Plastic failure of nonlocal beams

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This paper questions the mode of collapse of some simple softening nonlocal structural systems comprising the classical cantilever beam. Nanobeams can be concerned by such an elementary model. The homogeneous cantilever beam loaded by a concentrated force at its extremity is first considered as a structural paradigm. A nonlocal plasticity model is developed in order to control the localization process induced by microcracking phenomena. An implicit gradient plasticity model equivalent to a nonlocal integral plasticity model is used in this paper. It is shown that the regularized problem is well posed. Closed-form solutions of the elastoplastic deflection are finally derived. The length of the plastic zone grows during the softening process until an asymptotic limited value, which depends on the characteristic length of the material. Scale effects are clearly obtained for these static bending tests. Other structural cases are also presented, including the simply supported beam under uniform transverse loading. It is concluded that the mode of collapse is firmly a nonlocal phenomenon.

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I. INTRODUCTION

In the early 21st century, there was considerable interest in the application of nonlocal continuum mechanics for the modeling and analysis of microrods or nanorods [1–9]. Nonlocal continuum mechanics allow one to account for the small scale effect that becomes significant when dealing with microstructures or nanostructures. These articles presented simplified nonlocal elastic model, for the bending, buckling, and vibration analyses of small-scale rods. Integral type or gradient nonlocal models abandon the classical assumption of locality and admit that stress depends not only on the state variables at that point. The first models of this type were applied in the 1960's to the modeling of elastic waves dispersion in crystals [10–12]. A systematic rational procedure of nonlocal elasticity framework was established in the papers of Eringen [13–16]. Nonlocal field theory of mechanics has been applied to some various engineering problems, such as dispersion of phonon, Rayleigh wave, stress concentration at the crack tip, etc. The distinction between gradient elastic models (the stress is defined explicitly from the local strain and its derivative) and integral elastic models (the stress is obtained implicitly from an integral operator of the local strain) can be established for nonlocal elastic models. Moreover, it can be shown that some integral elastic models can be cast in a differential form [15]. The relevancy of both gradient and integral elastic models is often discussed in the literature. Gradient models can be considered as a "weakly" nonlocal model, and the need to employ an integral ("strongly" nonlocal) model can be discussed for specific structural cases. Despite the numerous works recently devoted to nonlocal modeling of elastic beams [1–9], very few papers have been published on nonlocal elastoplastic beams. Plasticity phenomena could, however, be predominant in the localization of the failure process induced by microcracking at small scale. Softening plasticity is typically observed for large deformations. The study of softening behavior, which is characterized by a negative stiffness or a loss in strength after reaching a critical load-carrying capability, has benefited from extensive coverage in the research literature over the last three decades [17].

This paper is devoted to the static response of a bended beam composed of softening material through a nonlocal plasticity model. Such a study can be understood as an inelastic extension of previous elastic studies devoted to nonlocal microrods or nanorods. The main applications of such theoretical study may be found in the field of nanomechanics, where scale effects may be typically sensitive (see, for instance, Refs. [1–9]), but also for large scale structural members where nonlocal effects may control the failure process (reinforced concrete structural members, composite beams, wood beams, etc.). The paper questions the inelastic bending mode of failure of a beam in simple structural configurations, through a plasticity model. The mode of failure of beams can also be a shear mode, but only bending is treated in this paper (see Ref. [18] for the use of a shear beam model to model failure of interfaces—see also Refs. [19–22] for the modeling of failure interfaces). We also mention that the beam model investigated in this paper is a time-independent model (no delayed effect). The studies of Refs. [23–27] give the specific modeling of the creep failure phenomenon and the modeling of creep failure of a simply supported beam [28]. Furthermore, the failure phenomenon is assumed to be statically controlled, and no inertia effects affect the failure process (the dynamic failure of a brittle beam is investigated in Ref. [29]). Finally, no nonlinear geometrical phenomena are introduced in the model: The beam is assumed to be sufficiently restraint in order to prevent lateral-torsional buckling (see Refs. [30,31] for the modeling of lateral-torsional buckling of cantilever beams). Hence, no geometrical instability may arise in the softening beam treated in this paper (only material nonlinearity governs the mode of failure driven by a nonlocal plasticity model).

Nonlocal elasticity was further extended to nonlocal elastoplasticity by Eringen [32,33] in the early 1980's. Nonlocal inelastic models (damage or plasticity models) were later successfully used as a localization limiter with a regularization effect on softening structural response [17]. The nonlocal character of the constitutive law, generally introduced through an internal length, is restricted to the loading function (damage loading function or plasticity loading function). Pijaudier-Cabot and Bažant [34] elaborated a nonlocal damage theory, based on the introduction of the nonlocality in the damage loading function. This theory has the advantage to leave the initial elastic behavior unaffected and to control the localization process in the post-peak regime. It is worth mentioning that this idea was already used before to model shear bands [35,36]. As for nonlocal elastic models, gradient plasticity models (also called explicit gradient plasticity models), and integral plasticity models may be distinguished. Gradient plasticity models first initiated in the 1970's [37], were included at the beginning of the 1990's in a variational formulation with a view to computational analyses [38,39]. Following earlier results obtained for elastic models [15], some integral plasticity models can be cast in a differential form, and are called implicit gradient plasticity models [40,41]. The theoretical challenges related to these nonlocal inelastic theories (plasticity or damage) were mainly oriented towards the relevancy of an integral or a gradient-based formulation, the justification of relevant boundary conditions associated to the nonlocal nature of the constitutive law, or the thermodynamically background of these models [42].

Despite the numerous papers devoted to the modeling of softening media with a nonlocal constitutive law, very few works have been published on the application of such models at the beam scale, or for simple structural members (see also Ref. [43] for this problem). Historically, moment-curvature relationships with softening branch were first introduced for reinforced concrete beams. Wood [44] did point out some specific difficulties occurring during the solution of the evolution problem for plastic softening models. More precisely, he highlighted the impossibility of the plastic softening beam to flow in presence of moment gradient, a phenomenon sometimes called Wood's paradox (see also Ref. [45]). It is expected that the nonlocal moment-curvature relationship can overcome Wood's paradox. An explicit gradient plasticity model, similar to the one developed in Refs. [38,39], has been considered by Challamel for the beam solicited by a bending moment [46]. However, Wood's paradox is also encountered for such gradient plasticity models in presence of moment gradient, except in some specific inhomogeneous beams [47]. We show in this paper that Wood's paradox can be overcome with an implicit gradient plasticity model (integral-based nonlocal plasticity model). The homogeneous cantilever beam loaded by a concentrated force at its extremity is first considered. The cantilever beam can be considered as a structural paradigm. An implicit gradient plasticity model is developed in order to control the localization process induced by microcracking phenomena. It is shown that the regularized problem is well posed. Closed-form solutions of the elastoplastic deflection are finally derived. The length of the plastic zone grows during the softening process until an asymptotic limited value, which depends on the characteristic length of the material. Other structural cases are also presented, including the simply supported beam under uniform transverse loading. It is concluded that the mode of collapse is firmly a nonlocal phenomenon.

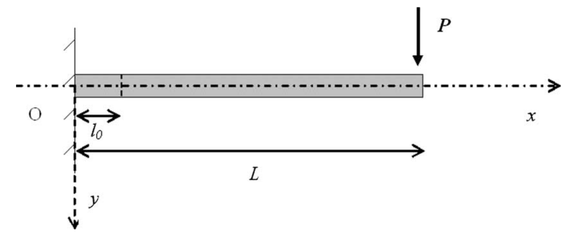

FIG. 1. Beam model—Cantilever case.

II. LOCAL CONSITUTIV LAW: WOOD'S PARADOX

The homogeneous cantilever beam of length L is loaded by a vertical concentrated load P at its end (Fig. 1). The cantilever beam loaded by a concentrated force can be viewed as a typical case of plastic beams with nonconstant bending moment. The axial and transversal coordinates are denoted by x and y, respectively, and the transverse deflection denoted by w. Further, the beam is assumed to be sufficiently restraint in order to prevent lateral-torsional buckling. The symmetrical section has a constant second moment of area denoted by I (about the z axis). We assume that plane cross sections remain plane and normal to the deflection line and that transverse normal stresses are negligible (Euler-Bernoulli assumption). According, the curvature χ is related to the deflection through

$$\chi(x) = w''(x), \tag{1}$$

where a prime denotes a derivative with respect to x. The problem being statically determinate, equilibrium equations directly give the moment distribution along the beam

$$M(x) = P(L-x) \text{ with } P \geqslant 0 \text{ and } x \in [0;L]. \tag{2}$$

At the end of the beam, the displacement $v = w(L)$ of concentrated force P is used to control the loading process. The local moment-curvature relationship (M,χ) considered is bilinear with a linear elastic part and a linear strain-softening part (Fig. 2). This model is first considered in a local form, i.e., classical plastic model with negative hardening. The

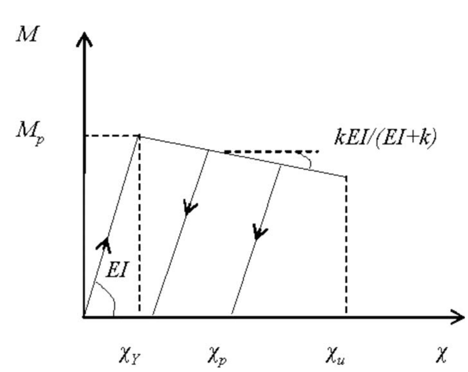

FIG. 2. Plastic softening moment-curvature law.

nonlocal extension will be investigated later in the paper. M_P is the limit elastic moment and χ_Y is the limit elastic curvature χ_Y, related through $M_p/\chi_Y=EI$, where E is the Young modulus of the homogeneous beam. In practice, the curvature cannot increase indefinitely and is limited by χ_u (the ultimate admissible curvature). However, such a limitation is not taken into account in the present study. The elastoplastic model represented in Fig. 2 is a classical plastic model with negative hardening (softening). The yield function f is given by

$$f(M,M^*) = |M| - (M_p + M^*), \tag{3}$$

where M^* is an additional variable which accounts for the loading history. The plastic curvature χ_p is obtained using the normality rule

$$\dot{\chi}_p = \dot{\lambda}\frac{\partial f}{\partial M}. \tag{4}$$

The overdot denotes the time derivative. As a rate-independent constitutive law is considered in this paper, it is equivalent to replace the time variable by a monotonic increasing variable such as the displacement at the tip of the beam $v=w(L)$. The plastic multiplier $\dot{\lambda}$ must satisfy the complementary conditions

$$\dot{\lambda} \geqslant 0, \quad f(M,M^*) \leqslant 0, \quad \dot{\lambda} f(M,M^*) = 0. \tag{5}$$

The softening being linear, the following relation holds:

$$M^*(\chi_p) = k\chi_p. \tag{6}$$

According to the sign of the plastic modulus k, we can have softening for $k<0$. Using the decomposition of the total curvature χ into an elastic part and a plastic part, the moment-curvature relation gives

$$M = EI(\chi - \chi_p). \tag{7}$$

The maximum bending moment occurs at $x=0$, where the beam is clamped. Plastic rotation starts as soon as the bending moment reach the plastic bending moment M_p. The maximum elastic displacement at the beam end v_Y and the corresponding load P_Y are given by

$$v_Y = \frac{M_p L^2}{3EI} \text{ and } P_Y = \frac{M_p}{L}. \tag{8}$$

For displacement v smaller than $v_Y(v \leqslant v_Y)$, the beam remains elastic and the deflection can be computed using the elastic solution

$$v \leqslant v_Y \Rightarrow EIw(x) = -\frac{P}{6}x^3 + PL\frac{x^2}{2} \text{ with } P = 3\frac{EI}{L^3}v. \tag{9}$$

The relationship (9) gives the deflection w as a function of the displacement at the end of the beam v which will be used to control the loading process. For $P=P_Y$, we obtain the characteristic deflection $w_Y(x)$:

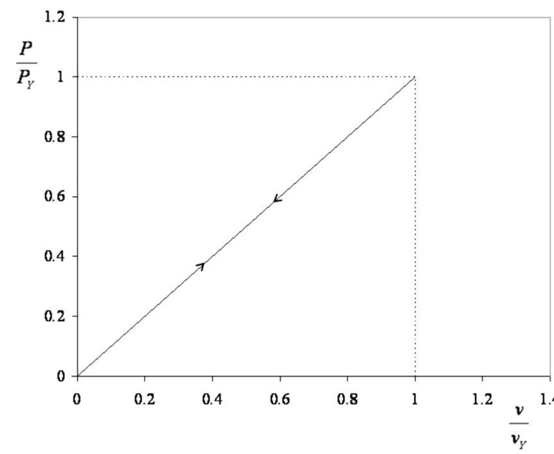

FIG. 3. Wood's paradox—local softening plasticity models.

$$v = v_Y \Rightarrow EIw_Y(x) = -\frac{M_p}{6L}x^3 + M_P\frac{x^2}{2}. \tag{10}$$

For v greater than v_Y $(v \geqslant v_Y)$, the plastic regime starts and the beam can be split into an elastic and a plastic domain. The size of the plastic domain is denoted by $l_0 \leqslant L$ (see Fig. 2). The governing equations in the plastic domain are

$$x \in [0;l_0]: \left| \begin{array}{l} EI[w^{-''}(x) - \chi_p(x)] = P(L-x), \\ \chi_p(x) = \dfrac{P(L-x) - M_p}{k}, \end{array} \right. \tag{11}$$

where w^- denotes the deflection in the plastic region. The elastic adjacent domain is governed by

$$x \in [l_0;L]: EIw^{+''}(x) = P(L-x). \tag{12}$$

w^+ is the deflection in the elastic region. The boundary conditions can be summarized as

$$\left| \begin{array}{l} w^-(0) = 0, \\ w^{-'}(0) = 0 \end{array} \right.$$

and

$$\left| \begin{array}{l} w^-(l_0) = w^+(l_0), \\ w^{-'}(l_0) = w^{+'}(l_0). \end{array} \right. \tag{13}$$

The deflection $w(x)$ and the rotation $w'(x)$ must be continuous functions of x (in particular at the intersection of the elastic and the plastic domains).

Enforcing that χ_p is a continuous function of x $[\chi_p(l_0)=0]$ leads to

$$\left| \begin{array}{l} P(L-l_0) = M_p, \\ PL \leqslant M_p, \end{array} \right. \Rightarrow l_0 = 0. \tag{14}$$

This additional assumption gives the Wood paradox. The unloading elastic solution is the only possible solution of the softening problem (Fig. 3). In this paper, an implicit gradient plasticity model (equivalent to an integral nonlocal plastic model) is developed, in order to overcome Wood's paradox.

III. NONLOCAL CONSTITUTIVE LAW

For the implicit gradient plasticity model, M^* is related to a nonlocal plastic curvature variable $\tilde{\chi}_p$ through the linear model

$$M^* = k\tilde{\chi}_p \text{ with } \tilde{\chi}_p = \chi_p + m(\overline{\chi_p} - \chi_p), \tag{15}$$

where the plastic modulus k is negative for softening models. $\tilde{\chi}_p$ is defined as a linear combination of the local curvature χ_p and the nonlocal plastic curvature $\overline{\chi_p}$. m is a dimensionless combination parameter. Such a combination of local and nonlocal plastic variables was initially proposed by Vermeer and Brinkgreve [48] (see also Ref. [42]). It is worth mentioning that a linear combination of local and nonlocal variable was already initiated by Eringen for a two-phase nonlocal elastic material [16]. For the implicit gradient plasticity model [40,41], the nonlocal plastic curvature $\overline{\chi_p}$ is defined as the solution of the differential equation

$$\overline{\chi_p} - l_c^2\overline{\chi_p}'' = \chi_p. \tag{16}$$

Therefore, a characteristic length l_c is introduced in the definition of the nonlocal plastic curvature $\tilde{\chi}_p$. As shown by Eringen for nonlocal elasticity [15], this differential equation clearly shows that the nonlocal plastic curvature $\overline{\chi_p}$ is a spatial weighted average of the variable χ_p. This spatial weighted average is calculated on the plastic domain

$$\overline{\chi_p}(x) = \int_0^{l_0} G(x,y)\chi_p(y)dy, \tag{17}$$

where the weighting function $G(x,y)$ is the Green's function of the differential system with appropriate boundary conditions. It can be shown that this nonlocal plastic softening constitutive law may be also expressed by

$$M^* - l_c^2 M^{*''} = k[\chi_p + a^2\chi_p''] \quad \text{with } a^2 = (m-1)l_c^2 \text{ for } m \geqslant 1. \tag{18}$$

This model comprises the purely nonlocal plastic softening model ($a=0$) and the gradient plasticity model ($l_c=0$). In case of the cantilever beam, the second derivative of the bending moment is vanishing ($M''=0 \Rightarrow M^{*''}=0$). Hence, for the cantilever beam, Eq. (18) shows that the nonlocal plasticity model looks similar to a gradient plasticity model, even if the boundary conditions differ for both models. The particular case $m=2$ leads to the simple equality $a=l_c$. In this last case, it can be observed that the nonlocal plastic curvature may also be defined as

$$m = 2 \Rightarrow \tilde{\chi}_p = \overline{\chi_p} + l_c^2\overline{\chi_p}''. \tag{19}$$

It is clear that such implicit gradient plasticity model is an integral nonlocal plasticity model. The apparently new definition of Eq. (19) is quite interesting, as the nonlocal plastic curvature $\tilde{\chi}_p$ appears to be very similar to the standard definition of explicit gradient plasticity models expressed with the nonlocal plastic curvature $\overline{\chi_p}$. It is worth mentioning that Eq. (7) is still valid for the elastoplastic moment-curvature relationship (see the discussion of Jirásek and Rolshoven [42] at the material scale or Appendix A).

The boundary conditions are expressed as (see also Appendix A)

$$\chi_p(l_0) = 0, \quad \overline{\chi_p}'(l_0) = 0, \quad \text{and } \overline{\chi_p}'(0) = 0. \tag{20}$$

An important difference with the implicit gradient plasticity model presented in Refs. [40,41], however, is that the extra boundary conditions are valid over the plastic domain, rather than over the entire domain. The system is now solved for the nonlocal plastic curvature $\overline{\chi_p}$:

$$\overline{\chi_p} + l_c^2\overline{\chi_p}'' = \frac{P(L-x) - M_p}{k} \tag{21}$$

with the three boundary conditions expressed in term of the unknown variable $\overline{\chi_p}$

$$\overline{\chi_p}(l_0) - l_c^2\overline{\chi_p}''(l_0) = 0, \quad \overline{\chi_p}'(l_0) = 0, \quad \text{and } \overline{\chi_p}'(0) = 0. \tag{22}$$

The general solution of the differential equation (21) is written as

$$x \in [0;l_0]: \overline{\chi_p}(x) = A\cos\frac{x}{l_c} + B\sin\frac{x}{l_c} + \frac{P(L-x) - M_p}{k}. \tag{23}$$

The nonlinear system of three equations with three unknowns A, B, and l_0 is finally obtained

$$\left|\begin{array}{l} 2A\cos\dfrac{l_0}{l_c} + 2B\sin\dfrac{l_0}{l_c} + \dfrac{P(L-l_0) - M_p}{k} = 0, \\ -\dfrac{A}{l_c}\sin\dfrac{l_0}{l_c} + \dfrac{B}{l_c}\cos\dfrac{l_0}{l_c} - \dfrac{P}{k} = 0, \\ \dfrac{B}{l_c} - \dfrac{P}{k} = 0. \end{array}\right. \tag{24}$$

The following dimensionless parameters may be introduced as

$$\beta = \left(1 - \frac{P_Y}{P}\right)\frac{L}{l_c} \leqslant 0 \text{ and } \xi = \frac{l_0}{l_c} \geqslant 0 \tag{25}$$

and the load-plastic zone relationship is finally written as

$$\beta = \xi - 2\frac{1-\cos\xi}{\sin\xi} \quad \text{for } \sin(\xi) \neq 0. \tag{26}$$

The solution ($\xi = 2n\pi$) has to be excluded, as this solution cannot be connected to the elastic solution. The asymptotic expansion for small values of ξ shows that in this last case

$$\xi \ll 1 \Rightarrow \beta \sim -\frac{\xi^3}{12}. \tag{27}$$

The third-order term of the asymptotic expansion may take negative values for positive values of the plastic length. For β being in the admissible range, ξ is inside its admissible range, and the plastic zone may spread (see also Fig. 4):

$$\xi \to 0^+ \Rightarrow \beta \to 0^-. \tag{28}$$

Therefore, the connection of the elastic and the plastic solution is implicitly fulfilled with this regularization method. In

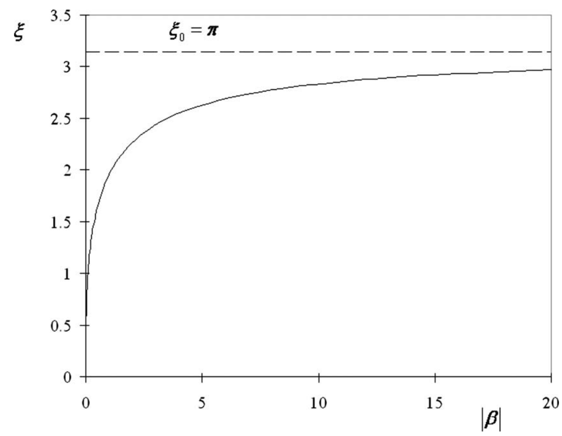

FIG. 4. Evolution of the plastic zone ξ versus the loading parameter β.

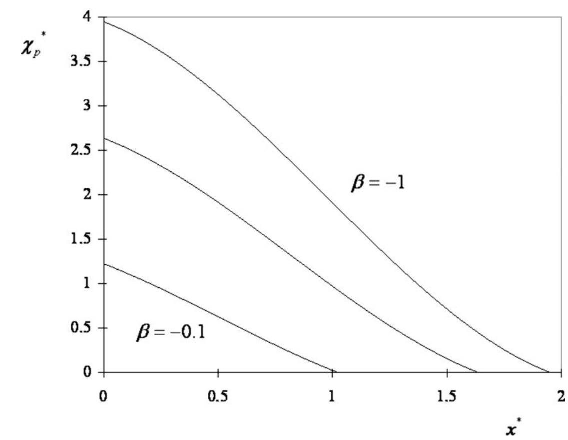

FIG. 5. Evolution of the plastic curvature χ_p^* for three loading cases.

particular, it is easy to check that the nonlocal plastic curvature χ_p is a positive parameter during the softening evolution. Furthermore, it is shown in Appendix B that no unloading could appear inside the plastic zone. Hence, the continuous bifurcation phenomenon recently observed by Benallal and Marigo [49] cannot occur for the beam with moment gradient. However, it is worth mentioning that the lost of uniqueness would be also observed for the beam under uniform bending moment, as observed by Benallal and Marigo [49] for the bar in tension. In other words, Wood's paradox is overcome for the cantilever cases and uniqueness prevails for the softening evolution considered in the paper. Figure 4 shows the evolution of the plastic zone ξ in term of the positive dimensionless parameter $|\beta|$. The parameter $|\beta|$ varies between 0 and tends towards an infinite value when P tends towards zero. Moreover, the size of the plastic zone tends towards an asymptotic value for large values of $|\beta|$ (and sufficiently small values of P)

$$\xi_0 = \pi. \tag{29}$$

$\xi_0 = \pi$ is the limiting value of the maximum width of the localization zone. The plastic zone evolves from a transitory regime towards a material scale that does not depend anymore on the loading range. The results reveal that the evolution tends towards one unique solution with a finite energy dissipation that depends only on the characteristic length. The maximum width of the localization zone l_0 directly depends on the characteristic length of the nonlocal model via the relation $l_0 = \pi l_c$ (for the cantilever beam). The determination of the characteristic length l_c (or the maximum width of the localization zone l_0) is related to the question of the finite-length hinge model, a central question of the present nonlocal model. Wood [44] inspired by the works of Barnard and Johnson [50] suggested the term of discontinuity length. Many papers have been published on the experimental or theoretical investigation of such a length [51–55]. Based on the experience with three-dimensional softening media, the characteristic length l_c must be bounded by the size of inhomogeneities in the material, and cannot be less than several aggregate sizes in concrete [55]. It seems, however, that this bound may be too small for bending problems. The value of l_c (or the maximum localization zone l_0) must be also related to the depth of the cross section h. In fact, in the beam theory, a limitation arises from Saint-Venant's principle and the basic assumption of plane cross sections. The bending theory that we are using is not valid for large deformations occurring within a length that is less than approximately the size of the cross section h [55]. Therefore, it is recommended that the maximum width of the localization zone l_0 is chosen in the order of magnitude of the depth of the cross section h (see also Ref. [52]). This implies for the cantilever beam that the characteristic length l_c is in the order of magnitude of h/π. The existence of this finite size fracture process zone leads to the specific structural size effect. The softening process is firmly associated to the deflection-controlled loading studied in this paper. A load-controlled test with monotonic increasing load would lead to a brittle response without softening state.

IV. RESOLUTION OF THE CANTILEVER CASE

The solution of the plastic curvature in the plastic zone can finally be written as

$$x^* \in [0;\xi]: \quad \chi_p{}^*(x^*) = -2\frac{\cos\xi - 1}{\sin\xi}\cos x^* - 2\sin x^* + x^* - \beta(\xi) \text{ with } \left| \begin{array}{l} x^* = \dfrac{x}{l_c}, \\ \chi_p{}^* = |k|\dfrac{\chi_p}{P l_c}, \end{array} \right. \tag{30}$$

where ξ is computed from β (or P) from Eq. (26). The size of the plastic zone is increasing as $|\beta|$ is increasing (Fig. 5). It should be mentioned that the non-local plastic curvature is not continuous at the boundary between the plastic and the elastic zones (see Fig. 6). The nonlocal plastic curvature also grows in the plastic zone:

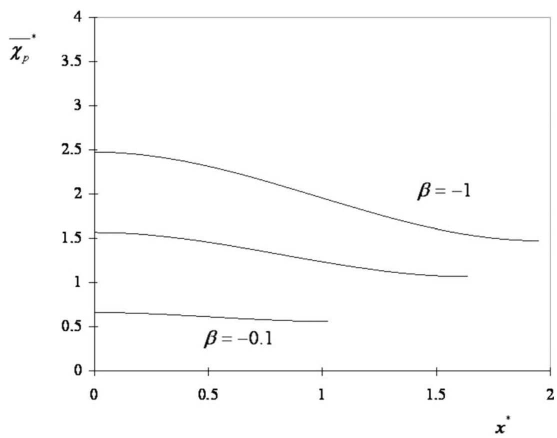

FIG. 6. Evolution of the nonlocal plastic curvature $\overline{\chi_p}^*$ for three loading cases.

$$x^* \in [0;\xi]: \quad \overline{\chi_p}^*(x^*) = -\frac{\cos\xi - 1}{\sin\xi}\cos x^* - \sin x^* + x^* - \beta(\xi) \text{ with } \overline{\chi_p}^* = \frac{\overline{\chi_p}}{Pl_c}|k|. \tag{31}$$

The rotation function $\theta(x)$ [or $w'(x)$] is calculated from integration of the differential equation

$$\theta'(x) = \chi_p(x) + \frac{P(L-x)}{EI} \text{ with } \theta(0) = 0. \tag{32}$$

The rotation in the plastic zone is obtained from

$$\theta^-(x) = P\left(\frac{1}{EI} + \frac{1}{k}\right)\left(Lx - \frac{x^2}{2}\right) - \frac{M_p}{k}x + 2\frac{Pl_c^2}{k}\frac{\cos\left(\frac{l_0}{l_c}\right) - 1}{\sin\left(\frac{l_0}{l_c}\right)}\sin\left(\frac{x}{l_c}\right) - 2\frac{Pl_c^2}{k}\left[\cos\left(\frac{x}{l_c}\right) - 1\right]. \tag{33}$$

The rotation in the elastic zone is derived from the continuity of the rotation along the elastoplastic boundary [see Eq. (13)]

$$\theta^+(x) = \frac{PL}{EI}(x - l_0) - \frac{P}{2EI}(x^2 - l_0^2) + \theta^-(l_0). \tag{34}$$

The rotation at the end of the beam is denoted by Θ (and $\Theta_Y = P_Y L^2/2EI$). The relationship between this normalized rotation and the loading parameter is simplified in

$$\frac{\Theta}{\Theta_Y} = \frac{P}{P_Y} + \frac{2EI}{k}\left[\frac{l_0}{L}\frac{P}{P_Y} - \frac{P}{2P_Y}\left(\frac{l_0}{L}\right)^2 - \frac{l_0}{L}\right]. \tag{35}$$

An example of softening response is shown in Fig. 7, for the load-rotation response. The regularization of the implicit gradient plasticity model is no more ambiguous. The deflection in the plastic zone is obtained by integrating the rotation, given by Eq. (33):

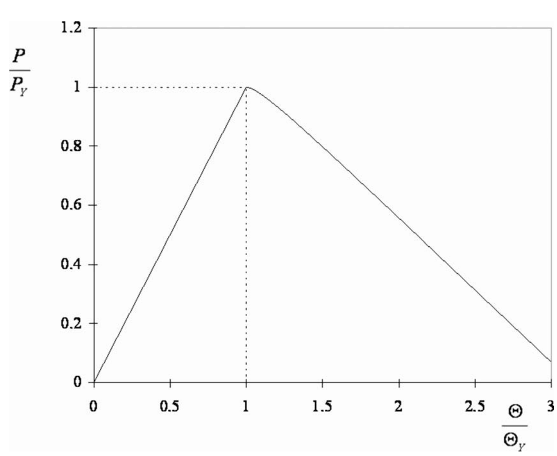

FIG. 7. Response of the elastoplastic beam $\frac{P}{P_Y}$ versus $\frac{\Theta}{\Theta_Y}$, $\frac{EI}{k} = -5$, $\frac{l_c}{L} = 0.1$.

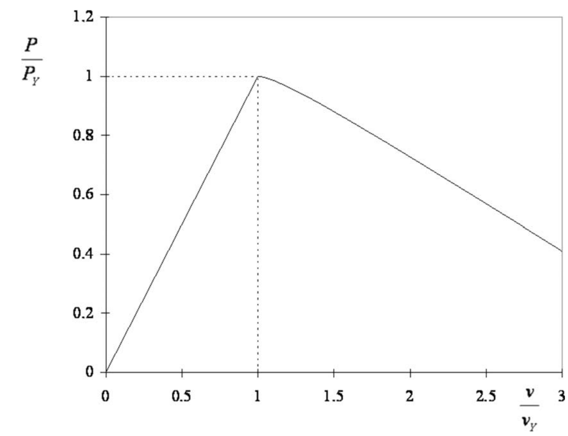

FIG. 8. Response of the elastoplastic beam $\frac{P}{P_Y}$ versus $\frac{v}{v_Y}$, $\frac{EI}{k} = -5$, $\frac{l_c}{L} = 0.1$.

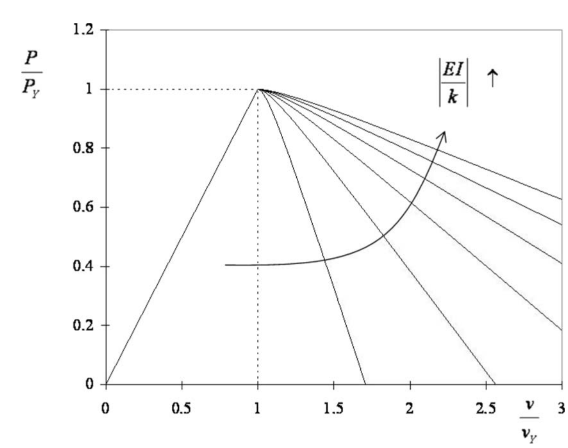

FIG. 9. Influence of the stiffness ratio on the response of the elastoplastic beam $\frac{P}{P_Y}$ versus $\frac{v}{v_Y}$, $\frac{l_c}{L} = 0.1$.

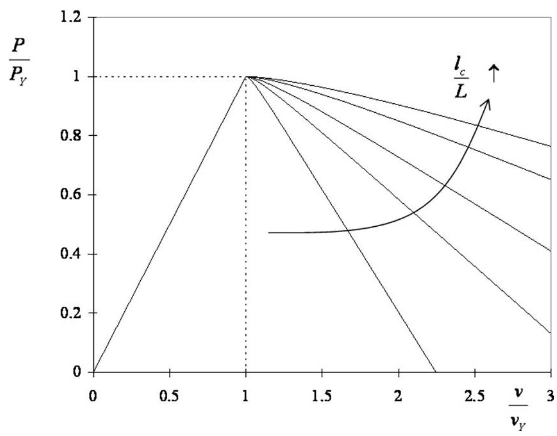

FIG. 10. Influence of the characteristic length on the response of the elastoplastic beam $\frac{P}{P_Y}$ versus $\frac{v}{v_Y}$, $\frac{EI}{k}=-5$.

$$w^{-}(x) = \left(\frac{PL}{EI} + \frac{PL - M_p}{k}\right)\frac{x^2}{2} - \left(\frac{P}{EI} + \frac{P}{k}\right)\frac{x^3}{6} - 2\frac{Pl_c{}^3}{k}\frac{\cos\left(\frac{l_0}{l_c}\right) - 1}{\sin\left(\frac{l_0}{l_c}\right)}\left[\cos\left(\frac{x}{l_c}\right) - 1\right] - 2\frac{Pl_c{}^2}{k}\left[l_c\sin\left(\frac{x}{l_c}\right) - x\right]. \quad (36)$$

The deflection in the elastic zone is derived from the continuity condition given by Eq. (13):

$$w^{+}(x) = \frac{PLx^2}{2EI} - \frac{Px^3}{6EI} + \left[w^{-\prime}(l_0) - \frac{PLl_0}{EI} + \frac{Pl_0{}^2}{2EI}\right]x + \left[w^{-}(l_0) - l_0 w^{-\prime}(l_0) + \frac{PLl_0{}^2}{2EI} - \frac{Pl_0{}^3}{3EI}\right]. \quad (37)$$

The evolutions of the deflection at the beam end are shown in Figs. 8–10. The global ductility increases as the stiffness ratio $|EI/k|$ increases, or the length ratio l_c/L increases. Evolution of the plastic zone during the plastic softening process is shown in Fig. 11.

V. SIMPLY SUPPORTED BEAM UNDER CONCENTRATED LOAD

Another simple statically determinate model is the simply supported beam under central concentrated load (Fig. 12). We will show that this case is very close to the cantilever case. More specifically, the plastic zone parameterized by the length l_0 is related to the loading parameter through a similar law. For symmetrical reasons, the bending moment is symmetric with respect to the central axis (bending moment is assumed to be positive in this case due to the change of the y axis):

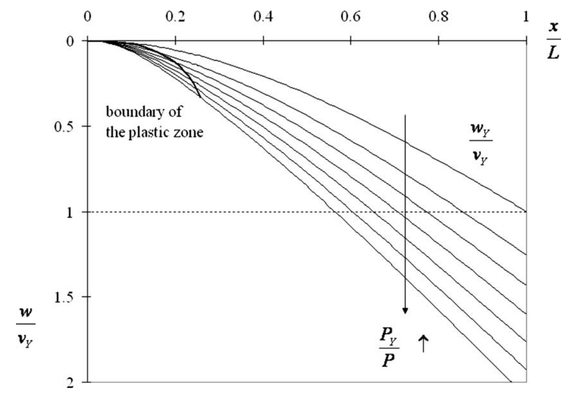

FIG. 11. Evolution of the plastic zone during the softening process, deflection of the cantilever beam, $\frac{EI}{k}=-5$, $\frac{l_c}{L}=0.1$.

$$M(x) = \frac{PL}{4} - \frac{P}{2}|x| \text{ with } P \geq 0 \text{ and } x \in \left[-\frac{L}{2};\frac{L}{2}\right]. \quad (38)$$

Using symmetrical considerations, it is sufficient to analyze half a structure, leading to the differential equation for the nonlocal plastic curvature

$$\overline{\chi_p} + l_c{}^2\overline{\chi_p}'' = \frac{\frac{PL}{4} - \frac{P}{2}x - M_p}{k} \text{ with } x \in \left[0;\frac{L}{2}\right]. \quad (39)$$

In this case, and using symmetrical arguments [41], the boundary conditions are written as

$$\chi_p\left(\frac{l_0}{2}\right) = 0, \quad \overline{\chi_p}'\left(\frac{l_0}{2}\right) = 0, \quad \text{and } \overline{\chi_p}'(0) = 0. \quad (40)$$

Let us introduce the change of variable

$$\hat{P} = \frac{P}{2}, \quad \hat{L} = \frac{L}{2}, \quad \hat{l}_0 = \frac{l_0}{2}, \quad \hat{P}_Y = \frac{M_p}{\hat{L}},$$

$$\hat{\xi} = \frac{\hat{l}_0}{l_c}, \quad \text{and } \hat{\beta} = \left(1 - \frac{\hat{P}_Y}{\hat{P}}\right)\frac{\hat{L}}{l_c}. \quad (41)$$

A new differential system is obtained:

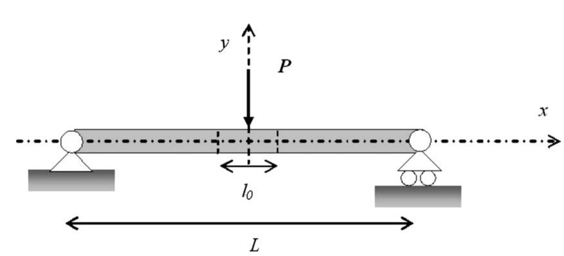

FIG. 12. Simply supported beam.

$$\overline{\chi_p} + l_c^2 \overline{\chi_p}'' = \frac{\hat{P}(\hat{L}-x) - M_p}{k} \text{ with } \chi_p(\hat{l}_0) = 0,$$

$$\overline{\chi_p}'(\hat{l}_0) = 0, \quad \text{and } \overline{\chi_p}'(0) = 0. \tag{42}$$

One recognizes the cantilever problem with the new variables introduced in Eq. (41). The load-plastic zone relationship is given by Eq. (26) corrected with the new variables

$$\hat{\beta} = \hat{\xi} - 2\frac{1-\cos\hat{\xi}}{\sin\hat{\xi}} \text{ for } \sin(\hat{\xi}) \neq 0. \tag{43}$$

Figures 6–9 obtained for the cantilever beam are still valid for the simply supported beam [with the new notation of Eq. (41)]. However, in the case of a simply supported beam, the boundary conditions dealing with the displacement function differ from the one of the cantilever case

$$\left| \begin{array}{l} w(\hat{L}) = 0, \\ w'(0) = 0. \end{array} \right. \tag{44}$$

This means that the solution of the cantilever case can be used for the rotation function, but not for the displacement function. The rotation in the plastic zone is obtained from

$$\theta^-(x) = \hat{P}\left(\frac{1}{EI}+\frac{1}{k}\right)\left(\hat{L}x - \frac{x^2}{2}\right) - \frac{M_p}{k}x + 2\frac{\hat{P}l_c^2}{k}\frac{\cos\left(\frac{\hat{l}_0}{l_c}\right)-1}{\sin\left(\frac{\hat{l}_0}{l_c}\right)}\sin\left(\frac{x}{l_c}\right) - 2\frac{\hat{P}l_c^2}{k}\left[\cos\left(\frac{x}{l_c}\right)-1\right] \text{ if } x \in [0;\hat{l}_0]. \tag{45}$$

The rotation in the elastic zone is derived from the continuity of the rotation along the elastoplastic boundary

$$\theta^+(x) = \frac{\hat{P}\hat{L}}{EI}(x-\hat{l}_0) - \frac{\hat{P}}{2EI}(x^2 - \hat{l}_0^2) + \theta^-(\hat{l}_0) \text{ if } x \in [\hat{l}_0;\hat{L}]. \tag{46}$$

The deflection in the elastic domain can be written as

$$w^+(x) = \frac{\hat{P}}{EI}\left(\hat{L}\frac{x^2}{2} - \frac{x^3}{6} - \frac{\hat{L}^3}{3}\right) + (x-\hat{L})\left(\frac{\hat{P}\hat{l}_0^2}{2EI} - \frac{\hat{P}\hat{L}\hat{l}_0}{EI} + \theta^-(\hat{l}_0)\right) \text{ if } x \in [\hat{l}_0;\hat{L}]. \tag{47}$$

VI. CANTILEVER BEAM UNDER DISTRIBUTED LATERAL LOAD

The failure process of the cantilever beam solicited by its own weight (or under uniform distributed lateral load) can be also studied (Fig. 13). In this case, the bending moment no more varies as a linear function, but as a parabolic function

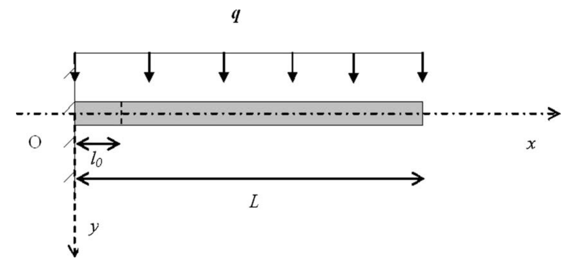

FIG. 13. Cantilever beam under uniformly distributed load.

$$M(x) = \frac{q}{2}(L-x)^2 \text{ with } P \geq 0 \text{ and } x \in [0;L]. \tag{48}$$

The differential equation for the nonlocal plastic curvature is now written as

$$\overline{\chi_p} + l_c^2 \overline{\chi_p}'' = \frac{\frac{q}{2}(L-x)^2 - M_p}{k}. \tag{49}$$

The solution of such a differential equation is obtained from

$$x \in [0;l_0]: \overline{\chi_p}(x) = A\cos\frac{x}{l_c} + B\sin\frac{x}{l_c} + \frac{q}{2k}(L-x)^2 - \frac{M_p}{k} - \frac{ql_c^2}{k} \tag{50}$$

with the three constants (A, B, l_0) identified from the three boundary conditions given by Eq. (22). The following dimensionless parameters may be introduced:

$$\beta = \left(1 - \frac{q_Y}{q}\right)\frac{L}{l_c} \leq 0, \quad \xi = \frac{l_0}{l_c}, \quad l_c^* = \frac{l_c}{L}, \quad \text{with } q_Y = \frac{2M_p}{L^2} \tag{51}$$

and the load-plastic zone relationship is finally written as (see Fig. 14)

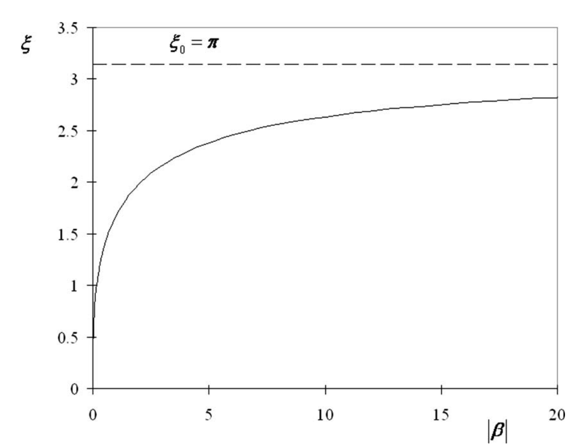

FIG. 14. Evolution of the plastic zone ξ versus the loading parameter β—cantilever beam under distributed load—$l_c^*=0.1$.

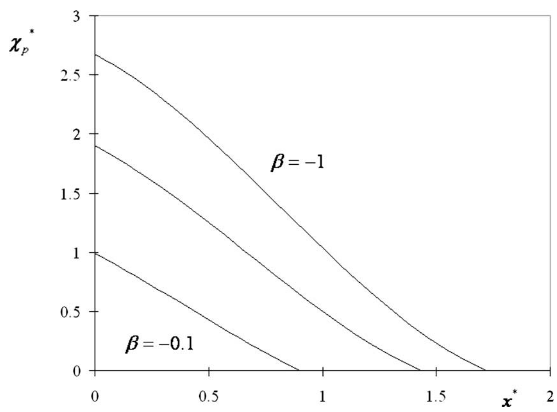

FIG. 15. Evolution of the plastic curvature χ_p^* for three loading cases—cantilever beam under distributed load—l_c^*=0.1.

$$\beta = -4\frac{1 + (l_c^*\xi - 1)\cos \xi}{\sin \xi} - l_c^*\xi^2 + 2\xi + 4l_c^* \text{ for } \sin(\xi) \neq 0. \tag{52}$$

The solution of the plastic curvature in the plastic zone can finally be written as

$$x^* \in [0;\xi]\text{: } \chi_p^*(x^*) = -2\frac{\cos \xi - 1 + \xi l_c^*}{\sin \xi}\cos x^* - 2\sin x^* - \frac{1}{2l_c^*}(1 - x^* l_c^*)^2 + \frac{1}{2}\left[\frac{1}{l_c^*} - \beta(\xi)\right] + 2l_c^*$$

with

$$\left| \begin{array}{l} x^* = \dfrac{x}{l_c}, \\ \chi_p^* = |k|\dfrac{\chi_p}{qLl_c}, \end{array} \right. \tag{53}$$

where ξ is computed from β (or P) from Eq. (52). The size of the plastic zone is increasing during the softening process (Fig. 15). The nonlocal plastic curvature slowly varies in the plastic region (see Fig. 16). This nonlocal variable also grows in the plastic zone during the softening process

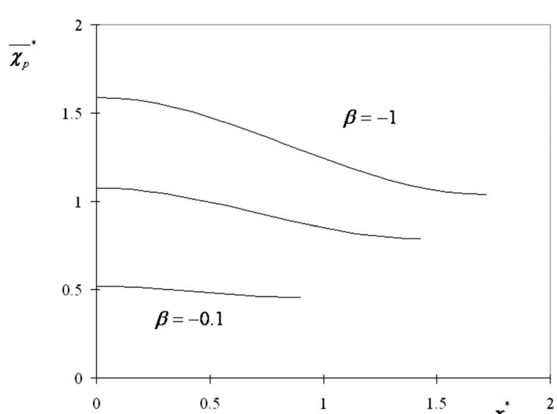

FIG. 16. Evolution of the plastic curvature $\overline{\chi_p}^*$ for three loading cases—cantilever beam under distributed load—l_c^*=0.1.

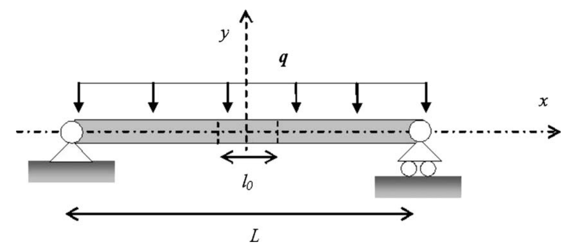

FIG. 17. Simply supported beam under uniform distributed load.

$$x^* \in [0;\xi]\text{: } \overline{\chi_p}^*(x^*) = -\frac{\cos \xi - 1 + \xi l_c^*}{\sin \xi}\cos x^* - \sin x^* - \frac{1}{2l_c^*}(1 - x^* l_c^*)^2 + \frac{1}{2}\left[\frac{1}{l_c^*} - \beta(\xi)\right] + l_c^* \text{ with } \overline{\chi_p}^* = \frac{\overline{\chi_p}}{qLl_c}|k|. \tag{54}$$

VII. SIMPLY SUPPORTED BEAM UNDER DISTRIBUTED LATERAL LOAD

Another simple statically determinate model is the simply supported beam under uniform distributed lateral load (Fig. 17). We will show that this case is not analogous to the cantilever case under distributed load. For symmetrical reasons, the bending moment is symmetric with respect to the central axis (bending moment is assumed to be positive in this case due to the change of the y axis):

$$M(x) = \frac{qL^2}{8} - \frac{qx^2}{2} \text{ with } x \in \left[-\frac{L}{2};\frac{L}{2}\right]. \tag{55}$$

Using symmetrical considerations, it is sufficient to analyze half a structure, leading to the differential equation for the nonlocal plastic curvature

$$\overline{\chi_p} + l_c^2\overline{\chi_p}'' = \frac{\dfrac{qL^2}{8} - \dfrac{qx^2}{2} - M_p}{k} \text{ with } \chi_p\left(\frac{l_0}{2}\right) = 0,$$

$$\overline{\chi_p}'\left(\frac{l_0}{2}\right) = 0 \text{ and } \overline{\chi_p}'(0) = 0. \tag{56}$$

The following change of variable can be adopted:

$$\hat{L} = \frac{L}{2}, \quad \hat{l}_0 = \frac{l_0}{2}, \quad \hat{q}_Y = \frac{2M_p}{\hat{L}^2}, \quad \hat{\xi} = \frac{\hat{l}_0}{l_c},$$

$$\hat{l}_c^* = \frac{l_c}{\hat{L}}, \quad \text{and } \hat{\beta} = \left(1 - \frac{\hat{q}_Y}{q}\right)\frac{\hat{L}}{l_c}. \tag{57}$$

A new differential system is obtained:

$$\overline{\chi_p} + l_c^2\overline{\chi_p}'' = \frac{\dfrac{q}{2}(\hat{L}^2 - x^2) - M_p}{k} \text{ with } \chi_p(\hat{l}_0) = 0,$$

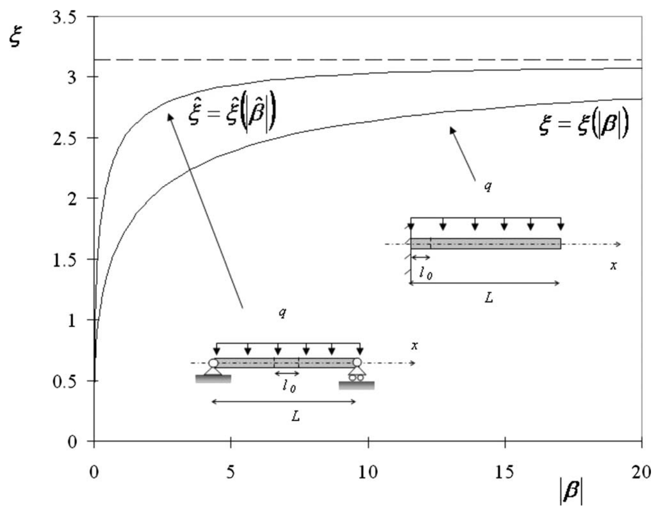

FIG. 18. Evolution of the plastic zone ξ (or $\hat{\xi}$) versus the loading parameter β (or $\hat{\beta}$)—Beam under distributed load—$l_c^*=0.1$.

$$\overline{\chi_p}'(\hat{l}_0) = 0 \text{ and } \overline{\chi_p}'(0) = 0. \tag{58}$$

This system cannot be cast in the form given by Eq. (49), and there is no equivalence between the cantilever case and the simply supported beam in case of uniform loading. The load-plastic zone relationship is finally written as (see Fig. 18)

$$\hat{\beta} = 4\hat{\xi}\hat{l}_c^* \frac{\cos\hat{\xi}}{\sin\hat{\xi}} + \hat{l}_c^* \hat{\xi}^2 - 4\hat{l}_c^* \text{ for } \sin(\xi) \neq 0. \tag{59}$$

The solution of the plastic curvature in the plastic zone can be finally written as

$$x^* \in [0;\xi]: \chi_p^*(x^*) = 2\frac{\hat{\xi}\hat{l}_c^*}{\sin\hat{\xi}}\cos x^* - \frac{\hat{\beta}(\hat{\xi})}{2} - 2\hat{l}_c^* + \frac{1}{2}\hat{l}_c^* x^{*2} \text{ with } \left| \begin{array}{l} x^* = \dfrac{x}{l_c}, \\ \chi_p^* = |k|\dfrac{\chi_p}{q\hat{L}l_c}. \end{array} \right. \tag{60}$$

The nonlocal plastic curvature also grows in the plastic zone as

$$x^* \in [0;\xi]: \overline{\chi_p}^*(x^*) = \frac{\hat{\xi}\hat{l}_c^*}{\sin\hat{\xi}}\cos x^* - \frac{\hat{\beta}(\hat{\xi})}{2} - \hat{l}_c^* + \frac{1}{2}\hat{l}_c^* x^{*2} \text{ with } \overline{\chi_p}^* = \frac{\overline{\chi_p}}{Pl_c}|k|. \tag{61}$$

VIII. CONCLUSIONS

This paper questions the mode of collapse of some simple softening structural systems, comprising the classical cantilever beam. Civil engineering, classical mechanical or nanomechanics are concerned by such an elementary model. The homogeneous cantilever beam loaded by a concentrated force at its extremity is first considered as a structural paradigm. A nonlocal plasticity model, also called implicit gradient plasticity model (equivalent to an integral nonlocal plasticity model), is developed in order to control the localization process induced by microcracking phenomena. It is shown that the regularized problem is well-posed (and, in particular, uniqueness of the evolution problem is shown). Such a result closes the discussion concerning explicit versus implicit gradient plasticity models (or integral-based plasticity models), as explicit gradient plasticity models may fail to capture the failure process of the cantilever beam [47,56]. Nonlocal plasticity models (implicit gradient plasticity models) are definitively needed to solve the modeling of beam failure.

Closed-form solutions of the elastoplastic deflection are finally derived. The scale effect linked to the nonlocal failure process is clearly highlighted. The length of the plastic zone grows during the softening process until an asymptotic limited value, which depends on the characteristic length of the material. The existence of this finite size fracture process zone leads to the specific structural size effect. As a consequence of this model, the plastic length evolves during the loading process, a phenomenon often noticed in structural design. The cantilever case can be understood at this stage as an elementary structural case, with moment gradient. Other structural cases are also presented, including the simply supported beam under uniform transverse loading. These results are valid for the beam bending problem, but also for the simple analogy of the bar subjected to distributed axial force [43,56].

It is concluded that the mode of collapse is firmly a nonlocal phenomenon. A local constitutive law cannot accurately describe the softening-induced microcracking phenomena. Structural collapse mobilizes a characteristic length introduced in the nonlocal constitutive law. Such phenomena may appear at the macrostructural level, but also when scale effects may be significant as it is typically observed in the field of nanomechanics. An extension of this study could be the necessary coupling between nonlocal elasticity and nonlocal plasticity.

APPENDIX A: DERIVATION OF THE NATURAL BOUNDARY CONDITIONS FROM A VARIATIONAL PRINCIPLE

Let us consider the energy functional

$$W[w,\chi_p] = \int_0^L \frac{1}{2}EI(w'' - \chi_p)^2 - \frac{k}{2}l_c^2\overline{\chi_p}'\chi_p' + M_p\chi_p + \frac{k}{2}\overline{\chi_p}\chi_p dx - Pw(L). \tag{A1}$$

The first variation of the energy functional leads to the extremal condition

$$\delta W[w,\chi_p] = \int_0^L EI(w'' - \chi_p)(\delta w'' - \delta\chi_p) - \frac{k}{2} l_c^2 (\overline{\delta\chi_p}' \chi_p' + \overline{\chi_p}' \delta\chi_p') + M_p \delta\chi_p + \frac{k}{2} (\overline{\delta\chi_p}\chi_p + \overline{\chi_p}\delta\chi_p) dx - P\delta w(L) = 0. \tag{A2}$$

Following a classical procedure also used for explicit gradient plasticity models (see Refs. [38,39]), the overall domain can be divided into a plastic domain and an elastic one:

$$\delta W[w,\chi_p] = \int_0^L EI(w'' - \chi_p)\delta w'' dx + \int_0^{l_0} - EI(w'' - \chi_p)\delta\chi_p - \frac{k}{2} l_c^2 (\overline{\delta\chi_p}' \chi_p' + \overline{\chi_p}' \delta\chi_p') + M_p \delta\chi_p + \frac{k}{2} (\overline{\delta\chi_p}\chi_p + \overline{\chi_p}\delta\chi_p) dx - P\delta w(L) = 0. \tag{A3}$$

Moreover, the following Green-type identity associated with the self-adjoint property of the regularized operator for relevant boundary conditions (see, for instance, Refs. [57,58]), the following identities hold:

$$\int_0^{l_0} \chi_p \overline{\delta\chi_p} dx = \int_0^{l_0} \overline{\chi_p}\delta\chi_p dx - l_c^2 [\overline{\chi_p}' \overline{\delta\chi_p}]_0^{l_0} + l_c^2 [\overline{\chi_p}\, \overline{\delta\chi_p}']_0^{l_0}. \tag{A4}$$

Unmixed boundary conditions and periodic boundary conditions lead to the self-adjoint property of the regularized operator [57]. The same reasoning can be applied for the following identity:

$$\int_0^{l_0} \chi_p' \overline{\delta\chi_p}' dx = \int_0^{l_0} \overline{\chi_p}' \delta\chi_p' dx - l_c^2 [\overline{\chi_p}'' \overline{\delta\chi_p}']_0^{l_0} + l_c^2 [\overline{\chi_p}' \overline{\delta\chi_p}'']_0^{l_0}. \tag{A5}$$

Using Eqs. (A4) and (A5), the extremal condition (A3) can then be simplified:

$$\delta W[w,\chi_p] = \int_0^L EI(w'' - \chi_p)\delta w'' dx + \int_0^{l_0} - EI(w'' - \chi_p)\delta\chi_p - k l_c^2 \overline{\chi_p}' \delta\chi_p' + M_p \delta\chi_p + k\overline{\chi_p}\delta\chi_p dx - \frac{k l_c^2}{2} ([\overline{\chi_p}' \overline{\delta\chi_p}]_0^{l_0} - [\overline{\chi_p}\, \overline{\delta\chi_p}']_0^{l_0}) + \frac{k l_c^4}{2} ([\overline{\chi_p}'' \overline{\delta\chi_p}']_0^{l_0} - [\overline{\chi_p}' \overline{\delta\chi_p}'']_0^{l_0}) - P\delta w(L) = 0. \tag{A6}$$

Moreover, the following integration by part is obtained for the plastic curvature:

$$\int_0^{l_0} \overline{\chi_p}' \delta\chi_p' dx = [\overline{\chi_p}' \delta\chi_p]_0^{l_0} - \int_0^{l_0} \overline{\chi_p}'' \delta\chi_p dx. \tag{A7}$$

The first variation of the functional W is then written as

$$\delta W[w,\chi_p] = \int_0^L M\delta w'' dx - \int_0^{l_0} [M - M_p - k(\overline{\chi_p} + l_c^2 \overline{\chi_p}'')]\delta\chi_p dx - k l_c^2 [\overline{\chi_p}' \delta\chi_p]_0^{l_0} - \frac{k l_c^2}{2} ([\overline{\chi_p}' \overline{\delta\chi_p}]_0^{l_0} - [\overline{\chi_p}\, \overline{\delta\chi_p}']_0^{l_0}) + \frac{k l_c^4}{2} ([\overline{\chi_p}'' \overline{\delta\chi_p}']_0^{l_0} - [\overline{\chi_p}' \overline{\delta\chi_p}'']_0^{l_0}) - P\delta w(L) = 0. \tag{A8}$$

The following integration by part can be considered for the deflection

$$\int_0^L M\delta w'' dx = [M\delta w']_0^L - [M'\delta w]_0^L + \int_0^L M'' \delta w dx \text{ with } M = EI(w'' - \chi_p). \tag{A9}$$

The extremal condition leads to the equilibrium equation and the yield condition

$$M'' = 0 \text{ and } M = M_p + k(\overline{\chi_p} + l_c^2 \overline{\chi_p}'') \tag{A10}$$

with the natural boundary conditions

$$M(L) = 0, \quad M'(L) = -P, \quad w(0) = w'(0) = 0,$$

$$\overline{\chi_p}'(0) = \overline{\chi_p}'(l_0) = \chi_p(l_0) = 0. \tag{A11}$$

The high-order boundary conditions considered in Eq. (20) are then obtained from a variational principle, which can be condensed as

$$[\overline{\chi_p}' \delta\chi_p]_0^{l_0} = 0. \tag{A12}$$

Some similar boundary conditions have also been considered for implicit gradient damage models [59]. The structure of the energy functional considered in Eq. (A1) can now be commented. Using Eq. (17), the nonlocal term of Eq. (A1) can be written in a more readable form

$$\int_0^{l_0} \chi_p(x)\overline{\chi_p}(x) dx = \int_0^{l_0}\int_0^{l_0} G(x,y)\chi_p(x)\chi_p(y) dx dy. \tag{A13}$$

A similar functional can be defined for the nonlocal term incorporating the gradient terms in Eq. (A1), based on the derivation of Eq. (16):

$$(\overline{\chi_p}') - l_c^2(\overline{\chi_p}')'' = \chi_p' \text{ and } \overline{\chi_p}'(0) = \overline{\chi_p}'(l_0) = 0 \Rightarrow \overline{\chi_p}'(x) = \int_0^{l_0} H(x,y)\chi_p'(y)dy. \quad (A14)$$

The nonlocal term incorporating the gradient terms can be written in a similar format:

$$\int_0^{l_0} \chi_p'(x)\overline{\chi_p}'(x)dx = \int_0^{l_0}\int_0^{l_0} H(x,y)\chi_p'(x)\chi_p'(y)dxdy. \quad (A15)$$

Finally, some other energy functional could be studied, such as,

$$W[w,\chi_p] = \int_0^L \frac{1}{2}EI(w'' - \chi_p)^2 + k\overline{\chi_p}\chi_p + M_p\chi_p - \frac{k}{2}\chi_p^2 dx - Pw(L). \quad (A16)$$

The stationarity of this functional also leads to Eq. (A10), except for the fact that the boundary conditions $\bar{\chi}_p'(0) = \bar{\chi}_p'(l_0) = 0$ are not necessarily derived from the variational principle (unmixed boundary conditions and periodic boundary conditions also lead to the self-adjoint property of the regularized operator [57]), but have to be postulated as an additional condition. The main advantage of the energy functional (A1) is that the high-order boundary conditions are rigorously derived from application of a variational principle. A similar discussion can be found for nonlocal elastic beam models [60].

APPENDIX B: UNIQUENESS PROOF FOR THE CANTILEVER CASE

We show in this appendix that unloading in the plastic zone is not possible if the plastic curvature is assumed to be a continuous function (and in particular the matching condition $\chi_p = 0$ at the end of the plastic zone). The nonlocal plastic curvature is given by Eq. (21):

$$x^* \in [0;\xi]: \overline{\chi_p}^*(x^*) = -\frac{\cos\xi - 1}{\sin\xi}\cos x^* - \sin x^* + x^* - \beta(\xi). \quad (B1)$$

The derivation of the nonlocal plastic curvature, with respect to the dimensionless spatial coordinate is written as

$$x^* \in [0;\xi]: \overline{\chi_p}^{*\prime}(x^*) = \frac{\cos\xi - 1}{\sin\xi}\sin x^* - \cos x^* + 1 \text{ with } \xi \in [0;\pi[. \quad (B2)$$

A necessary condition for unloading in the plastic zone is obtained from the boundary condition at the elastic-plastic boundary, i.e., the vanishing of the derivation of the nonlocal plastic curvature

$$\overline{\chi_p}^{*\prime}(x_0^*) = 0 \Leftrightarrow \sin(x_0^* - \xi) = \sin x_0^* - \sin\xi. \quad (B3)$$

Using trigonometric identities, Eq. (B3) can be solved with

$$\sin(x_0^* - \xi) = 2\cos\frac{x_0^* + \xi}{2}\sin\frac{x_0^* - \xi}{2} \Rightarrow \sin\left(\frac{x_0^* - \xi}{2}\right) = 0 \text{ or } \cos\left(\frac{x_0^* - \xi}{2}\right) = \cos\left(\frac{x_0^* + \xi}{2}\right). \quad (B4)$$

The only solution for $x_0^* \in [0;\xi]$ and $\xi \in [0;\pi]$ is the boundary of the plastic zone

$$x_0^* = 0 \text{ or } x_0^* = \xi. \quad (B5)$$

Therefore, no unloading can occur in the plastic zone, and uniqueness prevails for the softening evolution. A different conclusion would have been obtained if the continuity assumption of the plastic curvature would have been relaxed (uniqueness is no more guaranteed without the continuity assumption of the plastic curvature).

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