Crossing intervals of non-Markovian Gaussian processes

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We review the properties of time intervals between the crossings at a level M of a smooth stationary Gaussian temporal signal. The distribution of these intervals and the persistence are derived within the independent interval approximation (IIA). These results grant access to the distribution of extrema of a general Gaussian process. Exact results are obtained for the persistence exponents and the crossing interval distributions, in the limit of large $|M|$. In addition, the small-time behavior of the interval distributions and the persistence is calculated analytically, for any M. The IIA is found to reproduce most of these exact results, and its accuracy is also illustrated by extensive numerical simulations applied to non-Markovian Gaussian processes appearing in various physical contexts.

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I. INTRODUCTION

The persistence of a temporal signal X is the probability that $X(t')$ remains below (or above) a given level M, for all times $t' \in [0,t]$. This problem has elicited a large body of work by mathematicians [1–8], and physicists, both theorists [9–21] and experimentalists [22–26]. Persistence properties have been measured in systems as different as breath figures [22], liquid crystals [23], laser-polarized Xe gas [24], fluctuating steps on a Si surface [25], or soap bubbles [26].

Although the persistence is a very natural and easy quantity to define—in a sense, its physical meaning is easier to explain to the layman than the meaning of a two-point correlation function—this quantity turns out, in practice, to be extremely complex to deal with analytically. In fact, exact results have been obtained in a very limited number of cases, as far as non-Markovian processes are concerned [4,5,7].

The mathematical literature has mainly focused on evaluating the persistence

$$P_<(t) = \mathrm{Prob}\{X(t') < M,\ t' \in [0,t]\}, \tag{1}$$

mostly for Gaussian processes [i.e., processes $X(t)$ for which the joint distribution of $X(t_1),\ldots,X(t_n)$ is Gaussian] and for large $|M|$, a regime where efficient bounds or equivalents have been obtained [1,2,6]. Recently [8], and for Gaussian processes only, a numerical method to obtain valuable numerical bounds has been extended to all values of M, although the required numerical effort can become quite considerable for large t.

Physicists have also concentrated their attention on Gaussian processes [12–15,18,20], which are often a good or *exact* description of actual physical processes. For instance, the total magnetization in a spin system [16,17], the density field of diffusing particles [15], and the height profile of certain fluctuating interfaces [19,24,25] are true temporal Gaussian processes. Two general methods have been developed, focusing on the case $M=0$, which applies to many physical situations. The first one [12–14] is a perturbation of the considered process around the Markovian Gaussian process, which has been extended to small values of $|M|$ [13]. Within this method, only the large-time asymptotics of $P_<(t)$ is known, leading to the definition of the persistence exponent (see below). The alternative method, using the independent interval approximation (IIA) [15,20], gives very accurate results for "smooth" processes, that is, processes having a continuous velocity. Initially, the IIA remained restricted to the case $M=0$ [15], but it has been recently generalized to an arbitrary level M [20].

The persistence probability is also intimately related to another important physical quantity: the probability distribution of the extrema of the considered process, X. For instance, the quantity $P_<(t)$ can also be viewed as the probability that the *maximum* of X in the interval $[0,t]$ has remained below the level M. Thus, the distribution of the maximum of X is simply the derivative with respect to M of $P_<(t)$. The distribution of the extrema has been analytically obtained for the Brownian process [27,28], but its derivation remains a formidable task for general non-Markovian Gaussian processes. On this account, the persistence problem is also related to extreme value statistics [29,30], which is a notoriously difficult problem for correlated variables, and which has recently attracted a lot of attention among physicists [31].

Hence, the persistence problem has obvious applications in many other applied and experimental sciences, where one has to deal with data analysis of complex statistical signals. For instance, statistical bounds of noisy signals are extremely useful for image processing (for instance in medical imaging or astrophysics [32]), in order to obtain cleaner images by correcting spurious bright or dark pixels [1,8]. In general, it is important to be able to evaluate the maximum of a correlated temporal or spatial signal originating from experimental noise. The same question can arise when the signal lives in a more abstract space. For instance, in the context of genetic cartography, statistical methods to evaluate the maximum of a complex signal have been exploited to identify putative quantitative trait loci [33]. Finally, this same problem arises in econophysics or finance, where the probability for a generally strongly correlated financial signal to remain below or above a certain level is clearly of great interest [34].

In the present paper, we are interested in the persistence of a non-Markovian Gaussian process, which can be either

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stationary or scale invariant. More specifically, this study focuses on the properties of the distributions of time intervals during which the considered process remains below or above a given level M. We shall see that these distributions are simply related to the persistence itself, and contain valuable physical information.

We now summarize the content of the present work, and mention briefly our main results. In Sec. II, we introduce our main quantities of interest, and among them (i) the persistence $P_>(t)$ and $P_<(t)$ which measure the probability to remain above or below the level M up to time t; (ii) the associated distributions of $\pm$ time intervals, $P_\pm(t)$, during which the process remains above or below the level M. For a general level M, we then briefly review the IIA calculation introduced in [20], which leads to analytical approximate expressions for the Laplace transform of $P_\pm(t)$, $P_>(t)$, and $P_<(t)$. In fact, we present a simpler formulation compared to [20], as well as new additional results, which permit a fast and efficient numerical implementation of the IIA results. In Sec. III, we first introduce several examples of physically relevant smooth Gaussian processes, on which the general results of this paper are numerically tested. Within the IIA, we then present the calculation of the persistence exponents $\theta_\pm(M)$, defined, for stationary processes, by $P_\pm(t) \sim \exp(-\theta_\pm t)$, when $t \to +\infty$. We also obtain exact estimates for $\theta_\pm(M)$, for large $M>0$, including exact bounds for $P_>(t)$ and $\theta_+(M)$, and compute the exact asymptotic distributions $P_\pm(t)$. For large M, $P_+(t)$ takes the form of the Wigner distribution, whereas $P_-(t)$ becomes Poissonian. All these results are correctly reproduced by the IIA, except for the large-M asymptotics of $\theta_+(M)$. Finally, in Sec. IV, we obtain the exact small-time behavior of $P_\pm(t)$, $P_>(t)$, and $P_<(t)$, for "very smooth" processes (a term to be defined in the next section), and find that the IIA again reproduces these exact results. For marginally smooth processes, we also obtain exact results, but the IIA is not exact in this case, although its results remain qualitatively correct and even quantitatively accurate. Many of these results are also obtained by means of simple heuristic arguments, helping to understand them on physical grounds. This study is also illustrated by means of extensive numerical simulations, revealing a very satisfying accuracy of the IIA for moderate M, in a regime where, in contrast to the limit $M \to \pm\infty$, there are no available exact results for $P_\pm(t)$, $P_>(t)$, and $P_<(t)$.

II. INDEPENDENT INTERVAL APPROXIMATION

In this section, we introduce our main physical quantities of interest−interval distributions, persistence, sign autocorrelation, constrained densities of crossings—and relate their general properties. We then summarize the IIA calculation introduced in [20], and obtain more explicit expressions for the interval distributions and the persistence probability, in the case of a Gaussian process. These new results will prove useful in the next sections more specifically devoted to the interval distributions.

A. Introductory material and notations

One considers a stationary non-Markovian Gaussian process $X(t)$ of zero mean and unit variance. Its distribution at any time is then

$$g(X) = \frac{e^{-X^2/2}}{\sqrt{2\pi}}, \tag{2}$$

and we define its cumulative distribution as

$$G(X) = \int_{-\infty}^{X} g(x)dx = 1 - \bar{G}(X). \tag{3}$$

Due to its Gaussian nature, such a stationary process is entirely characterized by its two-point correlation function,

$$f(t) = \langle X(t+t')X(t')\rangle. \tag{4}$$

It is understood that the process starts from $t=-\infty$, so that the distribution of the initial condition $X(0)$ at $t=0$ is given by Eq. (2). In addition, all derivatives of $X(0)$, when they exists, are random Gaussian variables of zero average and second moment

$$\langle [X^{(n)}(0)]^2\rangle = (-1)^n f^{(2n)}(0), \tag{5}$$

where the superscript (n) refers to a derivative of order n.

The process is assumed to be smooth, although this constraint will be sometimes relaxed in the present paper. By smooth, we mean that the velocity of the process is a continuous function of time. Very smooth processes will have a differentiable velocity. In particular, the most general stationary *Markovian* Gaussian process defined by the equation of motion

$$\frac{dX}{dt} = -\lambda X + \sqrt{2\lambda}\,\eta \tag{6}$$

does not belong to this class of smooth processes [$\eta(t)$ is a Gaussian δ-correlated white noise]. In practice, a process is smooth if its correlator $f(t)$ is twice differentiable at $t=0$, and is very smooth if the fourth derivative of $f(t)$ exists at $t=0$. The process introduced in Eq. (6) has a correlator

$$f(t) = \exp(-\lambda|t|), \tag{7}$$

which has a cusp at $t=0$, and is thus not even differentiable at $t=0$.

Throughout this paper, we will be interested in the probability that the process X remains below or above a certain threshold M, for all times in the interval $[0,t]$. In particular, the smoothness of the process ensures that the M crossings [i.e., the times for which $X(t)=M$] are well separated, with a finite mean separation between them, denoted by τ. For the Markovian process mentioned above, the M crossings are distributed on a fractal set of dimension $1/2$, and the mean interval is then $\tau=0$. For a smooth process, τ can be computed by evaluating the mean number of M crossings during a time interval of length t,

$$N(t) = \left\langle \int_0^t |X'(t')|\,\delta(X(t')-M)dt' \right\rangle = \frac{t}{\tau}. \tag{8}$$

The correlation functions between the position X and the velocity X' is

$$\langle X(t)X'(t')\rangle = -f'(t-t'). \tag{9}$$

By time-reversal symmetry, and since $f(t)$ is twice differentiable at $t=0$, one has

$$\langle X(t)X'(t)\rangle = -f'(0) = 0, \tag{10}$$

so that the position and the velocity are uncorrelated at equal time. The distribution of the velocity $X'(t)$ is a Gaussian of zero mean and second moment

$$\langle X'^2(t)\rangle = -f''(0) = a_2, \tag{11}$$

so that

$$\langle |X'(t)|\rangle = \sqrt{\frac{2a_2}{\pi}}. \tag{12}$$

Finally, $N(t)=\langle |X'(t)|\rangle g(M)t$, which leads to

$$\tau = \frac{\pi}{\sqrt{a_2}} e^{M^2/2}. \tag{13}$$

We also define the distributions of time intervals between M crossings, $P_+(t)$ and $P_-(t)$, during which the process remains respectively above and below the level M. The means of these two kinds of intervals are defined by $\tau_\pm = \int_0^{+\infty} tP_\pm(t)dt$, and are related to τ. Indeed, since there are as many + as − intervals, since they simply alternate, one has

$$\tau = \frac{\tau_+ + \tau_-}{2}. \tag{14}$$

In addition, τ_+/τ_- is also equal to the ratio of the times spent by the process X above and below the level M, i.e., $\bar{G}(M)/G(M)$. Finally, we obtain

$$\tau_- = 2\tau G(M), \quad \tau_+ = 2\tau \bar{G}(M). \tag{15}$$

We now introduce the persistence of the process X, defined as the probability that it does not cross the level M during the time interval $[0,t]$. More precisely, we define $P_>(t)$ and $P_<(t)$ as the persistence, knowing that the process started, respectively, above and below the level M. In other word, $P_>(t)$ is also the probability that the process remains above the level M during the considered time interval, and $P_<(t)$ is the probability that X remains below the threshold M. Note that the persistence probes the entire history of the process, and is therefore an *infinite-point* correlation function: for instance $P_<(t)$ is the probability that the process remains below the level M between the times 0 and dt, dt and $2dt, \ldots,$ and $t-dt$ and t. For non-Markovian processes, it is thus understandable that this quantity is extremely difficult to treat analytically, and there are very few examples where $P_>(t)$ or $P_<(t)$ can be actually computed [2,7,8].

$P_>(t)$ and $P_<(t)$ are intimately related to the interval distributions $P_+(t)$ and $P_-(t)$ by the relations [20]

$$P_>(t) = \tau_+^{-1}\int_t^{+\infty} (t'-t)P_+(t')dt', \tag{16}$$

$$P_<(t) = \tau_-^{-1}\int_t^{+\infty} (t'-t)P_-(t')dt'. \tag{17}$$

Indeed, if $X(t)$ has remained below the level M up to time t, it belongs to a − interval of duration $t'>t$, starting at an initial position uniformly distributed between 0 and $t'-t$, an interpretation which leads to Eq. (17). The above expressions can also be differentiated twice, giving

$$P_+(t) = \tau_+ P''_>(t), \tag{18}$$

$$P_-(t) = \tau_- P''_<(t). \tag{19}$$

If $M=0$ [15], by symmetry of the process under the transformation $X\to -X$, one has $P_>(t)=P_<(t)$ and $P_+(t)=P_-(t)$. In general, we have the symmetrical relations

$$P_>(t,M) = P_<(t,-M), \quad P_+(t,M) = P_-(t,-M), \tag{20}$$

so that we will restrict ourselves to the case $M\geq 0$.

The knowledge of $P_>(t)$ and $P_<(t)$ also provides valuable information on the distribution of the extrema of the considered process. The statistical properties of extremal events are an active field of research among mathematicians [29,30] and physicists [31]. By definition, $P_<(t)$ is also the probability that the maximum of the process in a time interval of duration t is less than M, and $P_>(t)$ is the probability that the minimum of the process remains bigger than M for all times in $[0,t]$. Hence, defining $P_{\max}(M,t)$ and $P_{\min}(M,t)$ as the distribution of the maximum and minimum of the considered process, one has

$$P_{\min}(M,t) = -\frac{\partial}{\partial M}P_>(t), \tag{21}$$

$$P_{\max}(M,t) = \frac{\partial}{\partial M}P_<(t). \tag{22}$$

Note finally that Eqs. (16)–(19), (21), and (22) are in fact valid for *any stationary process*, not necessarily Gaussian.

Before presenting an approximate method leading to analytical expressions for the different quantities introduced above, we need to define two quantities which will prove useful in the following. We start with the autocorrelation function of $\theta[M-X(t)]$ [θ is Heaviside's function: $\theta(x)=1$ if $x>0$; $\theta(x)=0$ if $x<0$; $\theta(0)=1/2$],

$$A_>(t) = \langle \theta[X(t)-M]\theta[X(0)-M]\rangle, \tag{23}$$

$$A_<(t) = \langle \theta[M-X(t)]\theta[M-X(0)]\rangle = 2G(M) - 1 + A_>(t), \tag{24}$$

where the last relation is obtained by using $\theta(x)=1-\theta(-x)$. In addition, since the process is invariant under the transformation $X\to -X$, one also has

$$A_>(M,t) = A_<(-M,t). \tag{25}$$

For a Gaussian process, these quantities can be explicitly expressed in terms of the correlation function $f(t)$ [35],

$$A_>(t) = \int_M^{+\infty} g(x)\bar{G}\left(\frac{M - xf(t)}{\sqrt{1-f^2(t)}}\right)dx, \qquad (26)$$

$$A_<(t) = \int_{-\infty}^{M} g(x)G\left(\frac{M - xf(t)}{\sqrt{1-f^2(t)}}\right)dx. \qquad (27)$$

For $M=0$, these integrals can be explicitly performed [15], giving

$$A_>(t) = A_<(t) = \frac{1}{4} + \frac{1}{2\pi}\arcsin[f(t)]. \qquad (28)$$

Finally, the time derivative of $A_>(t)$ and $A_<(t)$ can be simply written as

$$A'_>(t) = A'_<(t) = \frac{1}{2\pi}\frac{f'(t)}{\sqrt{1-f^2(t)}}\exp\left(-\frac{M^2}{1+f(t)}\right). \qquad (29)$$

We now introduce $N_>(t)$ and $N_<(t)$, the mean number of M crossings in the time interval $[0,t]$, knowing that the process started from $X(0)>M$ and $X(0)<M$, respectively. These quantities satisfy the sum rule

$$G(M)N_<(t) + \bar{G}(M)N_>(t) = N(t) = \frac{t}{\tau}, \qquad (30)$$

which expresses the fact that an M crossing is crossed from either above or below M. Again, it is clear that

$$N_>(M,t) = N_<(-M,t). \qquad (31)$$

In addition, for large time t,

$$N_>(t) \sim N_<(t) \sim N(t) = \frac{t}{\tau}, \quad t \to +\infty, \qquad (32)$$

since the initial position of $X(t)$ becomes irrelevant when $t \to +\infty$. On the other hand, for short times,

$$N_>(t) \sim \frac{t}{\tau_+}, \quad N_<(t) \sim \frac{t}{\tau_-}, \quad t \to 0, \qquad (33)$$

which expresses the fact that the probability per unit time to meet the first M crossing is, respectively, τ_+^{-1} and τ_-^{-1}, for $+$ and $-$ intervals. Note that both asymptotics of Eqs. (32) and (33) are consistent with the sum rule of Eq. (30). For a Gaussian process, $N_>(t)$ and $N_<(t)$ can be calculated after introducing the correlation matrix of the Gaussian vector $(X(t),X(0),X'(t))$, which reads

$$\mathcal{C}(t) = \begin{pmatrix} 1 & f(t) & 0 \\ f(t) & 1 & f'(t) \\ 0 & f'(t) & -f''(0) \end{pmatrix}. \qquad (34)$$

For instance, in the same spirit as Eq. (8), one has

$$N_<(t) = G^{-1}(M)\int_0^t \langle|X'(t')|\rangle_< dt', \qquad (35)$$

where $\langle|X'(t)|\rangle_<$ is the average of the velocity modulus, knowing that $X(t)=M$, and averaged over $X(0)<M$:

$$\langle|X'(t)|\rangle_< = \int_{-\infty}^{M} dx_0 \int_{-\infty}^{+\infty} dv \frac{|v|e^{-\mathbf{U}^\dagger\mathcal{C}^{-1}\mathbf{U}/2}}{(2\pi)^{3/2}\sqrt{\det\mathcal{C}}}, \qquad (36)$$

where $\mathbf{U}=(M,x_0,v)$. $N_>(t)$ is similarly defined as

$$N_>(t) = \bar{G}^{-1}(M)\int_0^t \langle|X'(t')|\rangle_> dt', \qquad (37)$$

with

$$\langle|X'(t)|\rangle_> = \int_M^{+\infty} dx_0 \int_{-\infty}^{+\infty} dv \frac{|v|e^{-\mathbf{U}^\dagger\mathcal{C}^{-1}\mathbf{U}/2}}{(2\pi)^{3/2}\sqrt{\det\mathcal{C}}} \qquad (38)$$

$$= \tau^{-1} - \langle|X'(t)|\rangle_<. \qquad (39)$$

$\langle|X'(t)|\rangle_<$ can be written more explicitly,

$$\tau\langle|X'(t)|\rangle_< = G(b) + a\left(\frac{1}{2} - G(ab)\right)e^{-(M^2/2)(1-f)/(1+f)}, \qquad (40)$$

with

$$a(t) = \frac{|f'(t)|}{\sqrt{a_2(1-f^2(t))}}, \qquad (41)$$

$$b(t) = M\frac{1-f(t)}{\sqrt{1-f^2(t)-f'^2(t)/a_2}}, \qquad (42)$$

where $a_2=-f''(0)$. Using Eq. (39), one finds a similar expression for $\langle|X'(t)|\rangle_>$,

$$\tau\langle|X'(t)|\rangle_> = \bar{G}(b) - a\left(\frac{1}{2} - G(ab)\right)e^{-(M^2/2)(1-f)/(1+f)}. \qquad (43)$$

When $t\to+\infty$, $a(t)\to 0$ and $b(t)\to M$, so that [36]

$$\langle|X'(t)|\rangle_< \sim \frac{G(M)}{\tau}, \quad \langle|X'(t)|\rangle_> \sim \frac{\bar{G}(M)}{\tau}. \qquad (44)$$

Using Eqs. (35) and (37), one recovers the asymptotics of Eq. (32). On the other hand, when $t\to 0$, we have $a(t)\to 1$ and $f(t)\to 1$, which leads to

$$\langle|X'(t)|\rangle_< \sim \langle|X'(t)|\rangle_> \sim \frac{1}{2\tau}, \qquad (45)$$

and we recover Eq. (33).

B. Derivation of the IIA distributions

In the previous section, we introduced the so far unknown interval distributions $P_+(t)$ and $P_-(t)$ which are intimately related to the persistence probabilities $P_>(t)$ and $P_<(t)$ [through Eqs. (16)–(19)] and the distribution of extrema of the process [see Eqs. (21) and (22)]. On the other hand, for a Gaussian process, we have computed explicitly the autocorrelation $A_>(t)$ [and $A_<(t)$] and the constrained number of M crossings $N_>(t)$ [and $N_<(t)$] as a function of the correlator

$f(t)$ [see Eqs. (29), (40), and (43)]. We will now try to relate the two unknown interval distributions to these two known quantities.

Let us define $P_<(N,t)$ and $P_>(N,t)$ as the probabilities that there are exactly NM crossings in the interval $[0,t]$, starting, respectively, from $X(0)<M$ and $X(0)>M$. By definition of $A_>(t)$, one has

$$A_>(t) = G(M)\sum_{n=0}^{+\infty} P_>(2n,t), \tag{46}$$

$$A_<(t) = \bar{G}(M)\sum_{n=0}^{+\infty} P_<(2n,t), \tag{47}$$

since $X(t)$ is on the same side of M as $X(0)$ if and only if the number of M crossings in the interval $[0,t]$ is even. $N_>(t)$ and $N_<(t)$ can also be simply written as

$$N_>(t) = \sum_{n=0}^{+\infty} nP_>(n,t), \tag{48}$$

$$N_<(t) = \sum_{n=0}^{+\infty} nP_<(n,t). \tag{49}$$

Note that, by definition, one has $P_>(0,t)=P_>(t)$ and $P_<(0,t)=P_<(t)$.

Our central approximation now consists in assuming that the interval lengths between M crossings are uncorrelated [15,20]. This will lead to closed relation between $(P_<(N,t),P_>(N,t))$ and $P_\pm(t)$. Using Eqs. (46)–(49), we will then obtain an explicit expression of $P_\pm(t)$ as a function of $A_>(t)$ [or $A_<(t)$] and $N_>(t)$ [or $N_<(t)$].

Let us consider an odd value of $N=2n-1 (n\geq 1)$. Using the IIA, we obtain

$$\begin{aligned} P_<(2n-1,t) = \tau_-^{-1}\int_0^t dt_1 Q_-(t_1)\int_{t_1}^t dt_2 P_+(t_2-t_1) \\ \times \int_{t_2}^t dt_3 P_-(t_3-t_2)\times\cdots \\ \times \int_{t_{2n-3}}^t dt_{2n-2}P_+(t_{2n-2}-t_{2n-3}) \\ \times \int_{t_{2n-2}}^t dt_{2n-1}P_-(t_{2n-1}-t_{2n-2})Q_+(t-t_{2n-1}), \end{aligned} \tag{50}$$

where

$$Q_\pm(t) = \int_t^{+\infty} P_\pm(t')dt'. \tag{51}$$

is the probability that a $\pm$ interval is larger than t. Equation (50) expresses the fact that to find $2n-1$ crossings between 0 and t starting from $X(0)<M$, we should find a first crossing at t_1 [and hence an initial—interval of length bigger than t_1, with probability $\tau_-^{-1}Q_-(t_1)$], followed by a + interval of length t_2-t_1, and so on, up to a last crossing time t_{2n-1}, associated with a − interval of length $t_{2n-1}-t_{2n-2}$. Finally, there should not be any further crossing between t_{2n-1} and t; hence the last factor $Q_+(t-t_{2n-1})$. All these events have been treated as independent, so that there probabilities simply factor; this is the core assumption of the IIA. For even $N=2n$ $(n\geq 1)$, one obtains a similar expression,

$$\begin{aligned} P_<(2n,t) = \tau_-^{-1}\int_0^t dt_1 Q_-(t_1)\int_{t_1}^t dt_2 P_+(t_2-t_1)\int_{t_2}^t dt_3 P_-(t_3-t_2) \\ \times\cdots\times\int_{t_{2n-2}}^t dt_{2n-1}P_-(t_{2n-1}-t_{2n-2}) \\ \times\int_{t_{2n-1}}^t dt_{2n}P_+(t_{2n}-t_{2n-1})Q_-(t-t_{2n}). \end{aligned} \tag{52}$$

For a given function of time $F(t)$, one defines its Laplace transform $\hat{F}(s)=\int_0^{+\infty}F(t)e^{-st}dt$. The convolution products in Eqs. (50) and (52) take a much simpler form in the Laplace variable s:

$$\hat{P}_<(2n-1,s) = \tau_-^{-1}\hat{Q}_+\hat{Q}_-(\hat{P}_+\hat{P}_-)^{n-1}, \tag{53}$$

$$\hat{P}_<(2n,s) = \tau_-^{-1}\hat{Q}_-^2 P_+(\hat{P}_+\hat{P}_-)^{n-1}, \tag{54}$$

where

$$\hat{Q}_\pm(s) = \frac{1-\hat{P}_\pm(s)}{s} \tag{55}$$

is the Laplace transform of Eq. (51). If we express the conservation of probability,

$$P_<(t) + \sum_{N=1}^{+\infty} P_<(N,t) = 1, \tag{56}$$

after summing simple geometric series, we obtain

$$\hat{P}_<(s) = \frac{1}{s} - \frac{1-\hat{P}_-(s)}{\tau_- s^2}. \tag{57}$$

This relation is nothing but the Laplace transform of Eq. (17). It is certainly satisfying, and also reassuring, that the IIA reproduces this exact relation, as well as the equivalent relation between $P_>(t)$ and $P_+(t)$ of Eq. (16). Of course, $P_>(2n-1,t)$ and $P_>(2n,t)$ satisfy similar equations as Eqs. (50), (52)–(54), (56), and (57), obtained by exchanging the indices $+\leftrightarrow-$ and $<\leftrightarrow>$.

Using Eqs. (46)–(49), we can now write explicitly the Laplace transform of the known quantities $A_<(t)$ and $N_<(t)$ in terms of the Laplace transform of $P_\pm(t)$:

$$\hat{N}_<(s) = \frac{(1+\hat{P}_+)(1-\hat{P}_-)}{\tau_- s^2(1-\hat{P}_+\hat{P}_-)}, \tag{58}$$

$$\hat{A}_<(s) = G(M)\left(\frac{1}{s} - \frac{1-\hat{P}_+}{1+\hat{P}_+}\hat{N}_<(s)\right). \tag{59}$$

Again, $A_>(t)$ and $N_>(t)$ are given by similar expressions, after the substitution $+\leftrightarrow-$ and $<\leftrightarrow>$, and $G(M)\leftrightarrow\bar{G}(M)$.

Using $\hat{P}'_\pm(0)=-\int_0^{+\infty} tP_\pm(t)dt=-\tau_\pm$ and Eq. (15), one obtains the following estimates when $s\to 0$:

$$\hat{N}_<(s) \sim \frac{1}{\tau s^2}, \quad \hat{A}_<(s) \sim \frac{G^2(M)}{s}. \tag{60}$$

The first expression in Eq. (60) is equivalent to the general result of Eq. (32), whereas the second relation expresses that, for large t, $A_<(t)\sim G^2(M)$. For large s,

$$\hat{N}_<(s) \sim \frac{1}{\tau_- s^2}, \quad \hat{A}_<(s) \sim \frac{G(M)}{s}, \tag{61}$$

which corresponds to the small-time behavior of Eq. (33) for $N_<(t)$, whereas the second relation is equivalent to $A_<(0)=G(M)$.

Finally, writing

$$\hat{F}_<(s) = \frac{G(M) - s\hat{A}_<(s)}{G(M)s\hat{N}_<(s)}, \tag{62}$$

and using Eqs. (58) and (59), the interval distributions are given by

$$\hat{P}_+(s) = \frac{1-\hat{F}_<(s)}{1+\hat{F}_<(s)}, \tag{63}$$

$$\hat{P}_-(s) = \frac{2-\tau_- s^2\hat{N}_<(s)[1+\hat{F}_<(s)]}{2-\tau_- s^2\hat{N}_<(s)[1-\hat{F}_<(s)]}. \tag{64}$$

Inserting these expressions of $\hat{P}_\pm(s)$ in Eq. (57), one obtains $\hat{P}_<(s)$ [and $\hat{P}_>(s)$] from the sole knowledge of $A_<(t)$ and $N_<(t)$ (or their Laplace transforms), which are known explicitly for a Gaussian process. Alternative expressions for $\hat{P}_\pm$ in terms of the Laplace transform of $A_>(t)$ and $N_>(t)$ are readily obtained after the substitution $+\leftrightarrow-$ and $<\leftrightarrow>$, and $G(M)\leftrightarrow\bar{G}(M)$ in Eqs. (60)–(64). Finally, due to the symmetry of the process under the transformation $X\to -X$, the following symmetric relations hold:

$$P_\pm(-M,t) = P_\mp(M,t), \quad P_>(-M,t) = P_<(M,t). \tag{65}$$

Using Eq. (29), we find that the dimensionless function $W(t)$,

$$W(t) = -\tau A'_>(t) = -\tau A'_<(t)$$
$$= -\frac{\tau}{2\pi}\frac{f'(t)}{\sqrt{1-f^2(t)}}\exp\left(-\frac{M^2}{1+f(t)}\right) \tag{66}$$

$$= \frac{a}{2}e^{-(M^2/2)[(1-f)/(1+f)]}, \tag{67}$$

has a simpler analytical form than $A_>(t)$ or $A_<(t)$. Similarly, we define the dimensionless auxiliary function $V(t)$ by

$$\tau\bar{G}(M)N'_>(t) = \tau\langle|X'(t)|\rangle_> = \bar{G}(M) + V(t), \tag{68}$$

$$\tau G(M)N'_<(t) = \tau\langle|X'(t)|\rangle_< = G(M) - V(t). \tag{69}$$

Using Eq. (40), we find that $V(t)$ takes an explicit form in contrast to $N_>(t)$ or $N_<(t)$:

$$V(t) = G(M) - G(b) + a\left(G(ab) - \frac{1}{2}\right)e^{-(M^2/2)(1-f)/(1+f)}, \tag{70}$$

where $a(t)$ and $b(t)$ are simple functionals of the correlator $f(t)$, which have been defined in Eqs. (41) and (42). We give below the behavior of $W(t)$ and $V(t)$, in the limit $t\to+\infty$,

$$V(t) \sim \frac{M}{\sqrt{2\pi}}e^{-M^2/2}f(t), \tag{71}$$

$$W(t) \sim -\frac{1}{2\sqrt{a_2}}e^{-M^2/2}f'(t), \tag{72}$$

while one has $V(0)=G(M)-1/2$ and $W(0)=1/2$.

The Laplace transform of $W(t)$ and $V(t)$ can be explicitly written as

$$\tau^{-1}\hat{W}(s) = G(M) - s\hat{A}_<(s) = \bar{G}(M) - s\hat{A}_>(s), \tag{73}$$

and

$$\tau\bar{G}(M)s\hat{N}_>(s) = \frac{\bar{G}(M)}{s} + \hat{V}(s), \tag{74}$$

$$\tau G(M)s\hat{N}_<(s) = \frac{G(M)}{s} - \hat{V}(s). \tag{75}$$

In terms of $\hat{W}(s)$ and $\hat{V}(s)$, the interval distributions take the symmetric form

$$\hat{P}_+(s) = \frac{G(M) - s\hat{V}(s) - s\hat{W}(s)}{G(M) - s\hat{V}(s) + s\hat{W}(s)}, \tag{76}$$

$$\hat{P}_-(s) = \frac{\bar{G}(M) + s\hat{V}(s) - s\hat{W}(s)}{\bar{G}(M) + s\hat{V}(s) + s\hat{W}(s)}. \tag{77}$$

In practice, the explicit forms of $W(t)$ and $V(t)$ obtained in Eqs. (67) and (70) permit a fast and efficient numerical implementation of the IIA.

Note that for a smooth *non-Gaussian* process, all the above results of the IIA remain unaltered, $G(M)$ now being the cumulative sum of the associated distribution of X, and τ being given by the general form [20]

$$\tau^{-1} = g(M)\langle|X'(t)|\rangle, \tag{78}$$

whereas $\tau_\pm$ are still given by Eq. (15). Applying the IIA results to a general non-Gaussian process requires only the prior knowledge of $A_<(t)$ and $N_<(t)$ [or $A_>(t)$ and $N_>(t)$]. In general, these time-dependent functions should be given *a priori*, analytically, or extracted from numerical or experimental data.

Finally, let us briefly address the validity of the IIA. First, crossing intervals are strictly *never* independent, except in the particular case of a Markovian process [see Eq. (6)], for which the IIA does not apply, due to the singular nature of this process. The IIA is also an uncontrolled approximation which seems almost impossible to improve systematically, by introducing interval correlations. However, in practice, the IIA is found numerically to be a surprisingly good approximation, especially for "very smooth" processes for which $f(t)$ is analytic [15,20] [see the counterexample of $f_3(t)$ in Eq. (90) below]. We will even show in the next sections that some of the predictions of the IIA are in fact exact for smooth Gaussian processes.

III. PERSISTENCE EXPONENTS

A. General properties and physical applications

In many contexts, one is interested in the large-time behavior of the persistence probabilities $P_<(t)$ and $P_>(t)$. It has been rigorously established that, if $|f(t)| \leq C/t$, for sufficiently large time t (C is some arbitrary constant), then the persistence decays exponentially [6]. Hence, we define the two persistence exponents, by the asymptotics

$$P_<(t) \sim e^{-\theta_- t}, \quad P_>(t) \sim e^{-\theta_+ t}, \tag{79}$$

valid when $t \to +\infty$. Due to the symmetry relation of Eq. (65), we have

$$\theta_\pm(-M) = \theta_\mp(M). \tag{80}$$

Hence, from now on, we consider only the case $M \geq 0$. From Eqs. (16)–(19), we find that the interval distributions $P_\pm(t)$ decays in the same way as their associated persistence, for $t \to +\infty$.

The name persistence "exponent" (instead of "decay rate") arises from its numerous applications in out-of-equilibrium physics [9–26]. Indeed, in many cases, the normalized two-times correlation function of the relevant physical variable $Z(T)$ obeys *dynamical scaling*,

$$\frac{\langle Z(T)Z(T')\rangle}{\sqrt{\langle Z^2(T)\rangle\langle Z^2(T)\rangle}} = F(T/T'), \tag{81}$$

where T is the physical time, and F is the scaling correlation function. Defining

$$t = \ln T, \quad X(t) = \frac{Z(T)}{\sqrt{\langle Z^2(T)\rangle}}, \tag{82}$$

the resulting process $X(t)$ becomes *stationary* in the new effective time t [12,13], with correlator

$$f(t) = F(\exp t). \tag{83}$$

Hence, the persistence $P^X_>(M,t)$ for the process $X(t)$ is equal to the probability $P^Z_>(M,T)$ that the process $Z(T)$ remains above the level $M\sqrt{\langle Z^2(T)\rangle}$ up to time $T = \exp t$ [12,18]. Since $P^X_>(M,t)$ decays exponentially, the persistence of the process $Z(T)$ decays as a power law, hence the name persistence "exponent,"

$$P^Z_>(M,T) = P^X_>(M,t) \sim e^{-\theta_+ t} \sim T^{-\theta_+}. \tag{84}$$

Similarly, one has

$$P^Z_<(M,T) = P^X_<(M,t) \sim e^{-\theta_- t} \sim T^{-\theta_-}. \tag{85}$$

In particular, for $M=0$, the persistences of the processes X and Z are both equal to the probability that the associated process does not change sign up to the time $t = \ln T$.

In order to illustrate the dynamical scaling of the correlator resulting from Eq. (81), let us give three physical examples. In the next sections, our analytical results will be tested on the correlators $f_1(t)$, $f_2(t)$, and $f_3(t)$, introduced below in Eqs. (87)–(90).

(1) Consider a d-dimensional ferromagnetic system (for instance, modeled by the Ising model) quenched from its equilibrium state above the critical temperature $\mathcal{T}_c$, down to $\mathcal{T}_c$ (critical quench) or below $\mathcal{T}_c$ (subcritical quench). As time T proceeds, correlated domains of linear size $L(T) \sim T^{1/z}$ grow, and this coarsening dynamics leads to dynamical scaling for the total magnetization, of the form Eq. (81). Initially the spins have only short-range spatial correlations, and as the domains grow, the correlation length remains finite, of order $L(T)$. If $L(T)$ remains much smaller than the linear size of the system, the law of large numbers ensures that the magnetization $Z(T)$, which is the sum of the individual spins, is a true Gaussian variable. For $M=0$, the persistence is equivalent to the probability that the magnetization never changes sign from the time of the quench ($T=0$), up to time T. For a critical quench, the persistence decays as a power law with a persistence exponent θ_c, which is a universal critical exponent of spin systems, independent of the familiar ones ($\beta, \eta, z, \ldots$), due to the non-Markovian nature of the magnetization [14,16]. For a subcritical quench, the magnetization persistence also decays as a power law with a universal d-dependent persistence exponent controlled by a zero-temperature fixed point [17], and the dynamical exponent is $z=2$.

(2) If the field $Z(\mathbf{x},T)$ evolves according to the d-dimensional diffusion equation (or more sophisticated interface model equations [19])

$$\frac{\partial Z}{\partial T} = \nabla^2_{\mathbf{x}} Z, \tag{86}$$

starting from an initial random configuration of zero mean, the process becomes Gaussian for large times, another consequence of the law of large numbers. For any fixed $\mathbf{x}$, the normalized two-time correlator of $Z(\mathbf{x},T)$ obeys dynamical scaling, and the probability that $Z(\mathbf{x},T)$ does not change sign decays as a power law with a d-dependent exponent computed approximately in [15]. The associated stationary correlator in the variable $t = \ln T$ is

$$f_1(t) = \frac{1}{\cosh^{d/2}(t/2)}. \tag{87}$$

Moreover, in $d=1$ and at a fixed time T, the process $Z(x,T)$ is a stationary Gaussian process in the spatial variable x. The variable $X(x) = Z(x,T)/\sqrt{\langle Z^2(x,T)\rangle_x}$ (where $\langle\cdot\rangle_x$ denotes the average over the spatial variable x) has a Gaussian correlator.

Hence, we shall later illustrate our results using the correlator

$$f_2(t) = e^{-t^2/2}. \quad (88)$$

(3) The random acceleration process [7] is defined by its equation of motion

$$\frac{d^2Z}{dT^2} = \eta(T), \quad (89)$$

where $\eta(T)$ is a δ-correlated white noise. Again, its two-time correlator obeys dynamical scaling, and the associated stationary correlator is [16]

$$f_3(t) = \frac{3}{2}e^{-|t|/2} - \frac{1}{2}e^{-3|t|/2}. \quad (90)$$

For $M=0$, this process is a rare case for which the exact value of the persistence exponent is known exactly [7],

$$\theta_\pm(M=0) = \frac{1}{4}. \quad (91)$$

Note that for the random acceleration process, the correlator $f_3(t)$ is not analytic. Although, twice differentiable at $t=0$, its third derivative is not defined at $t=0$. For this process, it is not surprising to find that the IIA is not as precise as for smoother processes [15,20]. Indeed, one finds the IIA result $\theta_\pm(M=0)=0.2647\ldots$, off by 6% compared to Eq. (91), a relative error much bigger than usually observed for $M=0$ persistence exponents obtained by means of the IIA.

B. Persistence exponents within the IIA

Within the IIA, the persistence exponents $\theta_\pm$ are obtained as the first pole on the negative real axis of the Laplace transform of the associated interval distribution $P_\pm(t)$, since the Laplace transform of $\exp(-\theta_\pm t)$ is $1/(s+\theta_\pm)$. Using Eqs. (63) and (64), we find that $\theta_\pm$ satisfies

$$G(M)[1 + \theta_- \hat{N}_<(-\theta_-)] + \theta_- \hat{A}_<(-\theta_-) = \frac{1}{\tau\theta_-}, \quad (92)$$

$$G(M)[1 - \theta_+ \hat{N}_<(-\theta_+)] + \theta_+ \hat{A}_<(-\theta_+) = 0, \quad (93)$$

or the equivalent relations in terms of $\hat{A}_>$ and $\hat{N}_>$,

$$\bar{G}(M)[1 + \theta_+ \hat{N}_>(-\theta_+)] + \theta_+ \hat{A}_>(-\theta_+) = \frac{1}{\tau\theta_+}, \quad (94)$$

$$\bar{G}(M)[1 - \theta_- \hat{N}_>(-\theta_-)] + \theta_- \hat{A}_>(-\theta_-) = 0. \quad (95)$$

In terms of the auxiliary functions $V(t)$ and $W(t)$ introduced in Eqs. (66), (68), and (69), the defining equations of $\theta_\pm$ take a simpler form

$$\hat{W}(-\theta_+) - \hat{V}(-\theta_+) = \frac{G(M)}{\theta_+}, \quad (96)$$

$$\hat{W}(-\theta_-) + \hat{V}(-\theta_-) = \frac{\bar{G}(M)}{\theta_-}. \quad (97)$$

The residues $R_\pm$ associated with $\theta_\pm$, and defined by

$$P_\pm(t) \sim R_\pm e^{-\theta_\pm t}, \quad (98)$$

can easily be extracted from Eqs. (76) and (77), using the identity

$$R_\pm^{-1} = \frac{d\hat{P}_\pm^{-1}}{ds}(s=-\theta_\pm). \quad (99)$$

Before addressing the limit $M \to \pm\infty$ in the next section, we present some numerical results for moderate M, and for the processes associated with the correlators $f_1(t)$, $f_2(t)$, and $f_3(t)$, introduced in Eqs. (87)–(90). We recall the symmetry relation $\theta_\pm(-M)=\theta_\mp(M)$, so that we restrict ourselves to the case $M \geqslant 0$. In order to compare the values of $\theta_\pm$ for the different correlators, it is instructive to multiply the persistence exponents (of dimension $[t]^{-1}$) by the time scale τ_0, which is the mean crossing time interval for $M=0$,

$$\tau_0 = \tau(M=0) = \frac{\pi}{\sqrt{a_2}}. \quad (100)$$

We have performed extensive numerical simulations of the processes associated with the correlators $f_1(t)$, $f_2(t)$, and $f_3(t)$, and measured the persistence and the crossing interval distributions, and in particular, the persistence exponents. Let us briefly describe how to generate long trajectories of a stationary Gaussian process solely characterized by its two-time correlator [12]. In real time, the most general form of such a process X reads

$$X(t) = \int_{-\infty}^{t} J(t-t')\,\eta(t')dt', \quad (101)$$

where $\eta(t)$ is a δ-correlated Gaussian white noise. The Gaussian nature of $\eta(t)$ and the linear form of Eq. (101) ensure that $X(t)$ is a Gaussian process. Moreover, the convolution product of the noise $\eta(t')$ with the kernel $J(t-t')$ [instead of a general kernel $J(t,t')$] ensures stationarity. Taking the Fourier transform of Eq. (101), we obtain

$$\tilde{X}(\omega) = \tilde{J}(\omega)\,\tilde{\eta}(\omega), \quad (102)$$

where $\tilde{X}(\omega)=\int_{-\infty}^{+\infty}X(t)\exp(-i\omega t)dt$, and the noise Fourier transform satisfies $\langle\tilde{\eta}(\omega)\tilde{\eta}(\omega')\rangle=2\pi\delta(\omega+\omega')$. The Fourier transform of the correlator of X is hence

$$\frac{\langle\tilde{X}(\omega)\tilde{X}(\omega')\rangle}{2\pi} = \tilde{f}(\omega)\,\delta(\omega+\omega') = |\tilde{J}(\omega)|^2\delta(\omega+\omega'), \quad (103)$$

which relates the kernel $J(t)$ to the correlation function $f(t)$, through their Fourier transform, $\tilde{f}(\omega)=|\tilde{J}(\omega)|^2$. Note that a necessary condition for $f(t)$ to be the correlator of a Gaussian process is that its Fourier transform $\tilde{f}(\omega)$ remains positive for all real frequencies ω. Finally, a trajectory of X is obtained after sampling

$$\tilde{X}(\omega) = \sqrt{\tilde{f}(\omega)}\,\tilde{\eta}(\omega) \quad (104)$$

on a frequency mesh, and performing an inverse fast Fourier transform of the obtained $\tilde{X}(\omega)$. The Fourier transform of the

TABLE I. Values of $\tau_0\theta_-(M)$ as obtained from the IIA calculation (θ_-^{IIA}) and simulations (θ_-^{sim}), for different values of M, calculated for the processes associated with the correlators $f_1(t)$ ($\tau_0 = 2\pi$) and $f_2(t)$ ($\tau_0 = \pi$), introduced in Eqs. (87) and (88).

M	$\tau_0\theta_{-,1}^{\mathrm{IIA}}$	$\tau_0\theta_{-,1}^{\mathrm{sim}}$	$\tau_0\theta_{-,2}^{\mathrm{IIA}}$	$\tau_0\theta_{-,2}^{\mathrm{sim}}$
0	1.1700	1.178(2)	1.2928	1.330(5)
1/2	0.6949	0.7008(7)	0.7587	0.7723(8)
1	0.3715	0.3743(6)	0.3994	0.4032(8)
3/2	0.1734	0.1750(4)	0.1831	0.1850(4)
2	6.813×10^{-2}	$6.834(6)\times 10^{-2}$	7.065×10^{-2}	$7.116(7)\times 10^{-2}$
5/2	2.177×10^{-2}	$2.180(3)\times 10^{-2}$	2.224×10^{-2}	$2.225(3)\times 10^{-2}$
3	5.509×10^{-3}	$5.510(2)\times 10^{-3}$	5.568×10^{-3}	$5.568(2)\times 10^{-3}$

TABLE II. Values of $\tau_0\theta_+(M)$ as obtained from the IIA calculation (θ_+^{IIA}) and simulations (θ_+^{sim}), for different values of M, calculated for the processes associated with the correlators $f_1(t)$ ($\tau_0 = 2\pi$) and $f_2(t)$ ($\tau_0 = \pi$), introduced in Eqs. (87) and (88).

M	$\tau_0\theta_{+,1}^{\mathrm{IIA}}$	$\tau_0\theta_{+,1}^{\mathrm{sim}}$	$\tau_0\theta_{+,2}^{\mathrm{IIA}}$	$\tau_0\theta_{+,2}^{\mathrm{sim}}$
0	1.1700	1.178(2)	1.2928	1.330(5)
1/2	1.8164	1.865(6)	2.0232	2.125(7)
1	2.6475	2.67(2)	2.9606	3.11(2)
3/2	3.6677	3.74(3)	4.1031	4.36(4)
2	4.8926	5.06(5)	5.4338	5.90(7)
5/2	6.2651	6.6(1)	6.9152	7.6(2)

correlators $f_1(t)$, $f_2(t)$, and $f_3(t)$ having simple explicit expressions, this procedure for obtaining long trajectories is extremely efficient. Otherwise, one has to tabulate the Fourier transform of $f(t)$, before simulating Eq. (104).

In practice, our numerical results are obtained after averaging $10^6 - 10^7$ trajectories of length $\mathcal{T} = 1024\tau_0$ (sometimes $2048\tau_0$) and with a frequency mesh of typically 1024^2 points spaced by $\Delta\omega = \mathcal{T}^{-1}$. Since for a general M, $\tau = \tau_0 \exp(M^2/2)$, we obtain typically $(10^9 - 10^{10}) \times \exp(-M^2/2)$ M crossings, a number that decays rapidly for large M. Despite the loss of statistics for large positive M, we find that $\theta_-(M)$ can still be measured with great accuracy, since $P_-(t)$ and $P_<(t)$ becomes purely Poissonian in this limit (see next section). On the other hand, the error bars for $\theta_+(M)$ increase rapidly with M, due to the occurrence of fewer M crossings. In addition, the determination of $\theta_+(M)$ is also plagued by the fact that the exponential asymptotics of $P_+(t)$ and $P_>(t)$ develops only for increasingly large t, as M increases, which may produce uncontrolled systematic errors in the numerical estimates of $\theta_+(M)$ (see the next section).

Table I compares $\tau_0\theta_-(M)$ for the correlators $f_1(t)$ (for $d=2$) and $f_2(t)$, as obtained from numerical simulations and from the IIA calculation of Eq. (97). The agreement between the theory and the simulations is excellent for both correlators, and is even improving as M increases. In fact, we will show in the next section that the IIA becomes exact for $\theta_-(M)$ when $M \to \infty$, and we will obtain an analytic asymptotic expression for $\theta_-(M)$. Our theoretical and numerical results are also consistent with the numerical bounds computed in [8] for the correlator $f_1(t)$, for $M=1$ and $M=2$:

$$0.3681 < \tau_0\theta_{-,1}(M=1) < 0.4298, \qquad (105)$$

$$0.0666 < \tau_0\theta_{-,1}(M=2) < 0.0748. \qquad (106)$$

It is also clear that the IIA provides much better estimates of the persistence exponent $\theta_-(M)$ than these bounds, which are, however, exact, although they require a much bigger numerical effort than the IIA [8].

Table II compares $\tau_0\theta_+(M)$ for the correlators $f_1(t)$ and $f_2(t)$, as obtained from numerical simulations and from the IIA calculation of Eq. (96). The agreement between the theory and the simulations is satisfying, although we will show that the IIA ultimately fails in predicting the exact asymptotics of $\theta_+(M)$ as $M \to \infty$, although the limiting form of $P_+(t)$ will be given exactly by the IIA for t not too large.

For the process associated with the correlator $f_3(t)$ ($\tau_0 = 2\pi/\sqrt{3}$), $\tau_0\theta_\pm^{\mathrm{IIA}}(M=0) = 0.9602\ldots$, compared to the exact value $\tau_0\theta_\pm(M=0) = \pi/2\sqrt{3} = 0.9069\ldots$ obtained in [7]. The agreement between the IIA and numerical estimates of $\theta_-(M)$ greatly improves as M increases, as observed for the two other correlators in Table I, whereas $\theta_+(M)$ is only fairly reproduced for large M (as already observed in Table II), suggesting that the IIA somewhat fails to reproduces $\theta_+(M)$ in the limit $M \to +\infty$, as will be confirmed in the next section. Finally, we note that for the three processes considered here (and all other smooth Gaussian processes known to the author), one has $\theta_\pm(M=0) \sim \tau_0^{-1}$, with a proportionality constant close to unity.

C. Exact results in the limit $M \to \pm\infty$

In the limit $M \to +\infty$, and using

$$\bar{G}(M) \sim \frac{e^{-M^2/2}}{\sqrt{2\pi}M} \qquad (107)$$

and Eqs. (13) and (15), the means of the ± intervals are

$$\tau_- \sim \frac{2\pi}{\sqrt{a_2}} e^{M^2/2} \sim 2\tau, \qquad (108)$$

$$\tau_+ \sim \sqrt{\frac{2\pi}{a_2}} M^{-1}. \qquad (109)$$

Hence, as expected physically, the typical length of the − intervals is becoming increasingly large as $M \to +\infty$ (of order 2τ), whereas the typical length of the + is going slowly to zero.

1. Distribution of − intervals and θ_-

For large M, and using Eqs. (67) and (70), we obtain that

$$V(t) = W(t)[1 + O(e^{-M^2/2})]. \qquad (110)$$

Moreover, in this limit both functions can be approximated by developing $f(t)$ up to second order in t,

$$V(t) \sim W(t) \sim \frac{1}{2}e^{-M^2a_2t^2/8} = \frac{1}{2}e^{-(\pi/4)(t/\tau_+)^2}. \quad (111)$$

The constraint that

$$\tau^{-1}\int_0^{+\infty} W(t)dt = A_<(0) - A_<(+\infty) \quad (112)$$

$$= A_>(0) - A_>(+\infty) \quad (113)$$

$$= G(M)\bar{G}(M) \quad (114)$$

$$\sim \frac{e^{-M^2/2}}{\sqrt{2\pi}M} \quad (115)$$

is consistently recovered up to leading order, after integrating Eq. (111). Moreover, the explicit form of Eq. (111) implies that the Laplace transform of $W(t)$ can be approximated by its value at $s=0$, provided that

$$|s| \ll \tau_+^{-1}, \quad M \gg 1. \quad (116)$$

Finally, using Eqs. (15) and (77), we find that the Laplace transform of $P_-(t)$ is given by

$$\hat{P}_-(s) = \frac{1}{1+s\tau_-}, \quad (117)$$

under the conditions of Eq. (116). Hence, we conclude that for large M one has

$$\theta_-(M) \sim \frac{1}{\tau_-} \sim \frac{1}{2\tau}, \quad (118)$$

and, taking the inverse Laplace transform of Eq. (117), that the distribution of − intervals is essentially Poissonian,

$$P_-(t) = \tau_-^{-1}e^{-t/\tau_-}, \quad P_<(t) = e^{-t/\tau_-}, \quad (119)$$

except in a narrow region of time $0 \leq t \leq \tau_+ \ll \tau_-$, corresponding to the conjugate time domain of the condition of Eq. (116). The behavior of $P_-(t)$ and $P_<(t)$, for $t \ll \tau_+$, will be obtained exactly in Sec. IV. The fact that the distribution of the long − intervals is becoming Poissonian can be physically interpreted. Indeed, since $\tau_- \to +\infty$ when $M \to +\infty$, the process X can be considered to be Markovian at this time scale. In addition, this also shows that for the − intervals the IIA is in fact exact. In the next section, in the process of finding exact bounds for $\theta_+(M)$, we will prove the exact result, valid in the opposite $M \to 0$ limit,

$$\theta_-(M) = \theta_0 - \frac{\langle X(t)\rangle_{X>0}}{2\hat{f}(0)}M + O(M^2), \quad (120)$$

where $\theta_0 = \theta_-(M=0)$, $\langle X(t)\rangle_{X>0}$ is the average of X over all trajectories for which $X(t) > 0$ for all times, and $\hat{f}(0) = \int_0^{+\infty} f(t)dt$.

In Fig. 1, we plot $\theta_-(M)$ obtained by simulating the process associated with the correlator $f_2(t)$ of Eq. (88). We also plot the IIA result of Eq. (97), which is in perfect agreement with numerical simulations. In addition, we illustrate the

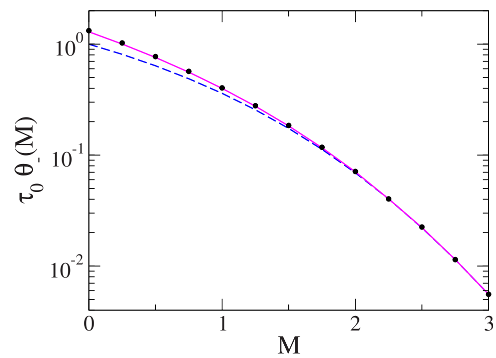

FIG. 1. (Color online) $\theta_-(M)$ (in unit of τ_0^{-1}), for the non-Markovian process associated with the correlator $f_2(t)$ of Eq. (88). Symbols are the results of numerical simulations [see text and Eq. (104)], while the full line is the result of the IIA approximation, Eq. (97). Finally, the dashed line corresponds to the exact asymptotic result, $\theta_-(M) \sim \tau_-^{-1}$. Some values of $\theta_-(M)$ are also reported in Table I.

rapid convergence of $\theta_-(M)$ to the exact asymptotics of Eq. (118). In Fig. 2, we plot $P_-(t)$ for $M=3$, which has already perfectly converged to its asymptotic Poissonian form of Eq. (119).

2. Distribution of + intervals and θ_+

For $M \to +\infty$, and using the asymptotics Eq. (110) and the IIA expression for $\hat{P}_+(s)$ of Eq. (76), we obtain

$$\hat{P}_+(s) = 1 - 2s\hat{W}(s). \quad (121)$$

In real time, Eq. (121) reads

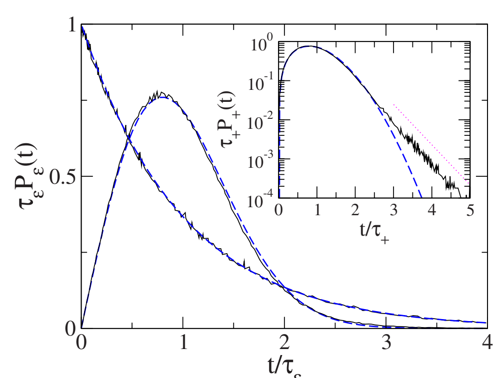

FIG. 2. (Color online) Distribution of $\varepsilon = \pm$ intervals, $P_\pm(t)$, for $M=3$, as a function of $t/\tau_\pm$ (full lines for $M=3$, $\tau_+ \approx 0.763\,50a_2^{-1/2}$, and $\tau_- \approx 564.83a_2^{-1/2}$), for the process associated with the correlator $f_2(t)$ of Eq. (88). The distribution of − intervals has converged to the Poissonian form of Eq. (119) (dashed line). Note that the linear regime of $P_-(t)$ predicted in Eqs. (163) and (168), for $t \ll \tau_+ \ll \tau_-$, cannot be seen at this scale. The distribution of + intervals is well described by the Wigner distribution of Eq. (122) (dashed line), but ultimately decays exponentially for $t \gg a_2^{-1/2}$ (inset: dotted line of slope $\tau_+\theta_+ \approx 2.35$).

$$P_+(t) = -2\tau A''_>(t) = \frac{\pi}{2}\frac{t}{\tau_+^2}e^{-(\pi/4)(t/\tau_+)^2}, \tag{122}$$

which is valid for $t \ll a_2^{-1/2}$. Using Eqs. (23) and (108), the relation $P_+(t) = -2\tau A''_>(t)$ can be explicitly written as

$$P_+(t) = -\tau_-\langle X'(t)X'(0)\delta(X(t)-M)\delta(X(0)-M)\rangle. \tag{123}$$

When the process X crosses the level M for the first time at time t, after the preceding crossing at time 0, one has $-X'(t)X'(0)>0$. Equation (123) states that, for small times $t \ll a_2^{-1/2}$, the probability of having a + interval of length t is of the same order as the probability of having an M crossing at time t, knowing that there is such a crossing at time 0, which follows a − interval (hence the factor τ_-). In other words, for short time t, a crossing at time t is very often the first crossing following the crossing at time 0.

The form of $P_+(t)$ found in Eq. (122) can be obtained from a simple physical argument, which will be our basis for obtaining exact results for the small-t behavior of $P_\pm(t)$ in Sec. IV. Indeed, let us consider a + interval of length $t \ll a_2^{-1/2}$. For $0 \leq t' \leq t$, one can expand the *smooth* process $X(t')$ in power of t' up to second order, starting from $X(0) = M$,

$$X(t') = M + X'(0)t' + \frac{X''(0)}{2}t'^2 + \cdots, \tag{124}$$

remembering that $X'(t)$ is a Gaussian variable of second moment $\langle X'^2(t)\rangle = a_2 = -f''(0)$, the probability distribution of the velocity $v = X'(0) > 0$, at an M crossing following a − interval, is given by

$$\rho(v) = \frac{\langle |X'(0)|\delta(X(0)-M)\delta(X'(0)-v)\rangle_{X'(0)>0}}{\langle |X'(0)|\delta(X(0)-M)\rangle_{X'(0)>0}}. \tag{125}$$

Since $X(0)$ and $X'(0)$ are uncorrelated [as $\langle X'(0)X(0)\rangle = f'(0) = 0$], these Gaussian averages are easily performed, leading to the exact result

$$\rho(v) = \frac{v}{a_2}e^{-v^2/2a_2}. \tag{126}$$

In addition, the distribution of $X''(0)$, conditioned to the fact that $X(0) = M$, is a Gaussian of mean

$$\langle X''(0)\rangle_{X(0)=M} = -Ma_2 \tag{127}$$

and mean square deviation $f^{IV}(0) - f''^2(0)$, which is of order a_2^2, and is independent of M. Hence, for large M, one can replace $X''(0)$ by its average, and the interval length t can be obtained by finding the first M crossing of the trajectory of Eq. (124),

$$t_v = \frac{2v}{Ma_2}. \tag{128}$$

Hence, for small time t, one has

$$P_+(t) = \int_0^{+\infty} \rho(v)\delta(t-t_v)dv. \tag{129}$$

Finally, using Eq. (126) and the asymptotic expression for τ_+ of Eq. (108), we obtain the distribution of + intervals given by Eq. (122).

Thus we find that the distribution of + intervals is given by the Wigner distribution for $t \ll a_2^{-1/2}$, a result also obtained in [3]. However, note that the ratio t/τ_+ can be arbitrarily large in the limit $M \to +\infty$, or $\tau_+ \to 0$. The probability distribution of Eq. (122) is correctly normalized to unity and has a mean equal to τ_+. Of course, for $t \gg a_2^{-1/2}$, the actual distribution of + intervals should decay exponentially [6], with a rate $\theta_+(M)$. Matching the two asymptotics at $t \sim a_2^{-1/2}$, we find that, up to a so far unknown multiplicative constant,

$$\theta_+(M) \sim \sqrt{a_2}M^2 \tag{130}$$

for large M. In addition, the above argument shows that the total probability contained in the exponential tail of $P_+(t)$ vanishes extremely rapidly as $M \to +\infty$, and is of order $\exp(-KM^2)$, where K is a constant of order unity. Moreover, the result of Eq. (122) implies that the persistence is given by

$$P_>(t) = e^{-(\pi/4)(t/\tau_+)^2} \tag{131}$$

for $t \ll a_2^{-1/2}$, and decays exponentially for $t \gg a_2^{-1/2}$.

In the limit $M \to +\infty$, the determination of $\theta_+(M)$ seems to be beyond the IIA. Indeed, let us assume for simplicity that $f(t)$ decays faster than any exponential, $f(t) \sim \exp(-ct^\gamma)$, with $\gamma > 1$. Anticipating that $\theta_+(M)$ is large, we need to evaluate $\hat{P}_+(s)$ for large negative s. In this limit, Eqs. (71) and (72) lead to

$$\hat{V}(s) \sim \frac{M}{\sqrt{2\pi}}e^{-M^2/2}\hat{f}(s), \tag{132}$$

$$\hat{W}(s) \sim -\frac{1}{2\sqrt{a_2}}e^{-M^2/2}s\hat{f}(s). \tag{133}$$

If $1 < \gamma < 2$, and using the IIA expression for $\hat{P}_+(s)$ of Eq. (76), we obtain

$$\hat{P}_+(s) \sim \frac{2\sqrt{2a_2/\pi}M}{s + \sqrt{2a_2/\pi}M}, \tag{134}$$

which leads to

$$\theta_+^{\mathrm{IIA}}(M) \sim \sqrt{\frac{2a_2}{\pi}}M, \tag{135}$$

which grossly underestimate the divergence of $\theta_+(M)$ when $M \to \infty$. If $f(t) \sim \exp(-ct^2)$ decays as a Gaussian ($\gamma = 2$), Eq. (76) leads to

$$\theta_+^{\mathrm{IIA}}(M) \sim \sqrt{c}M, \tag{136}$$

which, again, behaves linearly with M. Finally, if $\gamma > 2$, we find that

$$\theta_+^{\mathrm{IIA}}(M) \sim M^{2(\gamma-1)/\gamma}, \tag{137}$$

up to a computable multiplicative constant.

Let us now present exact bounds for $P_>(t)$ which will lead to an exact asymptotics for $\theta_+(M)$, fully consistent with Eq. (130). We discretize time $t_i=i\Delta t$, with $\Delta t=t/n$, and define $x_i=X(t_i)$. By definition,

$$P_>(t,M) = \int_M^{+\infty} \frac{\prod_{i=1}^n dx_i}{(2\pi)^{n/2}\sqrt{\det \mathbf{C}}} e^{-S(\{x_i\})}, \tag{138}$$

where the Gaussian "action" has the quadratic form

$$S(\{x_i\}) = \frac{1}{2}\sum_{i,j} D_{ij}x_ix_j, \tag{139}$$

and where the matrix $\mathbf{D}$ is the inverse of the correlation matrix $\mathbf{C}$ defined by its matrix elements

$$C_{ij} = \langle X(t_i)X(t_j)\rangle = f(t_i - t_j). \tag{140}$$

Making the change of variables $y_i=x_i+M \in [0,+\infty]$, and noting that

$$S(\{y_i\}) = \frac{1}{2}\sum_{i,j} D_{ij}y_iy_j + M\sum_{i,j} D_{ij}y_i + \frac{M^2}{2}\sum_{i,j} D_{ij}, \tag{141}$$

we obtain

$$P_>(t,M) = P_>(t,M=0)\left\langle \exp\left(-M\sum_i \sigma_i y_i\right)\right\rangle_{y>0} e^{-(M^2/2)n\bar{\sigma}}, \tag{142}$$

with

$$\sigma_i = \sum_{j=1}^n D_{ij}, \quad \bar{\sigma} = \frac{1}{n}\sum_{i=1}^n \sigma_i, \tag{143}$$

and where $\langle\cdot\rangle_{y>0}$ denotes the average over all processes for which $y(t_i)\geq 0$ for all i. If we assume that the process is periodic of period t, the vector $\mathbf{u}=(1,1,\ldots,1)$ is an exact eigenvector of $\mathbf{C}$ associated with the eigenvalue

$$\lambda = \sum_{i=-n/2}^{n/2} f(t_i), \tag{144}$$

and one has

$$\sigma_i = \sum_{j=1}^n D_{ij} \times 1 = (\mathbf{D}\cdot\mathbf{u})_i = \lambda^{-1}u_i = \lambda^{-1}. \tag{145}$$

For large time t, the periodic constraint should not affect the value of the persistence exponent. In fact, in this limit of large time and fine discretization ($\Delta t \to 0$), and even dropping the assumption that the process is periodic, one finds that

$$\sigma_i = \bar{\sigma} = \lambda^{-1} = \frac{\Delta t}{\int_{-\infty}^{\infty} f(t)dt} = \frac{\Delta t}{2\hat{f}(0)}, \tag{146}$$

where the discrete sum of Eq. (144) has been transformed into an integral when $\Delta t \to 0$, and the integral limits $\pm t/2$ extended to $\pm\infty$ for large t. Finally, defining $\theta_0=\theta_\pm(M=0)$ and using Eq. (142), we find the exact result

$$\theta_+(M) = \theta_0 + \Theta(M) + \frac{M^2}{4\hat{f}(0)}, \tag{147}$$

with

$$\Theta(M) = \lim_{t\to+\infty} -\frac{1}{t}\ln\left\langle \exp\left(-\frac{M}{2\hat{f}(0)}\int_0^t X(t')dt'\right)\right\rangle_{X>0}. \tag{148}$$

Using the convexity of the exponential function and the fact that the argument in the exponential in Eq. (148) is negative, we obtain the exact bounds

$$0 \leq \Theta(M) \leq \frac{\langle X(t)\rangle_{X>0}}{2\hat{f}(0)}M, \tag{149}$$

where $\langle X(t)\rangle_{X>0}$ is the average of the process X restricted to the trajectories which remain positive for all times, and is a constant strictly independent of M. Equations (147) and (149) lead to an exact bound for $\theta_+(M)$,

$$\theta_0 + \frac{M^2}{4\hat{f}(0)} \leq \theta_+(M) \leq \theta_0 + \frac{\langle X(t)\rangle_{X>0}}{2\hat{f}(0)}M + \frac{M^2}{4\hat{f}(0)}, \tag{150}$$

which implies that for large M

$$\theta_+(M) \sim \frac{M^2}{4\hat{f}(0)}, \tag{151}$$

with a subleading correction bounded by a linear term in M. The exact asymptotics of Eq. (151) confirms our heuristic argument of Eq. (130). In addition, for small M, it is clear from Eqs. (148) and (149) that one has the exact expansion,

$$\theta_+(M) = \theta_0 + \frac{\langle X(t)\rangle_{X>0}}{2\hat{f}(0)}M + O(M^2). \tag{152}$$

Since $\theta_-(M)=\theta_+(-M)$, we also get

$$\theta_-(M) = \theta_0 - \frac{\langle X(t)\rangle_{X>0}}{2\hat{f}(0)}M + O(M^2). \tag{153}$$

In Fig. 3, we plot $\theta_+(M)$ obtained by simulating the process associated with the correlator $f_2(t)$ of Eq. (88). We also plot the IIA result of Eq. (97), which underestimates the actual value of $\theta_+(M)$, as explained above. In addition, we plot the exact bounds of Eq. (150), as well as a convincing fit of $\theta_+(M)$, to the functional form $\theta_+(M)=a_0+a_1M+M^2/4\hat{f}(0)$, where the exact leading term was obtained in Eq. (151). In Fig. 2, we plot $P_+(t)$ for $M=3$, which follows the predicted Wigner distribution of Eq. (122), for $t \ll a_2^{-1/2}$, before decaying exponentially, with rate $\theta_+(M)$, for large t.

IV. DISTRIBUTIONS $P_\pm(t)$ FOR SMALL INTERVALS

The heuristic argument presented in the preceding section, below Eq. (124), can be adapted to provide the exact behav-

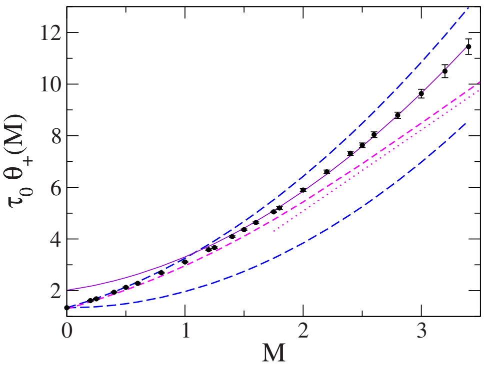

FIG. 3. (Color online) $\theta_+(M)$ (in unit of τ_0^{-1}) for the non-Markovian process associated with the correlator $f_2(t)$ of Eq. (88). Symbols correspond to results of numerical simulations. The upper and lower dashed lines are the exact bounds of Eq. (150) (the upper bound being exact up to order M, for small M). The ratio of these two bounds goes to 1 when $M\to+\infty$. The full line is a quadratic fit for $M>3/2$, to the functional form $\theta_+(M)=a_0+a_1M+[M^2/4\hat{f}(0)]$, with $a_0\approx 2.0$ and $a_1\approx 0.66$. Finally, the middle dashed line is the IIA result [along with its asymptotic slope given by Eq. (136); dotted line], which underestimates the quadratic growth of $\theta_+(M)$. Some values of $\theta_+(M)$ are also reported in Table II.

ior of $P_\pm(t)$, $P_>(t)$, and $P_<(t)$ for small time t. For a smooth process with $f(t)$ at least four times differentiable, we will show that the IIA surprisingly reproduces these exact results. However, for a marginally smooth process, such that for small t and $2<\alpha<4$,

$$f(t) = 1 - \frac{a_2}{2}t^2 + a_\alpha|t|^\alpha + \cdots , \tag{154}$$

the fourth derivative of $f(t)$ is not defined at 0. We will show that for such a process the IIA does not lead to exact results for the small-t behavior of $P_\pm(t)$, although it is in fact qualitatively and even quantitatively accurate. Note that the process associated with the correlator $f_3(t)$ defined in Eq. (90) satisfies the property of Eq. (154), with $\alpha=3$, which implies that its velocity is not differentiable.

A. Exact small-time behavior for very smooth processes

In the limit $t\to 0$, the trajectory of the very smooth process X inside a + or − interval is essentially parabolic,

$$X(t') = M + vt' + \frac{a}{2}t'^2 + O(t^3), \tag{155}$$

where the distribution of the velocity at a crossing time is given by

$$\rho(v) = \frac{|v|}{a_2}e^{-v^2/2a_2}, \tag{156}$$

and the distribution of the acceleration a, conditional on the fact that $X(0)=M$, is

$$\sigma(a) = \frac{1}{\sqrt{2\pi}a_2 z}e^{-(a+Ma_2)^2/2a_2^2z^2}, \tag{157}$$

with

$$z = \sqrt{\frac{f^{IV}(0)}{f''^2(0)} - 1}, \tag{158}$$

and $a_2=-f''(0)$. Note that the acceleration at $t'=0$ is independent of the velocity at $t'=0$, since $\langle X''(0)X'(0)\rangle=f'''(0)=0$. In addition, since

$$f^{IV}(0) - f''^2(0) = \int_{-\infty}^{+\infty}(\omega^2-a_2)^2\tilde{f}(\omega)\frac{d\omega}{2\pi} > 0, \tag{159}$$

z defined by Eq. (158) is indeed a positive real number.

Let us first consider the small-time behavior of $P_+(t)$. For an interval of length t to be small, and since $v>0$, the acceleration is necessarily negative. From Eq. (155), the crossing time is then given by

$$t = -\frac{2v}{a}, \tag{160}$$

which is valid only when t is small. Hence, for small t,

$$P_+(t) = \int_0^{+\infty}dv\int_{-\infty}^0 da\,\rho(v)\sigma(a)\delta\left(t+\frac{2v}{a}\right). \tag{161}$$

After integrating over v, we obtain

$$P_+(t) = \frac{1}{2}\int_{-\infty}^0\rho\left(\frac{at}{2}\right)\sigma(a)|a|da. \tag{162}$$

Using the explicit form of $\rho(v)$, and taking the limit $t\to 0$, we obtain

$$P_+(t) = c_+(M)a_2t + O(t^3), \tag{163}$$

where the dimensionless constant $c_+(M)$ is given by

$$c_+(M) = \frac{1}{4a_2^2}\int_{-\infty}^0\sigma(a)a^2da. \tag{164}$$

Performing explicitly the Gaussian integral above, we finally obtain the exact result,

$$c_+(M) = \frac{M^2+z^2}{4}G\left(\frac{M}{z}\right) + \frac{zM}{4}g\left(\frac{M}{z}\right), \tag{165}$$

where $g(X)$ is the Gaussian distribution and $G(X)$ its cumulative sum, both defined in Eqs. (2) and (3). In the limit $M\to+\infty$, we find $c_+(M)\sim M^2/4$, which leads to

$$P_+(t) \sim \frac{M^2a_2}{4}t \sim \frac{\pi}{2}\frac{t}{\tau_+^2}, \tag{166}$$

in agreement with our result of Eq. (122). Moreover, using Eq. (16), we obtain the small-time expansion of $P_>(t)$, up to third order in time,

$$P_>(t) = 1 - \frac{t}{\tau_+} + \frac{c_+a_2}{6\tau_+}t^3 + O(t^5). \tag{167}$$

Finally, the corresponding results for $c_-(M)$, $P_-(t)$, and $P_<(t)$ are obtained by the substitution $M\leftrightarrow -M$, and the exchange of the indices $+\leftrightarrow-$ and $>\leftrightarrow<$. In particular, we have

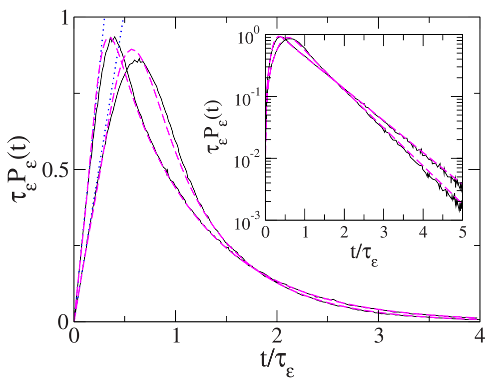

FIG. 4. (Color online) Distribution of $\varepsilon=\pm$ intervals, $P_\pm(t)$, for $M=1/2$, as a function of $t/\tau_\pm$ (full lines), for the process associated with the correlator $f_2(t)$ of Eq. (88) [$P_-(t)$ is the most peaked distribution]. The straight dotted lines have the predicted slopes at $t=0$, given by Eqs. (163), (165), and (168). The dashed lines are the distributions obtained from the IIA, after taking the inverse Laplace transform of Eqs. (76) and (77). The inset shows the same data on a semi logarithmic scale, illustrating the good accuracy of the IIA in predicting the persistence exponents $\theta_\pm(M)$ and their associated residue $R_\pm$.

$$c_-(M) = \frac{M^2+z^2}{4}\bar{G}\left(\frac{M}{z}\right) - \frac{zM}{4}g\left(\frac{M}{z}\right). \qquad (168)$$

In the limit $M\to+\infty$, we find that

$$c_-(M) \sim \frac{z^5}{2M^3}g\left(\frac{M}{z}\right). \qquad (169)$$

Thus, we find that $P_\pm(t)$ behaves linearly with time for small t, a result that will be shown in Sec. IV C to be specific to very smooth processes, for which the correlator $f(t)$ is at least four times differentiable at $t=0$.

In Fig. 4, for $M=1/2$, we plot $P_\pm(t)$ obtained from numerical simulations of the process associated with the correlator $f_2(t)$, illustrating their linear behavior for small time t. The initial slope at $t=0$ is in perfect numerical agreement with the exact results of Eqs. (163), (165), and (168). In addition, we also plot the full distributions $P_\pm(t)$, obtained by taking the inverse Laplace transform of Eqs. (76) and (77). For this moderate value of M, far from the large-M regime where the IIA becomes exact, the good agreement between the IIA results and numerical simulations is certainly encouraging.

B. IIA results

We now derive the small-time behavior of $P_\pm(t)$ by using the IIA. Expanding the explicit form of $V(t)$ an $W(t)$ of Eqs. (67) and (70), we find

$$W(t) = \frac{1}{2} - \frac{M^2+z^2}{16}a_2t^2 + O(t^4), \qquad (170)$$

$$V(t) = G(M) - \frac{1}{2} - \left\{\frac{M^2+z^2}{8}\left[G\left(\frac{M}{z}\right) - \frac{1}{2}\right] + \frac{zM}{8}g\left(\frac{M}{z}\right)\right\}a_2t^2 + O(t^4). \qquad (171)$$

Taking the Laplace transform of these expression and inserting them into the IIA expression for $\hat{P}_+(s)$ of Eq. (76), we obtain, for large s,

$$\hat{P}_+(s) = \left[(M^2+z^2)G\left(\frac{M}{z}\right) + zMg\left(\frac{M}{z}\right)\right]\frac{a_2}{4s^2} + O(s^{-4}), \qquad (172)$$

which is exactly the Laplace transform of Eqs. (163) and (165). Thus, we find that the IIA reproduces the exact small-time behavior of $P_\pm(t)$.

C. Marginally smooth processes

In this section, we study marginally smooth processes characterized by a correlator $f(t)$ having a small-time expansion of the form of Eq. (154), so that $f^{IV}(0)$ does not exist. Since $f^{IV}(0)$ appeared explicitly in our results of the preceding sections, this suggests that the small-time behavior of $P_\pm(t)$ should be affected by the weak singularity in $f(t)$.

Let us first apply the IIA in this marginal case. The small-time expansion of $V(t)$ and $W(t)$ now read

$$W(t) \sim \frac{1}{2} - \frac{(\alpha-1)a_\alpha}{2a_2}t^{\alpha-2}, \qquad (173)$$

$$V(t) \sim G(M) - \frac{1}{2} - M\sqrt{\frac{(\alpha-1)a_\alpha}{4\pi}}t^{\alpha/2}. \qquad (174)$$

Note that, since $2<\alpha<4$, one has $\alpha-2<\alpha/2$. For large s, and using Eq. (76) and the above asymptotics, we obtain

$$\hat{P}_\pm(s) \sim \frac{(\alpha-1)\Gamma(\alpha-1)a_\alpha}{2a_2s^{\alpha-2}}. \qquad (175)$$

In real time, this leads to the small-time behavior

$$P_\pm(t) \sim \frac{(\alpha-1)\Gamma(\alpha-1)a_\alpha}{2\Gamma(\alpha-2)a_2}t^{\alpha-3}, \qquad (176)$$

which is independent of M. In particular, for the quite common case $\alpha=3$, which corresponds to the correlator $f_3(t)$ introduced in Eq. (90), we find that $P_\pm(t)$ should be constant at $t=0$, with

$$P_\pm(0) = \frac{a_3}{a_2}. \qquad (177)$$

For the correlator $f_3(t)$, one has $a_2=3/4$ and $a_3=1/4$, so that $P_\pm(0)=1/3$.

Let us now derive an exact expression for $P_+(t)$ for a marginally smooth process with $\alpha=3$. For small t, the correlator of the velocity is

$$\langle X'(t)X'(0)\rangle = -f''(t) = a_2 - 6a_3|t| + O(t^2), \quad (178)$$

which coincides with the small-time behavior of the correlator of a Markovian process [see Eq. (7)]. Hence, for short time periods, the local equation of motion of $X'(t)$ is

$$X''(t) = -\frac{6a_3}{a_2}X'(t) + 2\sqrt{3a_3}\eta(t), \quad (179)$$

where $\eta(t)$ is a δ-correlated Gaussian noise. From the equation of motion Eq. (179), one indeed recovers Eq. (178), for short times. Now, using Eq. (179), a short-time trajectory of X, starting from $X(0)=M$ and $X'(0)=v$, takes the form

$$X(t) = M + vt + 2\sqrt{3a_3}Z(t) + O(t^2), \quad (180)$$

where $Z(t)=O(t^{3/2})$ is the random acceleration process introduced in Eq. (89),

$$Z(t) = \int_0^t dt_1 \int_0^{t_1} dt_2\, \eta(t_2). \quad (181)$$

Finally, for small t, the first M crossing of the process X corresponds to the first time for which $Z(t)/t$ crosses the level

$$Z_0 = -\frac{v}{2\sqrt{3a_3}}. \quad (182)$$

Introducing the probability distribution $\Psi(t_0,Z_0)$ that $Z(t)/t$ crosses Z_0 for the first time at time $t=t_0$, one has the scaling relation

$$\Psi(t_0,Z_0) = \frac{1}{Z_0^2}\psi\left(\frac{t_0}{Z_0^2}\right), \quad (183)$$

obtained by noticing that the scale-invariant process $Z(t)$ has dimension $[t]^{3/2}$. For small time t,

$$P_\pm(t) = \int_0^{+\infty} \rho(v)\psi\left(\frac{12a_3t}{v^2}\right)\frac{12a_3}{v^2}dv. \quad (184)$$

After making the change of variable $T=12a_3t/v^2$ and taking the limit $t\to 0$, while using the fact that $\rho(v)\sim v/a_2$ for small v, we obtain the final exact result,

$$P_\pm(0) = \frac{6a_3}{a_2}\int_0^{+\infty}\psi(T)\frac{dT}{T} = \frac{6a_3}{a_2}\left\langle\frac{1}{T}\right\rangle. \quad (185)$$

Up to a dimensional constant $6a_3/a_2$ depending on $f(t)$, $P_\pm(0)$ is proportional to the mean inverse first-passage time of the process $Z(t)/t$ at the level $Z_0=1$. By simulating the process Z, we have obtained

$$\left\langle\frac{1}{T}\right\rangle \approx 0.193(1), \quad (186)$$

which leads to

$$P_\pm(0) = 1.158(6)\frac{a_3}{a_2}. \quad (187)$$

This result has also been checked numerically for the process associated with $f_3(t)$. In fact, the constant appearing in Eq. (187) was obtained exactly by Wong [5], based on the study of the process $Z(t)$ of [4]. Their result leads to

$$P_\pm(0) = \frac{37}{32}\frac{a_3}{a_2} = 1.156\,25\frac{a_3}{a_2}. \quad (188)$$

Hence, we find that the IIA result of Eq. (177) is not exact for marginally smooth processes with $\alpha=3$, although it predicts correctly that $P_\pm(0)$ is a constant independent of M, leading to a reasonably accurate estimate of this constant. For general α, Eq. (176) is certainly correct dimensionally speaking, and probably fairly accurate in practice.

We end this section by an approximate calculation of $\langle 1/T\rangle$ for the process $Z(t)$, which does not reproduce the exact result of [5], obtained by a much more complex method [2,4]. We make the approximation

$$Z(t) = \int_0^t dt_1 \int_0^{t_1} dt_2\,\eta(t_2) \approx \frac{t}{\sqrt{3}}\int_0^t dt_1\,\eta(t_1), \quad (189)$$

where the factor $1/\sqrt{3}$ ensures that both processes have the same mean square displacement $\langle Z^2(t)\rangle = t^3/3$. Then, the original first-passage problem for $Z(t)/t$ becomes a standard first-passage problem for the usual Brownian motion $B(t)$ at the level $B_0=\sqrt{3}$, for which the first-passage time probability distribution is given by [21]

$$\Psi(T,B_0) = \frac{B_0}{\sqrt{2\pi}T^{3/2}}e^{-B_0^2/2T}, \quad (190)$$

for which

$$\left\langle\frac{1}{T}\right\rangle = \frac{1}{B_0^2}. \quad (191)$$

Finally, within this simple approximation, we find that $\langle 1/T\rangle = 1/3$, overestimating the value obtained in Eq. (186).

V. CONCLUSION

In this work, we have considered the M-crossing interval distributions and the persistence of a smooth non-Markovian Gaussian process. We have obtained exact results for the persistence exponents in the limit of a large crossing level M, including exact bounds for $\theta_+(M)$ and $P_>(t)$. In this limit, we have shown that the distributions of + and − intervals become universal, and are respectively given by the Wigner and Poisson distributions. For any value of M, we have obtained the exact small-time behavior of the interval distributions and the persistence. We have also derived these results within the independent interval approximation. Quite surprisingly, the IIA reproduces all these exact results, except for the large-M asymptotics of $\theta_+(M)$. In addition, the IIA fails in reproducing the exact small-time behavior of the interval distributions for marginally smooth processes, although it remains qualitatively correct and even quite accurate in this case. To the credit of the IIA, it is the only method to provide precise approximate expressions of the interval distributions and the persistence for all times, and all values of the level M, and thus to grant access to the distribution of extrema of a non-Markovian Gaussian process. In addition, the IIA can be straightforwardly applied to any *smooth non-Gaussian process*, for which the autocorrelation function $A_>(t)$ [or $A_<(t)$] and the conditional number of crossings $N_>(t)$ [or $N_<(t)$] are known analytically, or extracted from experimental or numerical data. For Gaussian processes, simple forms of the derivative of these quantities have been obtained, which permit a simple and fast numerical implementation of the IIA results.

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[36] Note that there was a typographical error in [20], and that one should replace $\langle |X'(t)| \rangle_<$ by $\langle |X'(t)| \rangle_< / G(M)$ in Eqs. (28) and (29) of [20].