pure states [18] not only in the F case, but also in the SG case. In fact, let us consider the correlation functions of order $k=1$—that is, $C^{(1)}=\overline{\langle\sigma_1\rangle}$ and $C^{(2)}=q_{EA}=\overline{\langle\sigma_1\rangle^2}$. We see that, for $C^{(1)}$, Eq. (116) in the SG phase gives $C^{(1)}=m^{(\mathrm{SG})}/2$. On the other hand, for any nonzero solution $m^{(\mathrm{SG})}$ of the self-consistent equation (15), there exists another solution $-m^{(\mathrm{SG})}$, and both solutions have $1/2$ probability to be realized in the random model. Since the SG phase is expected to be the phase characterized by having $q_{EA}\neq 0$ and $\overline{\langle\sigma_1\rangle}=0$, we see that if we introduce both the solutions $m^{(\mathrm{SG})}$ and $-m^{(\mathrm{SG})}$, we get $\overline{\langle\sigma_1\rangle}=0$ in the SG phase. Similarly, for $C^{(2)}$, Eq. (116) in the SG phase gives $C^{(2)}=(m^{(\mathrm{SG})})^2/2$, which at zero temperature gives $1/2$, whereas a completely frozen state with $q_{EA}=1$ is expected. Again, we recover the expected physical q_{EA} by using both the solutions $m^{(\mathrm{SG})}$ and $-m^{(\mathrm{SG})}$. Repeating a similar argument for any correlation function of order k and recalling that for k even (odd) the correlation function is an even (odd) function of the external magnetic field h, we arrive at Eqs. (25), (26), and (28).

B. Random models defined on unconstrained random graphs

Let us consider now more general random models in which the source of the randomness comes from both the randomness of the couplings and the randomness of the graph. Given an ensemble of graphs $\mathbf{g} \in \mathcal{G}$ distributed with some distribution $P(\mathbf{g})$, let us define

$$H_{\mathbf{g}}(\{\sigma_i\};\{J_b\}) \stackrel{def}{=} -\sum_{b\in\Gamma_{\mathbf{g}}} J_b\sigma_{i_b}\sigma_{j_b} - h\sum_i \sigma_i = -\sum_{b\in\Gamma_{\mathrm{full}}} g_b J_b\sigma_{i_b}\sigma_{j_b} - h\sum_i \sigma_i. \quad (119)$$

The free energy F and the physics are now given by

$$-\beta F \stackrel{def}{=} \sum_{\mathbf{g}\in\mathcal{G}} P(\mathbf{g}) \int d\mathcal{P}(\{J_b\})\ln[Z_{\mathbf{g}}(\{J_b\})],$$

and similarly for $\overline{\langle O\rangle^l}$, $l=1,2$. Here $Z_g(\{J_b\})$ is the partition function of the quenched system onto the graph realization $\mathbf{g}$ with bonds in $\Gamma_{\mathbf{g}}$,

$$Z_{\mathbf{g}}(\{J_b\}) = \sum_{\{\sigma_i\}} e^{-\beta H_{\mathbf{g}}(\{\sigma_i\};\{J_b\})},$$

and $d\mathcal{P}(\{J_b\})$ is again a product measure over all the possible bonds b given as defined in Eq. (104). Note that the bond variables $\{g_b\}$ are independent from the coupling variables $\{J_b\}$.

For unconstrained random graphs, or for random graphs having a number of constraints that grows sufficiently slowly with N, the probability $P(\mathbf{g})$, for large N, factorizes as

$$P(\mathbf{g}) = \prod_{b\in\Gamma_{\mathrm{full}}} p_b(g_b).$$

In such a case we can exploit the mapping we have previously seen for models over quenched graphs as follows. Let us define the effective coupling $\tilde{J}_b$:

$$\tilde{J}_b \stackrel{def}{=} J_b \cdot g_b.$$

Correspondingly, since the random variables J_b and g_b are independent, we have

$$d\tilde{\mu}_b(\tilde{J}_b) = d\mu_b(J_b)\cdot p_b(g_b),$$

with the sum rule

$$\int d\tilde{\mu}_b(\tilde{J}_b)f(J_b;g_b) = \sum_{g_b=0,1} p_b(g_b) \int d\mu_b(J_b)f(J_b;g_b).$$

As a consequence, if we define the following global measure

$$d\tilde{\mathcal{P}}(\{\tilde{J}_b\}) = P(\mathbf{g})\cdot d\mathcal{P}(\{J_b\}) = \prod_{b\in\Gamma_{\mathrm{full}}} d\tilde{\mu}_b(\tilde{J}_b),$$

we see that the mapping of the previous section can be applied as we had a single effective graph Γ_p given by

$$\Gamma_p \stackrel{def}{=} \{b \in \Gamma_{\mathrm{full}}: p_b(g_b=1) \neq 0\}.$$

In fact, we have

$$-\beta F = \int d\tilde{\mathcal{P}}(\{\tilde{J}_b\})\ln[Z_p(\{\tilde{J}_b\})],$$

where Z_p is the partition function of the model with Hamiltonian H_p given by

$$H_p(\{\sigma_i\};\{\tilde{J}_b\}) \stackrel{def}{=} -\sum_{b\in\Gamma_p} \tilde{J}_b\sigma_{i_b}\sigma_{j_b} - h\sum_i \sigma_i. \quad (120)$$

VIII. DERIVATION OF THE SELF-CONSISTENT EQUATIONS

By using the above results, we are now able to derive easily Eqs. (15)–(23). Sometimes to indicate a bond b we will use the symbol (i,j), or ij for short.

It is convenient to look formally at the coupling J_0 also as a random coupling with distribution

$$d\mu_0(J_0')/dJ_0' = \delta(J_0' - J_0). \tag{121}$$

Let us rewrite explicitly the Hamiltonian (2) as

$$H_{\boldsymbol{c}} = -\sum_{(i,j)\in\Gamma_0}(J_0 + c_{ij}J_{ij})\sigma_i\sigma_j - \sum_{i<j,(i,j)\notin\Gamma_0} c_{ij}J_{ij}\sigma_i\sigma_j - h\sum_i \sigma_i, \tag{122}$$

and let us introduce the random variables J_b', g_b', and $\tilde{J}_b'$, where

$$J_b' \overset{def}{=} \begin{cases} J_0 + c_b J_b, & b \in \Gamma_0, \\ J_b, & b \notin \Gamma_0, \end{cases}$$

$$g_b' \overset{def}{=} \begin{cases} 1, & b \in \Gamma_0, \\ c_b, & b \notin \Gamma_0, \end{cases}$$

and

$$\tilde{J}_b' \overset{def}{=} J_b' \cdot g_b'.$$

Taking into account that the random variable $J_0 + c_{ij}J_{ij}$, up to terms $O(1/N)$, is distributed according to $d\mu_0(J_0)$, the independent random variables J_b' and g_b' have distributions $d\mu_b'$ and p_b', respectively, given by

$$d\mu_b'(J_b') = \begin{cases} d\mu_0(J_b'), & b \in \Gamma_0, \\ d\mu(J_b'), & b \notin \Gamma_0, \end{cases}$$

and

$$p_b'(g_b') = \begin{cases} \delta_{g_b',1}, & b \in \Gamma_0, \\ p(g_b'), & b \notin \Gamma_0, \end{cases}$$

where the measures $d\mu$ and p are those of the model introduced in Sec. II. As a consequence, Eq. (122) can be cast in the form of Eq. (120) with the measure

$$d\tilde{\mu}_b'(\tilde{J}_b') = \begin{cases} d\mu_0(J_b')\delta_{g_b',1}, & b \in \Gamma_0, \\ d\mu(J_b')p(g_b'), & b \notin \Gamma_0. \end{cases} \tag{123}$$

Finally, since $p_b(g_b) \neq 0$ for any $b \in \Gamma_{\text{full}}$, we have also

$$\Gamma_p = \Gamma_{\text{full}}, \tag{124}$$

and due to the fact that $D(\Gamma_{\text{full}}) \propto N$, in the thermodynamic limit the mapping becomes exact.

According to Eqs. (105)–(107), the related Ising model of our small-world model has the following Hamiltonian with two free couplings: $J_0^{(I)}$, for Γ_0, and $J^{(I)}$, for Γ_{full}:

$$H_I = -J_0^{(I)}\sum_{(i,j)\in\Gamma_0}\sigma_i\sigma_j - J^{(I)}\sum_{i<j,(i,j)\notin\Gamma_0}\sigma_i\sigma_j - h\sum_i\sigma_i. \tag{125}$$

After solving this Ising (I) model the mapping allows us to come back to the random model by performing simultaneously for any $b \in \Gamma_{\text{full}}$ the reverse substitutions

$$\tanh(\beta J_b^{(I)}) \to \int d\tilde{\mu}_b'(\tilde{J}_b')\tanh^l(\beta\tilde{J}_b'), \tag{126}$$

where $l=1,2$ for $\Sigma=\text{F}$ or SG solution, respectively. Since the couplings $J_0^{(I)}$ and $J^{(I)}$ are arbitrary, we find it convenient to renormalize $J^{(I)}$ as $J^{(I)}/N$ and at the end of the calculation to put again $J^{(I)}$ instead of $J^{(I)}/N$. Note that for the mapping nothing changes if we do not make this substitution; the choice to use $J^{(I)}/N$ instead of $J^{(I)}$ is merely due to a formal convenience, since in this way the calculations are presented in a more standard and physically understandable form. In fact, according to Eqs. (123) and (126) what matters after solving the related Ising model with $J^{(I)}/N$ instead of $J^{(I)}$ is that, once for $\Sigma=\text{F}$ and once for $\Sigma=\text{SG}$, we perform, simultaneously in the two couplings, the following reverse mapping transformations ($l=1,2$ for $\Sigma=\text{F}$ or SG, respectively):

$$\tanh(\beta J^{(I)}/N) \to \int d\tilde{\mu}(\tilde{J}_{ij})\tanh^l(\beta\tilde{J}_{ij}), \tag{127}$$

for $(i,j) \notin \Gamma_0$, and

$$\tanh(\beta J_0^{(I)}) \to \int d\tilde{\mu}(\tilde{J}_{ij})\tanh^l(\beta\tilde{J}_{ij}), \tag{128}$$

for $(i,j) \in \Gamma_0$.

Explicitly, by applying Eqs. (123), (9), and (121), the transformations (127) and (128) become, respectively,

$$\beta J^{(I)} \to \beta J^{(\Sigma)} \tag{129}$$

and

$$\beta J_0^{(I)} \to \beta J_0^{(\Sigma)}, \tag{130}$$

where we have made use of the definitions (16)–(19) introduced in Sec. III.

Let us now solve the related Ising model. We have to evaluate the partition function

$$Z_I = \sum_{\{\sigma_i\}}\exp\left(\beta J_0^{(I)}\sum_{(i,j)\in\Gamma_0}\sigma_i\sigma_j + \beta\frac{J^{(I)}}{2N}\sum_{i\neq j}\sigma_i\sigma_j + \beta h\sum_i\sigma_i\right).$$

In the following we will suppose that $J^{(I)}$ (and then $J^{(\Sigma)}$) is positive. The derivation for $J^{(I)}$ (and then $J^{(\text{F})}$) negative differs from the other derivation just for a rotation of $\pi/2$ in the complex m plane and leads to the same result one can obtain by analytically continue the equations derived for $J^{(I)}>0$ to the region $J^{(I)}<0$.

By using the Gaussian transformation we can rewrite Z_I as

$$Z_I = c_N\sum_{\{\sigma_i\}}\exp\left(\beta J_0^{(I)}\sum_{(i,j)\in\Gamma_0}\sigma_i\sigma_j\right)\int_{-\infty}^{\infty}dm \times\exp\left(-\frac{\beta}{2}J^{(I)}m^2N + \beta(J^{(I)}m + h)\sum_i\sigma_i\right), \tag{131}$$

where c_N is a normalization constant,

$$c_N = \sqrt{\frac{\beta J^{(I)}N}{2\pi}},$$

and, in the exponent of Eq. (131), we have again neglected terms of order $O(1)$. For finite N we can exchange the integral and the sum over the σ's. By using the definition of the unperturbed model with Hamiltonian H_0, Eq. (1), whose free energy density, for given βJ_0 and βh, is indicated with $f_0(\beta J_0,\beta h)$, we arrive at

$$Z_I = c_N \int_{-\infty}^{\infty} dm e^{-NL(m)}, \qquad (132)$$

where we have introduced the function

$$L(m) = \frac{\beta}{2} J^{(I)} m^2 + \beta f_0(\beta J_0^{(I)}, \beta J^{(I)} m + \beta h). \qquad (133)$$

By using $\partial_{\beta h}\beta f_0(\beta J_0,\beta h) = -m_0(\beta J_0,\beta h)$ and $\partial_{\beta h} m_0(\beta J_0,\beta h) = \tilde{\chi}_0(\beta J_0,\beta h)$, we get

$$L'(m) = \beta J^{(I)}[m - m_0(\beta J_0^{(I)}, \beta J^{(I)} m + \beta h)],$$

$$L''(m) = \beta J^{(I)}[1 - \beta J^{(I)} \tilde{\chi}_0(\beta J_0^{(I)}, \beta J^{(I)} m + \beta h)].$$

If the integral in Eq. (132) converges for any N, by performing saddle point integration we see that the saddle point m^{sp} is a solution of the equation

$$m^{\text{sp}} = m_0(\beta J_0^{(I)}, \beta J^{(I)} m^{\text{sp}} + \beta h), \qquad (134)$$

so that, if the stability condition

$$1 - \beta J^{(I)} \tilde{\chi}_0(\beta J_0^{(I)}, \beta J^{(I)} m^{\text{sp}} + \beta h) > 0$$

is satisfied, in the thermodynamic limit we arrive at the following expression for the free energy density f_I of the related Ising model:

$$\beta f_I = \left[\frac{\beta}{2} J^{(I)} m^2 + \beta f_0(\beta J_0^{(I)}, \beta J^{(I)} m + \beta h) \right]_{m=m^{\text{sp}}}. \qquad (135)$$

Similarly, any correlation function C_I of the related Ising model is given in terms of the correlation function C_0 of the unperturbed model by the relation

$$C_I = C_0\,(\beta J_0^{(I)}, \beta J^{(I)} m + \beta h)|_{m=m^{\text{sp}}}. \qquad (136)$$

Of course, the saddle point solution m^{sp} represents the magnetization of the related Ising model, as can be checked directly by deriving Eq. (135) with respect to βh and by using Eq. (134).

If the saddle point equation (134) has more stable solutions, the "true" free energy and the "true" observable of the related Ising model will be given by Eqs. (135) and (136), respectively, calculated at the saddle point solution which minimizes Eq. (135) itself and that we will indicate with m_I.

Let us call $\beta_{c0}^{(I)}$ the inverse critical temperature of the unperturbed model with coupling $J_0^{(I)}$ and zero external field, possibly with $\beta_{c0}^{(I)} = \infty$ if no phase transition exists. As stressed in Sec. III B, for the unperturbed model we use the expression "critical temperature" for any temperature where the magnetization m_0 at zero external field passes from 0 to a nonzero value, continuously or not. Note that, as a consequence, if $J_0^{(I)} < 0$, we have formally $\beta_{c0}^{(I)} = \infty$, independently from the fact that some antiferromagnetic order may be not zero.

Let us start to make the obvious observation that a necessary condition for the related Ising model to have a phase transition at $h=0$ and for a finite temperature is the existence of some paramagnetic region P_I where $m_I=0$. We see from the saddle point equation (134) that, for $h=0$, a necessary condition for $m_I=0$ to be a solution is that be $\beta \leqslant \beta_{c0}^{(I)}$ for any β in P_I, from which we get also $\beta_c^{(I)} \leqslant \beta_{c0}^{(I)}$. In a few lines we will see, however, that the inequality must be strict if $\beta_{c0}^{(I)}$ is finite, which, in particular, excludes the case $J_0 < 0$ (for which the inequality to be proved is trivial).

Let us suppose for the moment that be $\beta_c^{(I)} < \beta_{c0}^{(I)}$. For $\beta < \beta_{c0}^{(I)}$ and $h=0$, the saddle point equation (134) has always the trivial solution $m_I=0$, which, according to the stability condition, is also a stable solution if

$$1 - \beta J^{(I)} \tilde{\chi}_0(\beta J_0^{(I)}, 0) > 0. \qquad (137)$$

The solution $m_I=0$ starts to be unstable when

$$1 - \beta J^{(I)} \tilde{\chi}_0(\beta J_0^{(I)}, 0) = 0. \qquad (138)$$

Equation (138), together with the constraint $\beta_c^{(I)} \leqslant \beta_{c0}^{(I)}$, gives the critical temperature of the related Ising model $\beta_c^{(I)}$. In the region of temperatures where Eq. (137) is violated, Eq. (134) gives two symmetrical stable solutions $\pm m_I \neq 0$. From Eq. (138) we see also that the case $\beta_c^{(I)} = \beta_{c0}^{(I)}$ is impossible unless $J^{(I)}=0$, since the susceptibility $\tilde{\chi}_0(\beta J_0^{(I)}, 0)$ must diverge at $\beta_{c0}^{(I)}$. We have therefore proved that $\beta_c^{(I)} < \beta_{c0}^{(I)}$. Note that for $J_0^{(I)} \geqslant 0$ and $\beta < \beta_{c0}^{(I)}$, Eq. (137) is violated only for $\beta > \beta_c^{(I)}$, whereas for $J_0^{(I)} < 0$, Eq. (137) in general may be violated also in finite regions of the β axis.

The critical behavior of the related Ising model can be studied by expanding Eq. (134) for small fields. However, we find it more convenient to expand $L(m)$ in series around $m=0$ since in this way everything can be cast in the standard formalism of the Landau theory of phase transitions. From Eq. (133), taking into account that the function $\tilde{\chi}_0(\beta J_0,\beta h)$ is an even function of βh, we have the following general expression valid for any m, β and small h:

$$L(m) = \beta f_0(\beta J_0^{(I)}, 0) - m_0(\beta J_0^{(I)}, 0)\beta h + \psi(m), \qquad (139)$$

where we have introduced the Landau free energy density $\psi(m)$ given by

$$\psi(m) = \frac{1}{2} a m^2 + \frac{1}{4} b m^4 + \frac{1}{6} c m^6 - m\beta\tilde{h} + \Delta(\beta f_0)(\beta J_0^{(I)}, \beta J^{(I)} m), \qquad (140)$$

where

$$a = [1 - \beta J^{(I)} \tilde{\chi}_0(\beta J_0^{(I)}, 0)]\beta J^{(I)}, \qquad (141)$$

$$b = -\frac{\partial^2}{\partial(\beta h)^2} \tilde{\chi}_0(\beta J_0^{(I)}, \beta h)|_{\beta h=0} \frac{(\beta J^{(I)})^4}{3!}, \qquad (142)$$

$$c = -\frac{\partial^4}{\partial(\beta h)^4} \tilde{\chi}_0(\beta J_0^{(I)}, \beta h)|_{\beta h=0} \frac{(\beta J^{(I)})^6}{5!}, \qquad (143)$$

$$\tilde{h} = m_0(\beta J_0^{(I)}, 0) J^{(I)} + \tilde{\chi}_0(\beta J_0^{(I)}, 0) \beta^{(I)} J^{(I)} \beta h. \qquad (144)$$

Finally, the last term $\Delta(\beta f_0)\,(\beta J_0^{(I)}, \beta J^{(I)} m)$ is defined implicitly to render Eqs. (139) and (140) exact, but terms $O(h^2)$ and $O(m^3 h)$; explicitly,

$$\Delta(\beta f_0)(\beta J_0^{(I)},\beta J^{(I)}m) = -\sum_{k=4}^{\infty}\frac{\partial^{2k-2}}{\partial(\beta h)^{2k-2}}\tilde{\chi}_0(\beta J_0^{(I)},\beta h)|_{\beta h=0}\frac{(\beta J^{(I)})^{2k}}{(2k)!}. \tag{145}$$

Finally, to come back to the original random model, we have just to perform the reversed mapping transformations (129) and (130) in Eqs. (133)–(145). As a result, we get immediately Eqs. (15)–(43), but Eq. (21).

IX. DERIVATION OF Eq. (21) AND Eqs. (64)–(70)

Concerning Eq. (21) for the full expression of the free energy density, it can be obtained by using Eqs. (108), (109), (115), and (117). Here φ_I is the high-temperature part of the free energy density of the related Ising model we have just solved:

$$-\beta f_I = \lim_{N\to\infty}\frac{1}{N}\sum_{(i,j)\in\Gamma_0}\ln[\cosh(\beta J_0^{(I)})] + \frac{N-1}{2}\times\ln[\cosh(\beta J^{(I)}/N)] + \ln[2\cosh(\beta h)] + \varphi_I, \tag{146}$$

where we have taken into account the fact that our related Ising model has $|\Gamma_0|$ connections with coupling $J_0^{(I)}$ and $N(N-1)/2$ connections with the coupling $J^{(I)}/N$. By using Eq. (135) calculated in m_I and Eq. (146), for large N we get

$$\varphi_I = -\frac{\beta}{2}J^{(I)}{m_I}^2 - \beta f_0(\beta J_0^{(I)},\beta J^{(I)}m_I+\beta h) - \lim_{N\to\infty}\frac{1}{N}\sum_{(i,j)\in\Gamma_0}\ln[\cosh(\beta J_0^{(I)})] - \ln[2\cosh(\beta h)] + O\left(\frac{1}{N}\right). \tag{147}$$

Therefore, on using Eq. (115), for the nontrivial part $\varphi^{(\Sigma)}$ of the random system, up to corrections $O(1/N)$, we arrive at

$$\varphi^{(\Sigma)} = -\frac{\beta}{2}J^{(\Sigma)}(m^{(\Sigma)})^2 - \ln[2\cosh(\beta h)] - \lim_{N\to\infty}\frac{1}{N}\sum_{(i,j)\in\Gamma_0}\ln[\cosh(\beta J_0^{(\Sigma)})] - \beta f_0(\beta J_0^{(\Sigma)},\beta J^{(\Sigma)}m^{(\Sigma)}+\beta h). \tag{148}$$

In terms of the function $L^{(\Sigma)}(m)$, Eq. (148) reads as

$$\varphi^{(\Sigma)} = -L^{(\Sigma)}(m^{(\Sigma)}) - \lim_{N\to\infty}\frac{1}{N}\sum_{(i,j)\in\Gamma_0}\ln[\cosh(\beta J_0^{(\Sigma)})] - \ln[2\cosh(\beta h)]. \tag{149}$$

By using Eqs. (108), (149), (115), and (117), with $l=1$ or 2 for $\Sigma=\mathrm{F}$ or $\Sigma=\mathrm{SG}$, respectively, we get Eq. (21).

For $h=0$, Eq. (149) can conveniently be rewritten also as

$$\varphi^{(\Sigma)} = \varphi_0(\beta J_0^{(\Sigma)},0) + [L^{(\Sigma)}(0) - L^{(\Sigma)}(m^{(\Sigma)})], \tag{150}$$

where

$$\varphi_0(\beta J_0,\beta h) = -\beta f_0(\beta J_0^{(\Sigma)},\beta h) - \lim_{N\to\infty}\frac{1}{N}\sum_{(i,j)\in\Gamma_0}\ln[\cosh(\beta J_0^{(\Sigma)})] - \ln[2\cosh(\beta h)] \tag{151}$$

is the high-temperature part of the free energy density of the unperturbed model with coupling $J_0^{(\Sigma)}$ and external field h. There are some important properties for the function $\varphi_0(\beta J_0,0)$: it is a monotonic increasing function of βJ_0; if the lattice $\mathcal{L}_0$ has only loops of even length, $\varphi_0(\beta J_0,0)$ is an even function of βJ_0; furthermore, if $d_0<2$ and the coupling-range is finite, or if $d_0=\infty$ at least in a wide sense [15], in the thermodynamic limit we have $\varphi_0(\beta J_0,0)=0$; if instead $2\leq d_0<\infty$, $\varphi_0(\beta J_0,0)\neq 0$. We see here therefore what anticipated in Sec. VII A: when $J_0\neq 0$, the symmetry among the random couplings is broken and for d_0 sufficiently high this reflects in a nonzero $\varphi^{(\Sigma)}$ also in the P region.

Next we prove Eqs. (64)–(70). To this aim we have to calculate Eq. (150) at the leading solution $\bar{m}^{(\Sigma)}$ and to compare $\varphi^{(\mathrm{F})}$ and $\varphi^{(\mathrm{SG})}$. Note that the term in the square brackets of Eq. (150) is non-negative since $\bar{m}^{(\Sigma)}$ is the absolute minimum of $L^{(\Sigma)}$. We recall that for critical temperature we mean here any temperature lying on the boundary P-F or P-SG, so that $\bar{m}^{(\Sigma)}|_\beta=0$ for any β in the P region.

A. $J_0 \geq 0$

If $J_0\geq 0$, for both solutions with labels F and SG, we have only one second-order phase transition so that $\bar{m}^{(\mathrm{F})}=0$ and $\bar{m}^{(\mathrm{SG})}=0$, respectively, are the stable and leading solutions even on the boundary with the P region.

Let us suppose $\beta_c^{(\mathrm{F})}<\beta_c^{(\mathrm{SG})}$. Let be $\varphi_0(\cdot,0)\neq 0$. From Eq. (150) and by using $J_0^{(\mathrm{F})}>J_0^{(\mathrm{SG})}$, we see that

$$\varphi^{(\mathrm{F})}|_{\beta_c^{(\mathrm{F})}} = \varphi_0(\beta_c^{(\mathrm{F})}J_0^{(\mathrm{F})},0) > \varphi^{(\mathrm{SG})}|_{\beta_c^{(\mathrm{F})}} = \varphi_0(\beta_c^{(\mathrm{F})}J_0^{(\mathrm{SG})},0). \tag{152}$$

Finally, by using this result and the general rule given by Eqs. (115) and (117), we see (and with a stronger reason, due to the factor 1/2 appearing in these equations for the SG solution) that the stable phase transition is the P-F one: $\beta_c=\beta_c^{(\mathrm{F})}$. Similarly, by using Eq. (150) for $\beta_c^{(\mathrm{F})}<\beta<\beta_c^{(\mathrm{SG})}$, we see that even for any β in the interval $(\beta_c^{(\mathrm{F})},\beta_c^{(\mathrm{SG})})$ the stable solution is that with label F. This last observation makes also clear that if $\varphi_0(\cdot,0)=0$ we reach the same conclusion: F is the stable phase in all the region $\beta_c^{(\mathrm{F})}<\beta<\beta_c^{(\mathrm{SG})}$ and in particular this implies also that the stable phase transition is the P-F one: $\beta_c=\beta_c^{(\mathrm{F})}$.

Let us suppose $\beta_c^{(\mathrm{F})}>\beta_c^{(\mathrm{SG})}$. If $\varphi_0(\cdot,0)\neq 0$, we arrive at

$$\varphi^{(\mathrm{F})}|_{\beta_c^{(\mathrm{SG})}} = \varphi_0(\beta_c^{(\mathrm{SG})}J_0^{(\mathrm{F})},0) > \varphi^{(\mathrm{SG})}|_{\beta_c^{(\mathrm{SG})}} = \varphi_0(\beta_c^{(\mathrm{SG})}J_0^{(\mathrm{SG})},0). \tag{153}$$

Finally, by using this result and the general rule given by Eqs. (115) and (117), we see (and with a stronger reason) that the stable phase on the boundary is that predicted by the F solution which has zero magnetization at $\beta_c^{(\mathrm{SG})}$. This does not imply that $\beta_c=\beta_c^{(\mathrm{F})}$, but only that $\beta_c^{(\mathrm{SG})}<\beta_c\leq\beta_c^{(\mathrm{F})}$. If instead $\varphi_0(\cdot,0)=0$, by using Eq. (150) for $\beta_c^{(SG)}<\beta<\beta_c^{(F)}$, we see that for any β in the interval $(\beta_c^{(SG)},\beta_c^{(F)})$ the stable solution is SG and then, in particular, the stable boundary is P-SG: $\beta_c=\beta_c^{(SG)}$.

B. $J_0<0$

If $J_0<0$, for the solution with label F, we may have both first and second-order phase transitions. In the first case we cannot in general assume that 0 is the stable and leading solution on the boundary with the P region: $m^{(F)}|_{\beta_c^{(F)}}\neq 0$ in general. As a consequence, for a first-order transition the term in square brackets of Eq. (150) may be nonzero even on the critical surface. Furthermore, as $J_0<0$, for the solution F we have at least two critical temperatures that we order as $\beta_{c1}^{(F)}\geqslant\beta_{c2}^{(F)}$. However, despite these complications, if we assume that $\mathcal{L}_0$ has only loops of even length, $\varphi_0(\cdot,0)$ turns out to be an even function and, due to the inequality $|J_0^{(F)}|>J_0^{(SG)}$, almost nothing changes in the arguments we have used in the previous case $J_0\geqslant 0$.

Let us consider first the surfaces $\beta_{c2}^{(F)}$ and $\beta_c^{(SG)}$. Independently of the kind of phase transition, first or second order, we arrive again at Eqs. (152) and (153), for $\beta_{c2}^{(F)}<\beta_c^{(SG)}$ and $\beta_{c2}^{(F)}>\beta_c^{(SG)}$, respectively, with the same prescription for the cases $\varphi_0(\cdot,0)\neq 0$, or $\varphi_0(\cdot,0)=0$.

Let us now consider the surfaces $\beta_{c1}^{(F)}$ and $\beta_c^{(SG)}$. If $\beta_{c1}^{(F)}<\beta_c^{(SG)}$ and $\varphi_0(\cdot,0)\neq 0$, for any β in the interval $[\beta_{c1}^{(F)},\beta_c^{(SG)}]$ we have

$$\varphi^{(F)}|_\beta=\varphi_0(\beta J_0^{(F)},0)>\varphi^{(SG)}|_\beta=\varphi_0(\beta J_0^{(SG)},0), \quad (154)$$

so that the interval $[\beta_{c1}^{(F)},\beta_c^{(SG)}]$ is a stable P region corresponding to the solution with label F. Similarly, we arrive at the same conclusion if $\varphi_0(\cdot,0)=0$. However, the interval of temperatures where the P region $[\beta_{c1}^{(F)},\beta_c^{(SG)}]$ is stable can be larger when $d_0\geqslant 2$. In fact (exactly as we have seen for $J_0\geqslant 0$), in this case the P-SG stable boundary may stay at lower temperatures. Finally, let us consider the case $\beta_{c1}^{(F)}>\beta_c^{(SG)}$. If $\varphi_0(\cdot,0)=0$, by using Eq. (150) we see that for any $\beta>\beta_c^{(SG)}$ we have that the stable solution corresponds to the SG one, so that there is no stable boundary with the P region. If instead $\varphi_0(\cdot,0)\neq 0$, due to the fact that $\varphi^{(SG)}|_\beta$ and $\varphi^{(F)}|_\beta$ grow in a different way with β, we are not able to make an exact comparison, and it is possible that the P-F boundary becomes stable starting from some β_{c1} with $\beta_{c1}\geqslant\beta_{c1}^{(F)}$. In general, as in the case $J_0<0$, we could have one (or even more) sectors where the P region corresponding to the solution with label F is stable.

X. CONCLUSIONS

In this paper we have presented a general method to analytically face random Ising models defined over small-world networks. The key point of our method is the fact that, at least in the P region, any such a model can be exactly mapped to a suitable fully connected model, whose resolvability is in general nontrivial for $d_0>1$, but still as feasible as a nonrandom model. As a main result, we then derive a general self-consistent equation, Eq. (15), which allows us to describe effectively the model once the magnetization of the unperturbed model in the presence of a uniform external field, $m_0(\beta J_0,\beta h)$, is known.

The physical interpretation of this general result is straightforward. From Eqs. (15) we see that, concerning the magnetization $m^{(F)}$, the effect of adding long-range Poisson-distributed bonds implies that the system, now perturbed, feels, besides the coupling J_0, also an effective external field $J^{(F)}$ shrunk by $m^{(F)}$ itself. Concerning $m^{(SG)}$, the effect is that the system now feels a modified effective coupling $J_0^{(SG)}$ and an effective external field $J^{(SG)}$ shrunk by $m^{(SG)}$ itself.

We are therefore in the presence of an *effective field theory*, which, as opposed to a simpler *mean-field theory*, describes $m^{(F)}$ and $m^{(SG)}$ in terms of not only an effective external field, but also through the nontrivial function $m_0(\beta J_0,\beta h)$, which, in turn, takes into account the correlations due to the nonzero short-range coupling J_0 or $J_0^{(SG)}$ felt by the unperturbed system. The combination of these two effects gives rise to the typical behavior of models defined over small-world networks: the presence of a nonzero effective external field causes the existence of a phase transition also at low d_0 dimension. However, the precise determination of both the critical surface and the correlation functions is obtained in a nontrivial way via the unperturbed magnetization $m_0(\beta J_0,\beta h)$.

We have used the method introduced to analyze the critical behavior of generic models with $J_0\geqslant 0$ and $J_0<0$ and showed that they give rise to two strictly different phase transition scenarios. In the first case, we have a mean-field second-order phase transition with a finite correlation length, whereas, in the second case, we obtain multiple first- and second-order phase transitions. Furthermore, we have shown that the combination of the F and SG solutions results in a total of four possible kinds of phase diagrams according to the cases (i) ($J_0\geqslant 0$; $d_0<2$, or $d_0=\infty$), (ii) ($J_0\geqslant 0$; $2\leqslant d_0<\infty$), (iii) ($J_0<0$; $d_0<2$, or $d_0=\infty$), and (iv) ($J_0<0$; $2\leqslant d_0<\infty$). One remarkable difference between systems with $d_0<2$, or $d_0=\infty$ and those with $2<d_0<\infty$, is that in the latter case we have, in principle, also first-order P-SG phase transitions and, moreover, reentrance phenomena are in principle possible even for $J_0\geqslant 0$.

In Secs. IV–VI we have applied the method to solve analytically those models for which the unperturbed magnetization $m_0(\beta J_0,\beta h)$ is known analytically—i.e., the small-world models in dimension $d_0=0,1,\infty$—corresponding to an ensemble of noninteracting units (spins, dimers, etc.), the one-dimensional chain, and the spherical model, respectively. In particular, we have studied in detail the small-world model defined over the one-dimensional chain with positive and negative short-range couplings, showing explicitly how, in the second case, multicritical points with first- and second-order phase transitions arise. Finally, the small-world spherical model—an exact solvable model (in our approach) with continuous spin variables—has provided us with an interesting case study to explore what happens as d_0 changes continuously from 0 to ∞. As expected on general grounds, unlike the nonrandom version of the model, the small-world model presents always a finite-temperature phase transition, even in the limit $d_0\to 0^+$. This latter result, besides being consistent with what we have found in the $d_0=0$ dimensional discrete models, has a simple physical explanation in our approach. In fact, it consists on mapping the small-world model (a random model) to a corresponding nonrandom model (no long-range bonds), but immersed in an effective uniform external field which is active as soon as the added random connectivity c is not zero [see Eqs. (15)–(17)].

Many interesting variants of the above models can be considered and are still analytically solvable by our approach (see also the generalizations considered in Sec. III E). However, our approach can be also applied numerically to study more complex small-world models for which the corresponding unperturbed model is not analytically available [29]. In fact, the numerical complexity in solving such small-world models is comparable to that required in solving a nonrandom model immersed in a uniform external field.

Models defined on complex small-world networks[3] are an interesting subject of future work.

ACKNOWLEDGMENTS

This work was supported by the FCT (Portugal) Grant Nos. SFRH/BPD/24214/2005, pocTI/FAT/46241/2002, and pocTI/FAT/46176/2003 and the Dysonet Project. We thank A. L. Ferreira, A. Goltsev, and C. Presilla for useful discussions.

APPENDIX: GENERALIZATION TO NONHOMOGENEOUS EXTERNAL FIELD

In this appendix we prove Eq. (33) calculating the $O(1/N)$ correction responsible for the divergence of the susceptibility of the random system at T_c. To this aim we first need to generalize our method to an arbitrary external field. Let us consider again a fully connected model having, as done in Sec. VIII, long-range couplings J (for brevity we will here omit the label I) and short-range couplings J_0, but now immersed in an arbitrary (nonhomogeneous) external field $\{h_n\}$, where $n=1,\ldots,N$. After using the Gaussian transformation we have the partition function

$$Z = c_N \int_{-\infty}^{\infty} dm e^{-NL(m)}, \tag{A1}$$

where we have introduced the function

$$L(m) = \frac{\beta}{2} J m^2 + \beta f_0(\beta J_0, \{\beta J m + \beta h_n\}), \tag{A2}$$

$f_0(\beta J_0, \{\beta h_n\})$ being the free energy density of the unperturbed model in the presence of an arbitrary external field $\{\beta h_n\}$. By using

$$\partial_{\beta h_i} \beta f_0(\beta J_0, \{\beta h_n\}) = -m_{0i}(\beta J_0, \{\beta h_n\}) \tag{A3}$$

and

$$\tilde{\chi}_{0;i,j}(\beta J_0, \{\beta h_n\}) \stackrel{def}{=} \langle \sigma_i \sigma_j \rangle_0 - \langle \sigma_i \rangle_0 \langle \sigma_j \rangle_0 = \partial_{\beta h_j} m_{0i}(\beta J_0, \{\beta h_n\}), \tag{A4}$$

we get

$$L'(m) = \beta J \left[m - \frac{1}{N} \sum_i m_{0i}(\beta J_0, \{\beta J m + \beta h_n\}) \right], \tag{A5}$$

$$L''(m) = \beta J \left[1 - \beta J \frac{1}{N} \times \sum_{i,j} \tilde{\chi}_{0ij}(\beta J_0, \{\beta J m + \beta h_n\}) \right]. \tag{A6}$$

By performing the saddle point integration we see that the saddle point m^{sp} is a solution of the equation

$$m^{\mathrm{sp}} = \frac{1}{N} \sum_i m_{0i}(\beta J_0, \{\beta J m^{\mathrm{sp}} + \beta h_n\}). \tag{A7}$$

Hence, by using

$$\tilde{\chi}_0(\beta J_0, \{\beta J m + \beta h_n\}) = \frac{1}{N} \sum_{i,j} \tilde{\chi}_{0ij}(\beta J_0, \{\beta J m + \beta h_n\}), \tag{A8}$$

we see that if the stability condition

$$1 - \beta J \tilde{\chi}_0(\beta J_0, \{\beta J m^{\mathrm{sp}} + \beta h_n\}) > 0 \tag{A9}$$

is satisfied, in the thermodynamic limit we arrive at the following expression for the free energy density f of the related Ising model immersed in an arbitrary external field:

$$\beta f = \left[\frac{\beta}{2} J m^2 + \beta f_0(\beta J_0, \{\beta J m + \beta h_n\}) \right]_{m=m^{\mathrm{sp}}}. \tag{A10}$$

On the other hand, by derivation with respect to βh_i and by using Eq. (A7), it is immediate to verify that

$$m_i \stackrel{def}{=} \langle \sigma_i \rangle = m_{0i}(\beta J_0, \{\beta J m^{\mathrm{sp}} + \beta h_n\}), \tag{A11}$$

and then also (from now on for brevity on we omit the symbol “sp”)

$$m = \frac{1}{N} \sum_i m_i. \tag{A12}$$

We want now to calculate the correlation functions. From Eq. (A11), by deriving with respect to βh_j we have

$$\tilde{\chi}_{ij} \stackrel{def}{=} \frac{\partial m_i}{\partial(\beta h_j)} = \sum_l \tilde{\chi}_{0;i,l}(\beta J_0, \{\beta J m + \beta h_n\}) \times \left(\beta J \frac{\partial m}{\partial(\beta h_j)} + \delta_{l,j} \right), \tag{A13}$$

which, by summing over the index i and using (A12), gives

$$\frac{\partial m}{\partial(\beta h_j)} = \frac{\frac{1}{N} \sum_i \tilde{\chi}_{0;i,j}(\beta J_0, \{\beta J m + \beta h_n\})}{1 - \beta J \tilde{\chi}_0(\beta J_0, \{\beta J m + \beta h_n\})}. \tag{A14}$$

We can now insert Eq. (A14) into the rhs of Eq. (A13) to get

[3] For a recent review on complex networks, see Ref. [27].

$$\tilde{\chi}_{ij} = \tilde{\chi}_{0;i,j}(\beta J_0,\{\beta J m + \beta h_n\}) + \frac{\beta J}{N}\frac{\Sigma_l \tilde{\chi}_{0;l,j}\Sigma_k \tilde{\chi}_{0;i,k}}{1-\beta J\tilde{\chi}_0(\beta J_0,\{\beta J m + \beta h_n\})}, \quad \text{(A15)}$$

where for brevity we have omitted the argument in $\tilde{\chi}_{0;l,j}$ and $\tilde{\chi}_{0;i,k}$, which is the same $\tilde{\chi}_0$ appearing in the denominator. If we now come back to choice a uniform external field $h_n=h$, $n=1,\ldots,N$, we can use translational invariance and for the related Ising model (fully connected) we obtain the correlation function

$$\tilde{\chi}_{ij} = \frac{\beta J}{N}\frac{[\tilde{\chi}_0(\beta J_0,\beta J m + \beta h)]^2}{1-\beta J\tilde{\chi}_0(\beta J_0,\beta J m + \beta h)} + \tilde{\chi}_{0;i,j}(\beta J_0,\beta J m + \beta h). \quad \text{(A16)}$$

Finally, by performing the mapping substitutions (129) and (130) we arrive at Eq. (33).

Similarly, any correlation function C of the related Ising model will be given by a similar formula with the leading term C_0 plus a correction $O(1/N)$ becoming important only near T_c.

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