Multicommunity weight-driven bipartite network model

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Community structure and rewiring phenomena exist in many complex networks, particularly in bipartite networks. We construct a model for the degree distribution of the rewiring problem in a multicommunity weight-driven bipartite network (MCWBN). The network consists of many interconnected communities, each of which holds a bipartite graph. The bipartite graph consists of two sets of nodes. We name each node in one set a "producer" and each node in the other set a "consumer." A weight value matrix defining the trade barrier between any two communities is used to characterize the structure of the communities, which ensures the higher preferential attachment probability in intracommunity than in intercommunity. The size of one producer is defined as the number of consumers connected to it. We find that the nonlinear dynamics of the scale of production, or the total size of all producers in each community is dependent only on the initial scale of production in each community, and independent of the distribution of the producer size. Furthermore, if the nonlinear system of the scale of production in each community is at an equilibrium state, the distribution of the producer size in each community of the MCWBN model is equivalent to that in a one-community model.

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I. INTRODUCTION

Complex networks play an important role in the fields of science, engineering, economy, and many others. The topology of networks has been extensively investigated [1–7]. One feature of many real-world networks is the scale-free nature, implying that the degree distribution of these complex networks follows the power law. The scale-free nature of complex networks can be considered e.g., by using a mechanism of growing with preferential attachment [2]. Another common feature of many real-world networks is the "community structure," that is, the tendency for vertices to divide into groups, with dense connections within each group and only sparse connections between the groups [8–10]. Social networks [11], biochemical networks [12–14], and information networks such as the Internet [15] have all been shown to possess strong community structure. Communities can be considered as regions or countries in a social network [16,17]; pages on the related topics in the internet [18]; some kinds of functional units in a biochemical network or neural network [12,19]; related papers on a topic in a citation network [20]. Studies on community structure mainly focus on the properties of the entire network and on algorithms for finding the community structure [21–24]. A number of recent results suggest that networks can possess properties at the community level that are quite different from their properties at the level of the entire network [24].

Studies on growing networks are attracting much attention [16,25–28]. However, in many real-world networks, the evolution of the network occurs not only through growing but also rewiring. Early stressing on the rewiring process is the small world network model of Watts and Strogatz [29]. The network rewiring has been studied in Refs. [30–32]. The zero range process [33–35], the diversity of genes [36,37], the popularity of minority game strategies [38], and the popular culture change [39] may also be cast in terms of network rewiring. Bipartite networks are an important kind of networks in the real world such as the movie actor network [29] consisting of actors and movies, the scientific collaboration network [25,40,41] consisting of scientists and articles, and the economic network [40,42]. The rewiring phenomena can also be found in many bipartite networks, for example, in the producer-consumer bipartite network with consumers changing the final product of one producer to the same kind of product from another producer, in the population-city network with people moving from one city to another city, in the netizen-website network with netizens shifting from one website to another website, and in the producer-bank network with producers switching from one bank to another bank. There have been some models concerning the rewiring dynamics of bipartite networks [30,42–44]. Some urn models are also kinds of rewiring processes in bipartite networks [45,46]. Although growing network model with community structure are proposed by some studies [16,17,47], there is relatively little attention paid to the rewiring dynamics in the bipartite network with community structure.

The present paper proposes a bipartite network model with the community structure to study the degree distribution of the rewiring problem, and gives an analysis of the degree distribution relationship between the model and a model without community structure that is almost identical to the model in Refs. [42–44]. First, a multicommunity weight-driven bipartite network (MCWBN) model is proposed in Sec. II. In Sec. III, we discuss the degree distribution of a one-community model, which describes the rewiring dynamics of one of communities in the multicommunity network model. After that, we analyze the dynamics of the scale of production in each community of the MCWBN in Sec. IV A

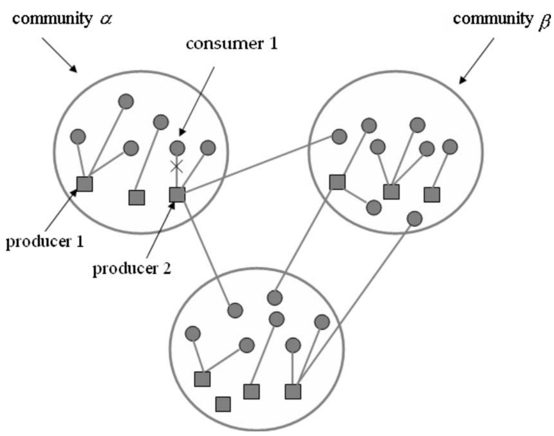

FIG. 1. The bipartite network with multicommunities; circles represent the consumers and squares represent the producers. $N_{i,\alpha}(t)$ is the expectation of the number of producers in the α^{th} community with i consumers from any community at time step t. The expectation of the number of producers after disconnecting a consumer is denoted by $N^{-}_{i,\alpha}(t)$. For example, in community α, $N_{3,\alpha}=1$ and $N^{-}_{3,\alpha}=2$.

and the degree distribution in each community in Sec. IV B. Finally, summary and discussion are presented in Sec. V.

II. THE MODEL

We will focus on a generic rewiring problem of a multicommunity bipartite network. The network consists of many interconnected communities, each of which holds a bipartite graph. The bipartite graph consists of two sets of nodes. We name each node in one set a "consumer" and each node in the other set a "producer." In this paper, the naming of the nodes is only used to facilitate the description of a generic bipartite graph, so we will keep our presentation abstract apart from the names [48].

The MCWBN is illustrated in Fig. 1, which consists of many communities with different numbers of consumers and producers, the numbers of which also differ community by community. Every consumer links with only one producer that may be either in the same community with the consumer or in another community. All the consumers have an equal quantity of consumption and the size of a producer is defined as the number of consumers that are connected to it. In this study, the number of the consumers and that of the producers in each community are maintained constant. However, the consumers may change their choice of producers. The consumers prefer choosing a producer from the same community to choosing a producer of the same size from another community due to the trade barrier (defined by a weight value) between communities [49]. Thus, the distribution of the producer size in each community evolves with the rewiring of the consumers.

Let R be the number of the communities; p_α, the number of producers in the $\alpha^{th}(\alpha=1,2,\ldots,R)$ community; and c_α, the number of consumers in the α^{th} community. Let P be the total number of all producers, i.e., $P=\Sigma_{\gamma=1}^{R}p_\gamma$. Let C be the total number of all consumers, i.e., $C=\Sigma_{\gamma=1}^{R}c_\gamma$.

Assume that $N_{i,\alpha}(t)$ is the expectation of the number of producers in the α^{th} community with i consumers from any community at time step t. The scale of production in community α is defined by

$$S_\alpha(t)=\sum_{i=1}^{C} iN_{i,\alpha}(t). \tag{1}$$

Beginning with any given initial distribution $N_{i,\alpha}(0)$, the MCWBN model evolves according to the following two steps. The first is a removing step, the second is a preferential attachment step.

(1) A consumer (see, e.g., consumer 1 in community α of Fig. 1) is selected randomly from the set of all consumers and is disconnected from its producer. Then, the expectation of the number of producers after disconnecting the consumer, which is denoted by $N^{-}_{i,\alpha}(t)$, can be calculated by

$$N^{-}_{i,\alpha}(t)=\frac{i+1}{C}N_{i+1,\alpha}(t-1)+\left(1-\frac{i}{C}\right)N_{i,\alpha}(t-1). \tag{2}$$

Similarly to Eq. (1), the scale of production in community α after disconnecting the consumer is defined by

$$S^{-}_\alpha(t)=\sum_{i=1}^{C} iN^{-}_{i,\alpha}(t). \tag{3}$$

(2) The producers compete with each other to win the consumer that is disconnected from its producer. According to the linear preferential attachment mechanism, the probability of the consumer choosing a producer with degree $i-1$ in community β is proportional to $\frac{(i-1+a)w_{\alpha,\beta}}{W_\alpha(t)}$ with $W_\alpha(t)$ defined by

$$W_\alpha(t)=\sum_{\gamma=1}^{R} w_{\alpha,\gamma}[S^{-}_\gamma(t)+p_\gamma a], \tag{4}$$

where a stands for the initial attractiveness of every producer and allows a nonzero probability for a producer with no consumer [50]. $w_{\alpha,\gamma}$ with $0\leqslant w_{\alpha,\gamma}<1$ is a coefficient arising due to the trade barrier that prevents the consumer in community α from choosing producers in community γ. We assume $w_{\alpha,\alpha}=1$ for any α in the present paper. $w_{\alpha,\gamma}<w_{\alpha,\alpha}$ ensures a higher preferential attachment probability in intracommunity than in intercommunity.

As the total probability of consumers in community β being selected to disconnect from their producers is c_β/C,

$$N_{i,\alpha}(t)=\sum_{\beta=1}^{R}\frac{c_\beta}{C}\frac{(i-1+a)w_{\beta,\alpha}}{W_\beta(t)}N^{-}_{i-1,\alpha}(t)+\left[1-\sum_{\beta=1}^{R}\frac{c_\beta}{C}\frac{(i+a)w_{\beta,\alpha}}{W_\beta(t)}\right]N^{-}_{i,\alpha}(t). \tag{5}$$

Since the total number of consumers is C, it is apparent that $N_{i,\alpha}=0$ when $i>C$. Therefore, $N_{C+1,\alpha}=N^{-}_{C+1,\alpha}(t)=0$ in the following analysis. Equations (2), (4), and (5) describe the dynamics of the distribution of the producer size in each community. When $R=1$, the model degenerates to a one-community model that is the same with that in Ref. [42] and is almost identical to that in Refs. [43,44].

III. THE DEGREE DISTRIBUTION OF THE BIPARTITE NETWORK IN A ONE-COMMUNITY MODEL

Before analyzing the MCWBN model, we first consider a one-community bipartite network model with rewiring dynamics, the community of which is one of communities in the multicommunity network model. The preferential attachment probability of a node in the one-community model is linearly proportional to the sum of the degree and the initial attractiveness of the node. The dynamics of the one-community model is represented as follows:

$$\tilde{N}_i^-(t) = \frac{(i+1)m}{\tilde{C}}\tilde{N}_{i+1}(t-1) + \left(1-\frac{im}{\tilde{C}}\right)\tilde{N}_i(t-1), \tag{6}$$

$$\tilde{N}_i(t) = \frac{(i-1+a)m}{\tilde{C}-m+\tilde{P}a}\tilde{N}_{i-1}^-(t) + \left[1-\frac{(i+a)m}{\tilde{C}-m+\tilde{P}a}\right]\tilde{N}_i^-(t), \tag{7}$$

where $i=0,1,2,\ldots,\tilde{C}$ and $0<m\leqslant 1$. The notations of $\tilde{N}_i(t), \tilde{N}_i^-(t), \tilde{C}, \tilde{P}$, and a correspond to $N_{i,\alpha}(t)$, $N_{i,\alpha}^-(t)$, c_α, p_α, and a with dropping index α for the community label for simplicity, respectively.

The parameter m represents that each consumer is selected to undergo rewiring with a probability $m/\tilde{C}$ at each time step [51].

Rewiring dynamics in one-community was also considered in some related models [30,42–45]. All these models concern a bipartite network, and assume that the size of the network does not change rapidly, but the structure of the network changes with time rapidly, and assume that a node is chosen randomly to be attached in the removing step. In Ref. [45], the fitness parameter is considered as the inverse local temperature and the attachment probability is a nonlinear function of the degree and fitness parameter of the node. The degree distribution of the bipartite network was discussed depending on the fitness parameter. The preferential attachment mechanism in Ref. [30] is the same with that in Ref. [45] except for the formulation of the fitness function.

The model in Refs. [42–44] is almost identical to our one-community model. If $m=1$ in the Eqs. (6) and (7), our one-community model is the same as that in Ref. [42], where a simulation result on the distribution of the producer size was presented. A time-dependent exact solution on the degree distribution in a one-community model was provided by Evans and Plato [43,44]. In the model of Evans and Plato, the attachment probability is a mixture of a random attachment probability ρ_r and the preferential attachment probability ρ_p, and the rewiring dynamics was organized into one step, in contrast to our two-step rewiring dynamics of Eq. (6) and Eq. (7). However, the two steps of the rewiring in our one-community model can be mapped to the one step rewiring in the model of Evans and Plato [44] by substituting Eq. (6) into Eq. (7) as follows:

$$\begin{aligned}\tilde{N}_i(t) - \tilde{N}_i(t-1) = {} & \tilde{N}_{i+1}(t-1)m\Pi_{R,i+1}(1-m\Pi_{A,i}) \\ & - \tilde{N}_i(t-1)m\Pi_{R,i}(1-m\Pi_{A,i-1}) \\ & - \tilde{N}_i(t-1)m\Pi_{A,i}(1-m\Pi_{R,i}) \\ & + \tilde{N}_{i-1}(t-1)m\Pi_{A,i-1}(1-m\Pi_{R,i-1}),\end{aligned} \tag{8}$$

where $\Pi_{R,i}=i/\tilde{C}$, $\Pi_{A,i}=\rho_r\frac{1}{\tilde{P}}+\rho_p\frac{i}{\tilde{C}-m}$ with $\rho_r=a\tilde{P}/(a\tilde{P}+\tilde{C}-m)$ and $\rho_p=1-\rho_r$. If $m=1$, the two models are almost identical. While in our one-community model, after a producer with degree i loses a consumer, the degree of the producer reduces to $i-1$ and the total degree of all the producers decreases to $\tilde{C}-1$.

In the present paper, we focus on the analysis of the relationship of the degree distribution between the one-community model and the multicommunity model. The exact solution of the one-community model and detailed discussion of the degree distribution depending on the parameter a can be obtained by the approach that was presented by Evans and Plato [44].

Based on the analytical results of Evans and Plato [44] and the simulation results in [42], a tendency of the variation of the degree distribution with a changing from zero to infinity occurs in the case that $\tilde{C}$ and $\tilde{P}$ are sufficiently large. When a is close to zero, all the consumers will link to one producer, which gives a condensate, or a monopoly in the present paper. With increasing of a from zero, the power law distribution with exponent 1 emerges. Exponential distribution can be found around $a=1$. When $a>1$, a Gamma-like distribution appears. To the extreme, if a is large enough, a dominates the attachment probability, thus the attachment probability of each producer becomes approximately equal, resulting in the Possion distribution.

IV. THE DYNAMICS OF THE DEGREE DISTRIBUTION OF THE BIPARTITE NETWORK IN THE MULTICOMMUNITY MODEL

In this section, we relate the degree distribution of the bipartite network in each community of the MCWBN model to that in the one-community model studied in the previous section. We first find that the dynamics of the scale of production $S_\alpha(t)$ [see Eq. (1)] in each community is independent of the distribution of the producer size and converges to a fixed point in Sec. IV A. Based on this result, we next show that the distribution of the producer size in each community of the MCWBN model is equivalent to that in the one-community model in Sec. IV B.

A. The dynamics of the scale of production in each community

Using Eqs. (1)–(3), the evolution of the scale of production in community α after the removing step of the rewiring is described as follows:

$$
\begin{aligned}
S_\alpha^-(t) &= \sum_{i=0}^{C+1} i\frac{i+1}{C}N_{i+1,\alpha}(t-1) + \sum_{i=0}^{C+1} i\left(1-\frac{i}{C}\right)N_{i,\alpha}(t-1)\\
&= \sum_{i=0}^{C+1} iN_{i,\alpha}(t-1) + \sum_{i=0}^{C+1}\left[\frac{(i+1)^2}{C}N_{i+1,\alpha}(t-1)\right.\\
&\quad \left. -\frac{i^2}{C}N_{i,\alpha}(t-1) - \frac{i+1}{C}N_{i+1,\alpha}(t-1)\right]\\
&= \left(1-\frac{1}{C}\right)S_\alpha(t-1),
\end{aligned}
\tag{9}
$$

because $N_{C+1,\alpha}(t-1)=0$.

Similarly, using Eqs. (1), (3), and (5), we obtain the evolution of the scale of production in community α as follows:

$$
\begin{aligned}
S_\alpha(t) &= \sum_{\beta=1}^{R}\frac{c_\beta w_{\beta,\alpha}}{CW_\beta(t)}\sum_{i=0}^{C+1} i(i-1+a)N_{i-1,\alpha}^-(t)\\
&\quad + \sum_{i=0}^{C+1} i\left[1-\sum_{\beta=1}^{R}\frac{c_\beta w_{\beta,\alpha}}{CW_\beta(t)}(i+a)\right]N_{i,\alpha}^-(t)\\
&= S_\alpha^-(t) + \sum_{\beta=1}^{R}\frac{c_\beta w_{\beta,\alpha}}{CW_\beta(t)}\sum_{i=0}^{C+1}[(i+1)(i+a)\\
&\quad - i(i+a)]N_{i,\alpha}^-(t)\\
&= S_\alpha^-(t) + \sum_{\beta=1}^{R}\frac{c_\beta w_{\beta,\alpha}}{CW_\beta(t)}[S_\alpha^-(t)+ap_\alpha],
\end{aligned}
\tag{10}
$$

because $p_\alpha=\Sigma_{i=0}^{C+1}N_{i,\alpha}^-(t)$.

Thus, from Eqs. (4), (9), and (10), we get the evolution dynamics of the scale of production in each community as follows:

$$
W_\beta(t) = \sum_{\gamma=1}^{R} w_{\beta,\gamma}\left[S_\gamma(t-1) - \frac{S_\gamma(t-1)}{C} + p_\gamma a\right],
\tag{11}
$$

$$
\begin{aligned}
S_\alpha(t) &= S_\alpha(t-1) - \frac{S_\alpha(t-1)}{C}\\
&\quad + \sum_{\beta=1}^{R}\frac{c_\beta w_{\beta,\alpha}}{CW_\beta(t)}\left[S_\alpha(t-1) - \frac{S_\alpha(t-1)}{C} + p_\alpha a\right],
\end{aligned}
\tag{12}
$$

where $w_{\beta,\alpha}$, c_β, C, p_α, and a are parameters in Eqs. (2), (4), and (5). It is shown from Eqs. (11) and (12) that the scale of production $S_\alpha(t)$ in community α at the time step t depends only on the scale of production in each community at the time step $t-1$.

To check whether the system of the scale of production [see Eqs. (11) and (12)] converges to a fixed point for various cases, we carry out 1000 runs of the system with five communities, in each run beginning with randomly generated initial conditions $S_1(0), S_2(0), \ldots, S_R(0)$ and randomly generated parameter values. Figure 2 shows the evolution of the average value and the standard deviation of $\Sigma_{\alpha=1}^{5}|S_\alpha(t)-S_\alpha(t-1)|$ over the 1000 runs. Since $\Sigma_{\alpha=1}^{5}|S_\alpha(t)-S_\alpha(t-1)|$ $\to 0$ when $t\to\infty$, the system actually converges to a fixed point for various values of the parameters and various initial conditions in the case of five communities. Additionally, the simulations for the cases of ten, twenty, and fifty communities are carried out respectively, and the system in these cases converges to a fixed point as well.

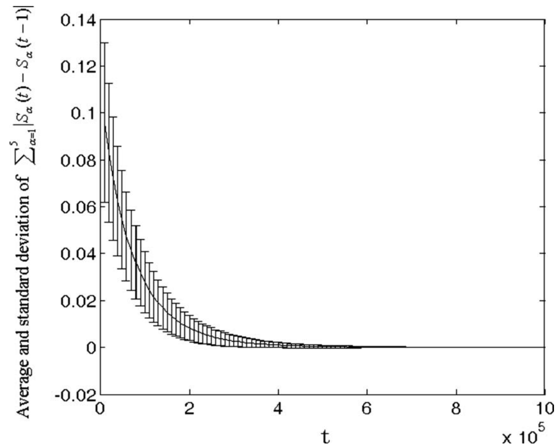

FIG. 2. Evolution of the average value and the standard deviation of $\Sigma_{\alpha=1}^{5}|S_\alpha(t)-S_\alpha(t-1)|$ with time t over the 1000 runs. Each run begins with randomly generated initial conditions $S_1(0), S_2(0), \ldots, S_\alpha(0)$ and randomly generated parameter values. $S_\alpha(0)$, $p_\alpha a$, and $w_{\beta,\alpha}$ with $\alpha,\beta=1,2,\ldots,5$ and $\alpha\neq\beta$ are uniformly distributed in the interval of (2000, 6000), (500, 2000), and (0.2, 0.8), respectively; $w_{\alpha,\alpha}=1$ and $c_\alpha=S_\alpha(0)$.

From Eqs. (11) and (12), the fixed point of the scale of production S_α^* is determined by

$$
W_\beta^* = \sum_{\gamma=1}^{R} w_{\beta,\gamma}\left[S_\gamma^* - \frac{S_\gamma^*}{C} + p_\gamma a\right],
\tag{13}
$$

$$
S_\alpha^* = C\sum_{\beta=1}^{R}\frac{c_\beta w_{\beta,\alpha}}{CW_\beta^*}\left[S_\alpha^* - \frac{S_\alpha^*}{C} + p_\alpha a\right].
\tag{14}
$$

We can see the influence of the community structure ($w_{\beta,\alpha}$) on the scale of production of community α (S_α^*) from Eqs. (13) and (14). If $w_{\beta,\alpha}(\beta=1,2,\ldots,R,\beta\neq\alpha)$ is large, i.e., there is little barrier for the consumers in other communities to choose the producers in community α, S_α^* will be large.

B. The relationship of the degree distribution in the bipartite network between the one-community model and the multicommunity model

According to Sec. IV A, the scale of production in each community converges to an equilibrium state S_α^*. At the equilibrium state, though the scale of the production in each community keeps constant, the rewiring dynamics continues, i.e., the consumers who are connected to the producers in community α may be reconnected to the producers in other communities and vice versa. We next study the evolution of the distribution of the producer size in each community of the MCWBN model. When the scale of production in each community is at an equilibrium state, the number of consumers that are connected to producers of each community keeps constant. Since the number of producers in each community does not change as well, we can consider the multicommunity model as R one-community models (see Appendix):

$$\widetilde{N}^{-}_{i,\alpha}(t) = \frac{(i+1)m}{\widetilde{C}}\widetilde{N}_{i+1,\alpha}(t-1) + \left(1 - \frac{im}{\widetilde{C}}\right)\widetilde{N}_{i,\alpha}(t-1), \quad (15)$$

$$\widetilde{N}_{i,\alpha}(t) = \frac{(i-1+a)m}{\widetilde{C}-m+\widetilde{P}a}\widetilde{N}^{-}_{i-1,\alpha}(t) + \left[1 - \frac{(i+a)m}{\widetilde{C}-m+\widetilde{P}a}\right]\widetilde{N}^{-}_{i,\alpha}(t), \quad (16)$$

where $\widetilde{C}=S^*_\alpha$, $\widetilde{P}=p_\alpha$, and $m=S^*_\alpha/\Sigma^R_{\gamma=1}S^*_\gamma=S^*_\alpha/C$ with $\alpha=1,2,\ldots,R$.

Equations (15) and (16) represent the dynamics of the distribution of the producer size in each community of the multicommunity model with each community holding dynamics like Eqs. (6) and (7). In other words, though all the communities in the MCWBN model connect with each other, the dynamics of the distribution of the producer size in R communities of the MCWBN model is equivalent to R mutually independent one-community models if the scale of production in each community converges to an equilibrium state. Furthermore, as the system for the distribution of the producer size in the one-community model is globally asymptotically stable [44], the distribution of the producer size in each community of the multicommunity model converges to a unique distribution that follows the exponential distribution, the power law distribution, the Gamma-like distribution, and the Possion distribution depending on different system parameters according to the discussion in Sec. III.

V. SUMMARY AND DISCUSSION

A multicommunity weight-driven bipartite networks (MCWBN) model was constructed to study the rewiring dynamics of the bipartite network with community structure. We first discussed the degree distribution of a one-community model which is taken out from the multicommunity model. Then we found that the evolution of the scale of production in each community of the MCWBN model depends only on the initial scale of production in each community and is independent of the distribution of the producer size. Furthermore, if the nonlinear system of the scale of production in each community is at an equilibrium state, the distribution of the producer size in each community of the MCWBN model is equivalent to that in a one-community model.

Many real-world networks are huge and exhibit community structure. It is difficult for empirical investigations on the degree distribution to cover the entire network or to consider the interaction of the communities. It is also very difficult to investigate whether the degree distribution in each community is the same as that in the whole network. In the multicommunity model, the community structure is represented by parameter $w_{\alpha,\beta}$ that only affects the scale of production in each community instead of the stationary distribution of the producer size in each community. After the scale of production converges to the fixed point, $w_{\alpha,\beta}$ can be neglected. Furthermore, if the initial attractiveness a is uniform in the whole network, the degree distribution in each community is identical. These results imply that we only need to investigate one of the communities without considering the community structure, i.e., we only need to carry out empirical investigations on the degree distribution in one community to estimate the degree distribution in other communities, when we investigate the stationary degree distribution in a huge multicommunity network with uniform initial attractiveness a. It is worthy of note that we cannot neglect the community structure before the system is fixed at the stationary distribution, as the scale of production in each community may change dynamically due to the parameter $w_{\alpha,\beta}$. Thus, the empirical investigations on the degree distribution without considering the community structure are to some extent reasonable. For instance, empirical investigations on the firm sizes carried out independently in the US and Japan show that the distribution of firm sizes vs. rank follows the identical power law distribution in the US and Japan, maybe as well as in other countries [52–55]. The identical power law distribution is also found for the distribution of city sizes in developed countries [56].

In some multicommunity real-world networks, the values of the parameter a may be different community by community. What kinds of distributions in each community and in the entire network emerge in such a situation is a future problem. Furthermore, the multicommunity model can be extended to consider the situation where growth and rewiring both exist.

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APPENDIX: DERIVATION OF EQS. (15) and (16)

Assume that the system of the scale of production in each community of the multicommunity model is at an equilibrium state at time step $t-1$, then $\Sigma^{C+1}_{i=0} iN_{i,\alpha}(t-1) = \Sigma^{\widetilde{C}+1}_{i=0} iN_{i,\alpha}(t-1) = S^*_\alpha$. According to Eq. (9),

$$S^{-*}_\alpha - S^*_\alpha = -\frac{S^*_\alpha}{C} = -m. \quad (A1)$$

As $\widetilde{C}=S^*_\alpha$, from Eq. (A1),

$$\sum_{i=0}^{\widetilde{C}+1}(i-1)N^{-}_{i-1,\alpha}(t) = \sum_{i=0}^{\widetilde{C}+1} iN^{-}_{i,\alpha}(t) = \widetilde{C} - m. \quad (A2)$$

Furthermore, from Eqs. (5) and (A1),

$$\begin{aligned} m &= S^*_\alpha - S^{-*}_\alpha \\ &= \sum_{i=0}^{\widetilde{C}+1}[iN_{i,\alpha}(t) - iN^{-}_{i,\alpha}(t)] \\ &= \sum_{\beta=1}^{R}\frac{c_\beta w_{\beta,\alpha}}{CW^*_\beta}\sum_{i=0}^{\widetilde{C}+1}[i(i-1+a)N^{-}_{i-1,\alpha}(t) - i(i+a)N^{-}_{i,\alpha}(t)]. \end{aligned} \quad (A3)$$

From Eq. (7),

$$\sum_{i=0}^{\tilde{C}+1}[i\tilde{N}_i(t)-i\tilde{N}_i^-(t)] = \frac{m}{\tilde{C}-m+\tilde{P}a}\sum_{i=0}^{\tilde{C}+1}[i(i-1+a)\tilde{N}_{i-1}^-(t)-i(i+a)\tilde{N}_i^-(t)] = \frac{m}{\tilde{C}-m+\tilde{P}a}\left[\sum_{i=0}^{\tilde{C}+1}(i-1)\tilde{N}_{i-1}^-(t)+\tilde{P}a\right]. \quad \text{(A4)}$$

From Eq. (6),

$$\sum_{i=0}^{\tilde{C}+1}(i-1)\tilde{N}_{i-1}^-(t) = \sum_{i=0}^{\tilde{C}+1}\frac{(i-1)im}{\tilde{C}}\tilde{N}_i(t-1) + \sum_{i=0}^{\tilde{C}+1}\left[i-1-\frac{(i-1)(i-1)m}{\tilde{C}}\right]\tilde{N}_{i-1}(t-1) = \tilde{C}-m. \quad \text{(A5)}$$

Combining Eqs. (A4) and (A5),

$$\sum_{i=0}^{\tilde{C}+1}[i\tilde{N}_i(t)-i\tilde{N}_i^-(t)] = \frac{m[\tilde{C}-m+\tilde{P}a]}{\tilde{C}-m+\tilde{P}a} = m. \quad \text{(A6)}$$

Using Eq. (A4) and Eq. (A6),

$$\frac{m}{\tilde{C}-m+\tilde{P}a}\sum_{i=0}^{\tilde{C}+1}[i(i-1+a)\tilde{N}_{i-1}^-(t)-i(i+a)\tilde{N}_i^-(t)] = m. \quad \text{(A7)}$$

Considering Eq. (A5), we conclude that Eq. (A7) holds for any values of $\tilde{N}_{i-1}^-(t)$ and $\tilde{N}_i^-(t)$ that satisfy $\Sigma_{i=0}^{\tilde{C}+1}i\tilde{N}_i^-(t) = \Sigma_{i=0}^{\tilde{C}+1}(i-1)\tilde{N}_{i-1}^-(t)=\tilde{C}-m$. Taking account of Eq. (A2) and letting $\tilde{N}_i^-(t)=N_{i,\alpha}^-(t)$ and $\tilde{N}_{i-1}^-(t)=N_{i-1,\alpha}^-(t)$, Eq. (A7) becomes

$$\frac{m}{\tilde{C}-m+\tilde{P}a}\sum_{i=0}^{\tilde{C}+1}[i(i-1+a)N_{i-1,\alpha}^-(t)-i(i+a)N_{i,\alpha}^-(t)] = m. \quad \text{(A8)}$$

Hence, it is shown from Eqs. (A3) and (A8) that

$$\sum_{\beta=1}^{R}\frac{c_\beta w_{\beta,\alpha}}{CW_\beta^*}\sum_{i=0}^{\tilde{C}+1}[i(i-1+a)N_{i-1,\alpha}^-(t)-i(i+a)N_{i,\alpha}^-(t)] = \frac{m}{\tilde{C}-m+\tilde{P}a}\sum_{i=0}^{\tilde{C}+1}[i(i-1+a)N_{i-1,\alpha}^-(t)-i(i+a)N_{i,\alpha}^-(t)].$$

Thus,

$$\frac{m}{\tilde{C}-m+\tilde{P}a} = \sum_{\beta=1}^{R}\frac{c_\beta w_{\beta,\alpha}}{CW_\beta^*}. \quad \text{(A9)}$$

When the system of the scale of production in each community of the multicommunity model is at an equilibrium state, $W_\beta(t)=W_\beta^*$. By Eq. (A9), the system of Eqs. (2), (4), and (5) becomes that of Eqs. (15) and (16), where $\tilde{C}=S_\alpha^*$, $\tilde{P}=p_\alpha$, and $m=S_\alpha^*/\Sigma_{\gamma=1}^R S_\gamma^*=S_\alpha^*/C$.

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[48] The name of nodes reflects possible applications in economy [42]. It could be "individual" and "artifact" describing the transmission of cultural artifacts such as pottery designs, dog breed, and baby name popularity [44]. Reference [44] discusses various possible applications of the generic rewiring problem such as cultural transmission, family names and wealth distributions. In such possible applications, the concept of communities reflects the regions or countries.
[49] The trade barrier in economics represents the effect of geographic instrument, culture, tariffs etc. on trade. In the context, due to the trade barrier with a weight value, the probability of the preferential attachment of consumers to producers in the same community is higher than that of consumers to producers in a different community.
[50] The initial attractiveness parameter a has the same meaning as the fitness parameter in Refs. [30,42,45]. In a producer-consumer bipartite network, initial attractiveness can be considered as the technology, the distinction, and other initial features for the producer. For a netizen-website network, it can be considered as the famousness, the topic, the style, and other initial features of the website.
[51] The purpose of the analysis of the one-community model is to provide a basis for the analysis of the multicommunity model. The parameter m is needed when we relate the rewiring dynamics in each community of the multicommunity model to the one-community model in Sec. IV B. For example, if we relate the rewiring dynamics of community α in the multicommunity model to the one-community model, $m=S_{\alpha}^{*}/C$, which is the ratio of the stationary scale of production in community α to the total number of consumers.
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