Transient multimodality in the presence of potential fluctuations

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When a stochastic system evolves from its unstable state the phenomenon of transient multimodality may appear, namely for a period of time, probability distribution develops a larger number of maxima than it possesses in the initial and in the stationary states. We investigate the influence of potential fluctuations generated by a Gaussian white noise on this effect. Some general conditions which suffice for the occurrence of noise induced transient multimodality are presented. The opposite phenomenon, namely the noise suppression of transient multimodality, is shown as well.

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I. INTRODUCTION

A widespread belief admits noise as a nuisance which blurs, garbles, or demolishes well-defined deterministic behavior of dynamical systems. During the last few decades, however, a number of phenomena have been identified in which noises also play constructive roles—they can increase some desired properties of systems, or even create some completely new features. The best known effects are noise induced transitions (NIT) [1], stochastic resonance [2], or directed transport in Brownian motors (ratchets) [3]. The first of these concerns the qualitative change of stationary properties of a system driven with a strong enough noise. In a single-variable case this effect is seen as the appearance or disappearance of some—one or more—new maxima of the stationary probability distribution function (PDF) while increasing noise. The noise alters the PDF's modality. The value of its intensity plays a similar role as the deterministic control parameter—when it exceeds its critical value some new stable states (maxima of PDF) arise and the existing ones lose their stability, e.g., a former maximum of PDF turns into a minimum.

In the standard description of one-dimensional Brownian motion [4] the modality of stationary PDF is determined by the number of minima of potential $U(x)$, which governs the deterministic part of evolution. The additive noise affects the width and the relative heights of the maxima, only. An abrupt change of the value of the control parameter by crossing its critical value involves the qualitative change of potential form and, consequently, stationary PDF—the new one has a different number of maxima and minima, or their locations are mutually converted. The question arises how the system evolves from its initial to final state, or how the transitions between the PDF's of different number of maxima go in time. The knowledge of those transient states may be of importance, e.g., for selection of states in ecological evolution or in species dynamics [5]; or for technical applications, if some of the ways, in which the switching on/off processes are being realized, may destroy a system [6], etc. Specially interesting is the case in which the change of one of the system's parameters implies a lack of stability of the initial state. The problem known as decay of an unstable state was analyzed extensively by many authors some 20–30 years ago [7–9]. The paradigm of the evolution states, as presented by Suzuki [7], that the PDF initially centered at the unstable state broadens in time until, at some time moment t_b, the wide hump breaks up into two peaks, which move away from the initial position of maximum, eventually forming the stationary bistable form. The evolution of PDF maxima resembles supercritical pitchfork bifurcation with time being the control parameter—beginning from a one-hump form the PDF changes into a two-hump one. The number of maxima is associated with the shape, first of the initial PDF, later of the stationary PDF. Or, in other words, with the number of minima of potential $U(x)$—the initial one has a single-well form, while the change of the control parameter alters it to the two-wells form.

Nonetheless, this is not the only possible scenario of evolving from one-to two-modal PDF. Namely, a transient stage with a three-hump form of PDF may appear before the final bimodal shape arises. This phenomenon, called transient multimodality (TM), resembles subcritical pitchfork bifurcation. It has been reported in [10–13] and later investigated systematically by the present author [14–16]. Particularly, it has been proven [14] that if near the unstable point the potential is flat enough the effect has to appear. It is worth mentioning that transient bimodality was also identified for stochastic evolution through a region of a very slow deterministic motion, e.g., in explosive chemical reactions and combustion [17–20], or in optical bistability [10,21–24]. Quite recently TM has been reported also for relaxation in monostable potentials [25].

It has been argued in [14], then confirmed analytically [15] and experimentally [16], that the occurrence of TM during the decay of an unstable state is of a deterministic nature, i.e., its appearance depends only on the deterministic properties of the system (potential). Although some randomness (either random initial conditions or noise driving the system) is required to identify its existence, however the very effect does not belongs to the class of noise-induced phenomena [26]. For further convenience we call it the deterministic transient multimodality (DTM).

In view of the above discussion on the number of maxima in the stationary state it seems natural to ask whether noise can also generate transient multimodality of PDF during relaxation from an unstable state. This question is addressed in

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the present paper. After beginning with a brief reminder of the main ideas used in the analysis of DTM (Sec. II), we formulate the problem and present general conditions which suffice for the appearance of noise induced transient multimodality (NITM) during the decay of an unstable state (Sec. III). Next (Sec. IV), we define all generic cases of the considered system and discuss the influence of different parameters of stochastic perturbation on the scenario of the PDF evolution. Some examples are presented and discussed in Sec. V and some final remarks complete the paper (Sec. VI).

II. DETERMINISTIC TRANSIENT MULTIMODALITY

Consider an overdamped Brownian particle moving in the field of a bistable potential $U(x)$ and driven by a (thermal) Gaussian white noise $\xi(t)$ of zero mean, correlation $\langle\xi(t)\xi(t')\rangle = 2q\delta(t-t')$ and intensity q. Its dynamics is given by the Langevin equation

$$\frac{dx}{dt} = -U'(x) + \xi(t), \tag{1}$$

or, equivalently, by the Fokker-Planck equation (FPE)

$$\frac{\partial}{\partial t}P(x,t) = \frac{\partial}{\partial x}U'(x)P(x,t) + q\frac{\partial^2}{\partial x^2}P(x,t), \tag{2}$$

for the PDF $P(x,t)$. Initially the particle is located either exactly at the top $x_{in}=x_t$ of the barrier separating potential wells, or with a probability distribution $P(x,0)=P_{in}(x)$ somewhere in its vicinity. For simplicity we assume symmetry of the problem, i.e., $U(x)=U(-x)$ and $P_{in}(x)=P_{in}(-x)$, which guarantees the space evenness of $P(x,t)$ during the entire evolution. Particularly, at any time moment PDF possesses an extremum at $x=x_t=0$: initially a maximum of $P_{in}(x)$ and finally a minimum of its stationary form $P_s(x)$. This means that initially the second x derivative of $P(x,t)$ at $x=0$ is negative, while finally positive. Thus a time moment $t_b>0$ exists at which this derivative changes its sign. To investigate this temporal transition from one-to two-humped form of PDF let us consider the second x derivative of the FPE (2) at $x=0$:

$$\dot{P}^{(2)}(t) = U^{(4)}P^{(0)}(t) + 3U^{(2)}P^{(2)}(t) + qP^{(4)}(t), \tag{3}$$

where $f^{(n)}(t) \equiv \partial^n/\partial x^n f(x,t)|_{x=0}$, $\dot{f}(t) \equiv \partial/\partial t f(x,t)|_{x=0}$, and we have exploited the evenness of $U(x)$ and $P(x,t)$. When $t=t_b$, then $P^{(2)}(t_b)=0$, but also $\dot{P}^{(2)}(t_b)\geq 0$. Since the maximum has still to be at $x=0$, so $P^{(4)}(t_b)\leq 0$. Comparing the signs of both sides of Eq. (3), one concludes that the necessary condition for their equality is

$$U^{(4)} \geq 0. \tag{4}$$

If not, i.e., if

$$U^{(4)} \leq 0, \tag{5}$$

the only possibility to ensure the fulfilment of the equality in (3) is $P^{(4)}(t_b)\geq 0$. This means that before the maximum of PDF at $x=0$ disappears, there already exist two other maxima at the points $x\neq 0$. So, for some interval of time ending at $t=t_b$ the PDF $P(x,t)$ has still possessed three humps. Thus, the *transient trimodality* phenomenon occurs [14].

The condition (4) is fulfilled, e.g., for the quartic potential (6)

$$U_4(x) = \frac{1}{4}bx^4 + \frac{1}{2}ax^2, \quad (b>0). \tag{6}$$

It displays supercritical pitchfork bifurcation as a alters from positive to negative values, thus for $a<0$ the state $x=0$ becomes unstable. Potential (6) is generic for the Suzuki's scheme [7] of temporal transition from the mono- to the bimodal form of $P(x,t)$. On the other hand, the sixth-order potential

$$U_6(x) = \frac{1}{6}cx^6 + \frac{1}{4}bx^4 + \frac{1}{2}ax^2, \quad (c>0), \tag{7}$$

is generic for the TM case. Namely, the bistable form of $U_6(x)$ is guaranteed by the negativeness of the quadratic term $a<0$, while b determines the sign of $U^{(4)}$. For $b<0$, the inequality (5) holds. The effect of TM during the decay of an unstable state has been demonstrated both numerically [14] and experimentally [16]. Assuming the weak noise approximation, the present author has obtained [15] some formulas for the points x_{cr} and time t_{cr} at which the two side maxima and minima originate. The agreement with numerics [15] and experiment [16] have been reasonably good. Let us mention that for the potential $U_6(x)$ the effect of TM can be seen also for $b>0$ up to a critical value b_c [15]. This is so because Eq. (4) is only a necessary and not a sufficient condition for the Suzuki's scheme of evolution. A more careful analysis of signs in Eq. (3) leads to the conclusion that $U^{(4)}$ must be positive enough to ensure the equality of both sides of this equation.

III. NOISE INDUCED TRANSIENT MULTIMODALITY

The aim of this paper is to investigate the evolution of an overdamped motion in the presence of both additive and multiplicative noises. So we generalize Langevin equation (1) to the following form:

$$\frac{dx}{dt} = -U'(x) - V'(x)\eta(t) + \xi(t). \tag{8}$$

The multiplicative term in this equation has the sense of a stochastic perturbation of the potential and the underlying dynamics can be understood as Brownian motion driven by the thermal noise $\xi(t)$ in the field of random potential $\mathcal{U}(x,t)=U(x)+V(x)\eta(t)$. Like $\xi(t)$, the multiplicative noise $\eta(t)$ is a Gaussian white one, it has zero mean, correlation $\langle\eta(t)\eta(t')\rangle = 2Q\delta(t-t')$, and intensity Q. We assume also that both noises are correlated $\langle\xi(t)\eta(t')\rangle = 2\lambda\sqrt{qQ}\delta(t-t')$, with the correlation rate λ taking values from +1 for complete correlation, through 0 for uncorrelated, to −1 for totally anticorrelated noises.

Since both noises are white, one can easily write FPE associated with (8). In the Stratonovich interpretation it reads [27]

$$\frac{\partial}{\partial t}P(x,t) = \frac{\partial}{\partial x}\left[U'(x) + \frac{1}{2}G'(x)\right]P(x,t) + \frac{\partial}{\partial x}G(x)\frac{\partial}{\partial x}P(x,t), \tag{9}$$

with the x-dependent diffusion function

$$G(x) = q + 2\lambda\sqrt{qQ}V'(x) + QV'(x)^2. \tag{10}$$

For simplicity, as in Sec. II, we consider only the symmetric case which requires that $G(x)=G(-x)$. This means that $V(x)$ has to have a well-defined parity property: it must be odd for $\lambda\neq 0$, and either odd or even for uncorrelated noises ($\lambda=0$). Finally, we assume that the bistable form of stationary PDF is preserved, i.e., we do not consider perturbations which could lead to noise induced transitions in the stationary state of the system. Hence, the stationary solution $P_s(x)$ of (9) has three extremes at the points given by the roots of equation

$$U'(x) + \frac{1}{2}G'(x) = U'(x) + \lambda\sqrt{qQ}V''(x) + QV'(x)V''(x) = 0. \tag{11}$$

By symmetry the minimum is located at $x=0$, which requires that

$$U^{(2)} + \frac{1}{2}G^{(2)} = U^{(2)} + (\lambda\sqrt{qQ} + QV^{(1)})V^{(3)} + Q(V^{(2)})^2 < 0. \tag{12}$$

We call (12) the unstability condition (USC). From the form of the second x derivative of the FPE (9) at the point $x=0$,

$$\dot{P}^{(2)}(t) = \left(U^{(4)} + \frac{1}{2}G^{(4)}\right)P^{(0)}(t) + \left(3U^{(2)} + \frac{9}{2}G^{(2)}\right)P^{(2)}(t) + GP^{(4)}(t), \tag{13}$$

one can infer, following similar arguments as in Sec. II, that if

$$U^{(4)} + \frac{1}{2}G^{(4)} = U^{(4)} + (\lambda\sqrt{qQ} + QV^{(1)})V^{(5)} + 4QV^{(2)}V^{(4)} + 3Q(V^{(3)})^2] < 0, \tag{14}$$

then the three-hump form of PDF—transient trimodality—does appear while evolving from the unstable state. If such a transient structure is absent in the unperturbed case (1) then (14) is the sufficient condition for the appearance of *the noise induced transient multimodality* (NITM). Conversely, if despite the condition (5) the sign of the inequality in (14) is opposite, one may expect that multiplicative noise destroys TM. In what follows we call inequality (14) the transient multimodality condition (TMC). Let us note, that any qualitative variation of a transient state caused by multiplicative noise is possible only for a sufficient degree of nonlinearity of the perturbation. For uncorrelated noises, $V(x)$ should have a nonvanishing third (for odd case) or fourth (for even case) derivative at the unstable point. The possible correlation of $\xi(t)$ and $\eta(t)$ becomes significant for an even higher derivative—the fifth one.

IV. GENERIC CASES

Both conditions (12) and (14) are expressed by means of the values of derivatives of potentials $U(x)$ and $V(x)$ at $x=0$, up to the fourth [for $U(x)$ and for even $V(x)$] or fifth [for odd $V(x)$] order. Thus, without loss of generality, one may consider only a system with polynomial shapes of potentials. We take the unperturbed potential $U(x)$ in the form of $U_4(x)$ (6) or $U_6(x)$ (7), and one of the forms

$$V_e(x) = \frac{1}{4}\beta x^4 + \frac{1}{2}\alpha x^2, \quad \text{for the even case}, \tag{15a}$$

$$V_o(x) = \frac{1}{5}\gamma x^5 + \frac{1}{3}\beta x^3 + \alpha x, \quad \text{for the odd case}, \tag{15b}$$

for the perturbing potential $V(x)$. Consequently, the inequalities (12) and (14) read

$$a + Q\alpha^2 < 0, \tag{16}$$

$$b + 4Q\alpha\beta < 0, \tag{17}$$

for even perturbation $V_e(x)$, and

$$a + 2\sqrt{qQ}\beta\lambda + 2Q\alpha\beta < 0, \tag{18}$$

$$b + 4\sqrt{qQ}\gamma\lambda + 2Q(2\alpha\gamma + \beta^2) < 0, \tag{19}$$

for odd perturbing potential $V_o(x)$. Further, for the even perturbation, Eq. (11) is given by a fifth-order polynomial which has only three solutions as far as the USC (16) is satisfied, which guarantees the bimodal form of stationary PDF. For the odd perturbation, Eq. (11) results in the seventh-order polynomial and the number of its zeros depend in a very complicated way on the system parameters. However, one can easily show, e.g., by means of the Descartes' rule of signs, that if both inequalities (18) and (19) are fulfilled, then the stationary state is also bistable. Thus, for any of the generic cases discussed here, the inequality (14) represented by (17) or (19) expresses the sufficient condition for the development of TM during the evolution from the unstable state at $x=0$.

As has been mentioned in Sec. II, TM can also appear with the opposite sign of inequality (14). For the evolution with a single additive noise (1), the theoretical analysis of this sort of TM presented in [15] has been based on the weak noise approximation and has been confirmed by numerics [15], as well as by an electronic analog experiment [16]. Unfortunately, a similar approach cannot be applied in the present problem since we consider effects caused by a noise of finite intensity Q. So, in what follows, we present only some numerical results for this case aiming to display some examples of different ways of evolution rather than to investigate this short-lived effect.

A. Uncorrelated noises

We begin the detailed analysis of the role of potential perturbation from the case of uncorrelated noises ($\lambda=0$) and even potential $V(x)=V_e(x)$. The USC (16) is fulfilled for a

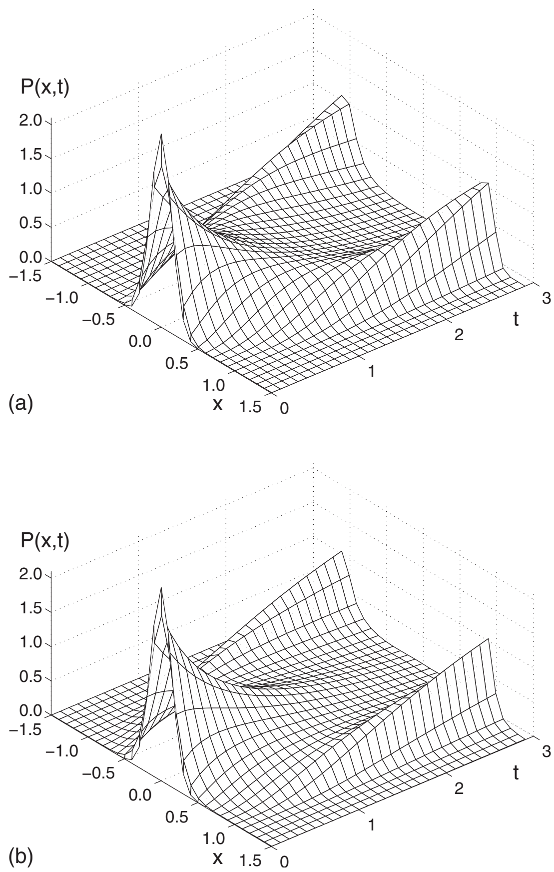

FIG. 1. Evolution in the field of potential $U_4(x)$ (6) for $a=-1$ and $b=1$ without (a) and with (b) potential perturbation which induces the TM effect. The perturbing potential has the even form (15a) with $\alpha=-1$ and $\beta=1$. The noises intensities are $q=0.02$ and $Q=0.4$, while the Gaussian initial distribution is centered at $x=0$ and has the variance 0.02.

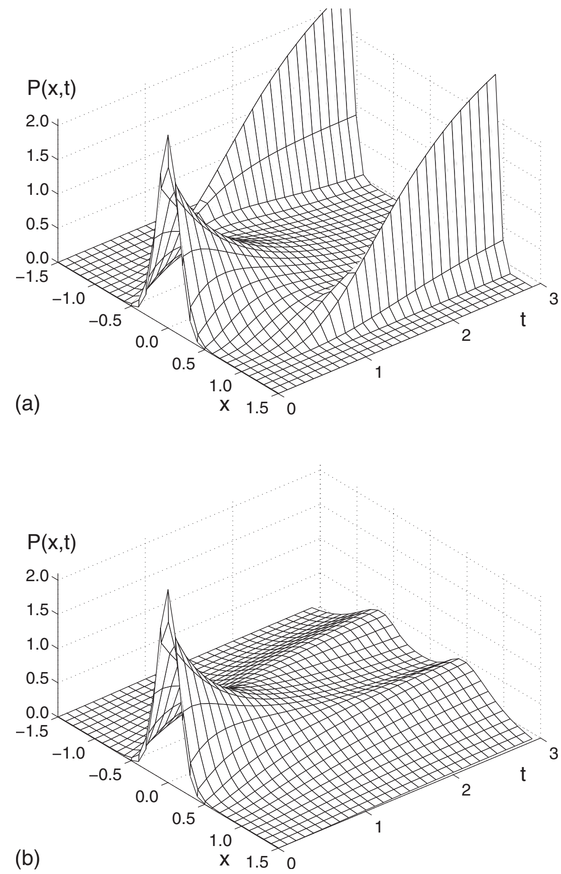

FIG. 2. Evolution in the field of potential $U_6(x)$ (7) for $a=-1$, $b=-1$, and $c=2$ without (a) and with (b) potential perturbation which suppresses the TM effect. The perturbing potential has the odd form (15b) with $\alpha=\gamma=0$, $\beta=1$, and the intensity of the perturbing noise is $Q=1$. The initial distribution and q are the same as in Fig. 1.

not too large perturbation $Q<|a|/\alpha^2$, while the TMC (17) yields that large enough potential perturbation is able to significantly modify the scenario of evolution. Thus, for some intermediate size of perturbation $1/|4b\alpha\beta|<Q\leqslant|a|/\alpha^2$, one expects either the appearance of a multihump form of PDF (for $b>0$ and $\alpha\beta<0$), or suppression of existing DTM (for $b<0$ and $\alpha\beta>0$). Since the sign of the inequality (17) depends on the product $\alpha\beta$, so for any case of transient evolution the unstability of $x=0$ can be easily preserved. In Fig. 1 we present an example of evolution in which the potential perturbation induces TM [Fig. 1(b)], while this effect is absent in the static potential $U_4(x)$ [Fig. 1(a)].

In order to observe any effect of random perturbation on the transient states, the potential $V(x)$ has to be a sufficiently complex function—both second and fourth derivatives of $V(x)$ should not vanish at $x=0$. The same signs of $V^{(2)}$ and $V^{(4)}$ lead to cancellation while opposite ones support the TM effect. In the first case, the two leading terms of perturbation act in the same way, increasing the rate of the dynamics (8). Consequently, most of the members of an ensemble of Brownian particles can leave the region of the instability around $x=0$ at a similar time moment, and soon enough, so that the fastest of them do not reach the region of the bottom of potential well. In the second case, these two leading terms act in opposition, which leads to prolongation of dwelling in the unstability region. Thus, some particles of the ensemble stay for a long time around the top of the barrier while the others, which have left this region earlier, begin to accumulate at the bottom of the well. Consequently, two new PDF humps build up while the initial one at $x=0$ still exists.

If the perturbing potential is odd $V(x)=V_o(x)$, the conditions (18) and (19) are more complex—up to three different derivatives of $V(x)$ calculated at $x=0$ can influence the way of evolution. The unstable character of $x=0$ is guaranteed if the first and third derivatives have opposite signs or one of them disappears. If their signs are the same, then the perturbing noise cannot be too intense, namely $Q<|a|/(2\alpha\beta)$. The value of the left-hand side of the inequality (19) increases for nonvanishing $V^{(3)}$ as well as for the same signs of $V^{(1)}$ and $V^{(5)}$. In these cases, TM is suppressed by the multiplicative noise, or it even disappears. The opposite signs of $V^{(1)}$ and $V^{(5)}$ cause an appearance or an increase of TM. As an example, in Fig. 2 we plot the case when DTM appears in a potential $U_6(x)$ [Fig. 2(a)], while a strong enough odd perturbation suppresses it [Fig. 2(b)].

In the last case, both the first and the third terms of $V_o(x)$ act in opposition, so, as for an even perturbation $V_e(x)$, their common effort is to slow the evolution around $x=0$. However, the existence of the second term $\beta x^3/3$ in $V_o(x)$ always increases the rate of evolution, even if β has the opposite sign to either α or γ.

B. Correlated noises

If the noises $\xi(t)$ and $\eta(t)$ are mutually correlated, then, in accordance to our assumption, $V(x)=V_o(x)$ and the previous discussion about the influence of an odd perturbation on the way of evolution should be completed by an analysis of the role of nonzero λ. In both UNC (18) and TMC (19), this correlation manifests itself through additional terms proportional to λ. In UNC this term is proportional to the value of the third, while in TMC to the value of the fifth derivative of $V(x)$ at $x=0$ [see (12) and (14)]. So the stochastic mutual dependence of additive and multiplicative noises can lead to a qualitative modification of the evolution from the unstable state only if the nonlinearity of the perturbation is of high enough degree. If $\beta\neq 0$, then the cross correlation of both noises may either increase or cancel an unstable character of $x=0$, but in any case it suppresses the TM effect. Similarly, it follows from Eq. (19) that depending on the sign of γ, either correlation ($\lambda>0$) or anticorrelation ($\lambda<0$) can either support or suppress the TM effect. This dependence of TMC on γ means that correlation between $\xi(t)$ and $\eta(t)$ becomes important for the way of evolution only far enough from the unstability. Inserting $V_o(x)$ into (8) one obtains

$$\frac{dx}{dt} = \ldots - \gamma x^4 \eta(t) + \xi(t), \tag{20}$$

where dots mean the terms irrelevant for the present discussion. The complete correlation ($\lambda=1$) or anticorrelation ($\lambda=-1$) means that $\eta(t)$ and $\xi(t)$ have the same or opposite signs, respectively. The negative sign of $\gamma\lambda$ means that the two terms on the right-hand side of (20) have the same sign so they cooperate in the dynamics. Thus far enough from $x=0$, the multiplicative noise increases the stochastic factor of evolution which enlarges the rate in which some trajectories leave the unstable region. The influence of mutual correlation of both noises on transient states is illustrated in Fig. 3, where we display the time dependence of the position of the extrema of PDF for different values of λ. When both noises are correlated strongly enough ($1\geqslant\lambda>\lambda_c$), the evolution of extrema is similar to the supercritical pitchfork bifurcation, while for weakly or anticorrelated noises ($\lambda_c>\lambda\geqslant-1$) it follows the subcritical pitchfork bifurcation scheme. For the data used the change of the way of evolution takes place for $\lambda_c\approx 0.3$.

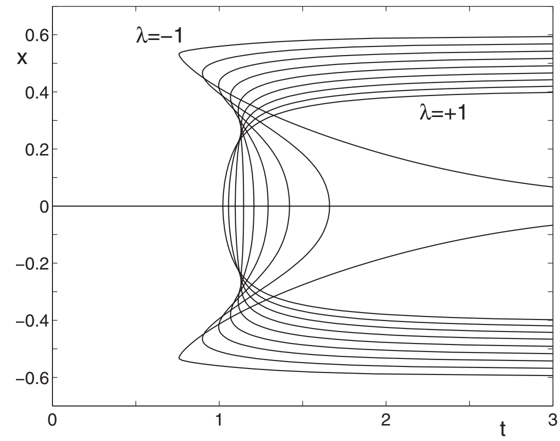

FIG. 3. Time dependence of the positions of the extremes of PDF for different values of the correlation rate $\lambda=-1.0, -0.75,\ldots,1.0$. The deterministic potential has the form $U_4(x)$ (6) with $a=-1$ and $b=1$, the perturbing one has the odd form (15b) with $\alpha=\beta=0$ and $\gamma=2$. The initial distribution and q are the same as in Fig. 1, while $Q=1$.

V. EXAMPLES

Following this general analysis, in this section we present some examples.

A. Additive noise

The simplest way of introducing a second noise $\eta(t)$ into the dynamics is to add it to the Langevin equation (1), so $V'(x)=1$ in (8). The only effect of $\eta(t)$ is a modification (an increase or decrease) of the diffusion constant $q\rightarrow G$ [see Eq. (10)]. It does not modify the position of extremes of $P_{st}(x)$ nor the sufficient condition for TM (5).

B. Linear perturbation

Very often while discussing the role of a multiplicative noise, one deals with a random perturbation linear in x, i.e. with $V(x)=1/2x^2$. This potential is an even function so we can consider only the case of uncorrelated noises. Since the extremes of $P_{st}(x)$ are given by the roots of the equation $(a+bx^2+cx^4+Q)x=0$, so the noise $\eta(t)$ can modify not only their location but also their number and character. This is a manifestation of the celebrated *noise induced transition* effect [1]. On the other hand $G^{(4)}(x)\equiv 0$, so the noise $\eta(t)$ does not influence the sign of the inequality (14) and has no impact on the TM effect.

C. Fluctuating barrier

Langevin equation (1) with a bistable form of potential $U(x)$ is a paradigm for the standard Kramers problem [28] of an escape over a potential barrier. As is well known, a stochastic perturbation of the barrier can modify the escape rate leading to the effect of resonant activation [29]. Inasmuch this effect is caused by colored perturbing noise, the correlation of two white noises can lead to giant enhancement of activation [30]. In any case, the activation problem considers an opposite way of particle movement (i.e., from the bottom to the top of the potential) than the one we are interested in here (from the top to the bottom). As a next example we consider a system used by the present author for an analysis of resonant activation [31]. Namely we assume that perturbation disturb the height of the barrier leaving its shape unchanged, i.e.,

$$V(x) = \begin{cases} U(x), & |x| \leqslant x_{min}; \\ 0, & |x| > x_{min}. \end{cases} \tag{21}$$

The limitation of $V(x)$ to the region between the minima $\pm x_{min}$ of $U(x)$ only, is insignificant for the present problem because we investigate the existence of a multihump structure of PDF in the vicinity of the top of the barrier. Since $U(x)$ is even, so is $V(x)$ and, in accordance with our assumption, we can consider only the case of uncorrelated noises $\lambda=0$. The UNC reads $a(1+aQ)<0$, so the perturbing noise cannot be too large $Q<1/|a|$. The TMC gives $b(1+4aQ)<0$. The expression on the left-hand side changes its sign when the noise intensity Q is greater than $Q_0=1/(4|a|)$. Thus, for $b<0$, a large enough noise can destroy transient multimodality, while for $b>0$ and $Q>Q_0$ the effect does exist. Let us note, however, that because TMC (14) is the sufficient condition for the appearance of TM, so this effect can exist also for some region of $Q>Q_0$ or $Q<Q_0$, respectively.

D. Laser with fluctuating pump parameter

As the last example we choose a model of laser in which the pumping parameter is subject to rapid time dependent fluctuations. The necessity of considering the pumping mechanism as a stochastic process had been identified in the beginning of the 1980s [32], while discussing some experimental results on the statistics of light emitted by a dye laser [33,34]. Since then, some hundred papers dealing with different aspects of the problem have been published significantly contributing to the theory of multiplicative noises. Leaving aside the detailed discussion about the results of the previous investigations we mention only that the crucial question of this problem concerns the process of evolution of the laser after switching it on. Its dynamics may be described by a Langevin equation for the amplitude x of the emitted light, while the switching event means a sudden change of the stability of the system at $x=0$. So we arrive just within the very topic of the present paper.

Namely, the mentioned Langevin equation reads [35]:

$$\frac{dx}{dt} = -x\left(1 - \frac{A}{1+x^2}\right) + \xi(t), \tag{22}$$

with A being the pumping parameter [36]. For $A<1$ the laser is switched off (the only stable state is $x=0$), while for $A>1$ the two stable states are $x=\pm\sqrt{A-1}$. The fraction which appears in (22) represents the so-called saturation effects in the lasing medium [37]. Close to the threshold region, i.e., for $A\approx 1$ and thus for small laser field x, this fraction may be expanded in a series yielding the simplified Langevin equation with $U_4(x)$ potential (6) with the parameters $a=1-A$ and $b=A$. If one assumes that the pump parameter is stochastic $A=A_0+\eta(t)$, then the problem reduces to the one with a linear perturbation mentioned in Sec. V B and consequently $\eta(t)$ shows no essential influence on the system evolution. On the contrary, far from the threshold the effect of $\eta(t)$ can be significant. Namely, the UNC (12) and TMC (14) give $Q<A_0-1$ and $Q>A_0/4$, respectively. Hence, one expects that for small pump fluctuations the evolution follows the Suzuki scenario, for intermediate noise $A_0/4<Q<A_0-1$ TM occurs, while for larger Q the noise induced transition in the stationary state takes place.

VI. CONCLUSION

We have discussed the way in which a multiplicative white noise can modify time development of PDF when the evolution starts from an unstable point. We have formulated the sufficient condition for the occurrence of the effect of transient multimodality and discussed different possible cases in which this noise either induces TM or demolishes it. The general conclusion from the discussion of the generic cases agrees with the explanation of TM for decaying from an unstable state without any multiplicative noise [14–16]. Namely, TM appears if the evolution proceeds in two regions of different time scales—the slow "diffusive" one in the vicinity of the unstable state, and the fast "deterministic" one apart from it. For a static potential these two regions are defined by the shape of potential $U(x)$, only. In the present problem, fast multiplicative perturbation $V(x)\eta(t)$ modifies the x-dependent force acting on the Brownian particle. The formulas (11), (12), and (14) suggest considering $\mathcal{U}(x)=U(x)+G(x)/2$ in Eq. (9) as a sort of effective potential, at least in the vicinity of the unstable state. This enables a similar interpretation of different scenarios of evolution as for the static case. Namely, it is the shape of the effective potential which constitutes two regions with different rates of evolution. The phenomenon of NITM results from the relative flattening of $\mathcal{U}(x)$ around its maximum due to the stochastic perturbation of potential. In contrary, the multiplicative noise cancels TM if it increases the steepness of the top region. In both cases the diffusion function $G(x)$ in the second term on the right-hand side of (9) is responsible for diffusion only, which, although state dependent, nevertheless is irrelevant for the qualitative picture of the evolution.

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