Vavilov-Cherenkov radiation in passive and active media with complex resonant dispersion

Andrey V. Tyukhtin and Sergey N. Galyamin
Radiophysics Department, St. Petersburg University, 1 Ulyanovskaya, St. Petersburg, 198504, Russia
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Some of the problems with the theory of moving charge radiation in media with frequency dispersion are analyzed. First, some general properties of the integrals for field components are described. The results are applied to the cases of passive and active media. In one instance, the field of a charge moving in passive media with an arbitrary number of resonances is considered. Components of the field have been presented as a sum of the "quasi-Coulomb" field, the wave field, and the "plasma trace." In another example, the case of an active medium with two resonant frequencies is considered. It has been demonstrated that radiation is amplified even with a purely real refractive index if the following conditions are fulfilled: the "lower" resonance is active, the "upper" one is passive, and the charge movement velocity lies within a certain range. Efficient algorithms for the computation of fields in the cases of passive and active media have been developed.

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I. INTRODUCTION

Starting from the second quarter of the 20th century, problems concerning electromagnetic wave radiation generated by sources moving in material media have attracted the attention of researchers [1,2]. A considerable number of experimental and theoretical works dedicated to these problems have been published, and the most important results have been presented in a number of monographs, reviews, and tutorials [3–9]. Currently, interest in this area is considerable. In particular, a detailed analysis of Vavilov-Cherenkov radiation (VCR) in a single-resonant dispersive medium was carried out recently [8,10,11]. VCR was also considered in more complex media, including, for example, media with periodic inhomogeneities such as photonic crystals [12,13].

The considerable interest in the field of VCR is connected, in particular, with many opportunities for its application in the physics of accelerators and detectors of charged particles as well as in microwave engineering. The prospective technique for accelerating charged particles is the so-called wakefield technique, which considers an ultrarelativistic bunch of charged particles accelerated in the electromagnetic field of another bunch [14–19]. There is, however, a range of difficulties in implementing this acceleration scheme, and its development requires a more detailed analysis of VCR in a medium with complex dispersion behaviors. Let us note that one of the latest ideas being discussed is the possibility of wakefield amplification in active media (such media have a frequency dispersion of an unusual nature [20–25]). Such subject matters are significant to a wider field of applied physics and microwave engineering—in particular, the creation of new sources of powerful electromagnetic radiation.

In this article, we analyze a series of questions concerning VCR theory in media with frequency dispersion. Section II uses methods of the theory of complex variable functions to investigate the general properties of known integrals for components of a moving charge field. In Sec. III, the results obtained are applied to the case of a passive medium with an arbitrary number of resonance frequencies. Section IV is dedicated to the analysis of VCR amplification through an active medium that has two resonant frequencies.

II. SOME GENERAL PROPERTIES OF INTEGRALS FOR MOVING CHARGE FIELD COMPONENTS

Let us assume that a point charge q moves with a constant velocity $\vec{V}=V\vec{e}_z$ (where $V>0$). The position of the charge at a moment in time t is determined by the relations $x=y=0$ and $z=Vt$. The medium is assumed to be isotropic, homogeneous, linear, and lacking spatial dispersion. The squared refractive index of the medium is equal to $n^2(\omega)=\varepsilon(\omega)\mu(\omega)$. As is known [3–9], in a cylindrical coordinate system, the ρ,ϕ,z components of an electromagnetic field of a charge can be written as follows:

$$\{E_\rho,E_z,H_\phi\}=\frac{q}{2c}\int_{-\infty}^{+\infty}\{e_\rho(\omega),e_z(\omega),h_\phi(\omega)\}d\omega, \qquad (2.1)$$

$$e_\rho(\omega)=\frac{is(\omega)}{\beta\varepsilon(\omega)}H_1^{(1)}(s(\omega)\rho)\exp\left[i\omega\frac{\zeta}{V}\right],$$

$$e_z(\omega)=\frac{\omega}{c}\frac{1-n^2\beta^2}{\varepsilon(\omega)\beta^2}H_0^{(1)}(s(\omega)\rho)\exp\left[i\omega\frac{\zeta}{V}\right],$$

$$h_\phi(\omega)=is(\omega)H_1^{(1)}(s(\omega)\rho)\exp\left[i\omega\frac{\zeta}{V}\right], \qquad (2.2)$$

where $\zeta=z-Vt$, $\beta=V/c$, c is the velocity of light in a vacuum, and $H_\nu^{(1)}(\xi)$ are the Hankel functions,

$$s^2(\omega)=\frac{\omega^2}{V^2}[n^2(\omega)\beta^2-1]. \qquad (2.3)$$

The contour of integration in formulas (2.1) is the real axis (we exclude from consideration here the case of an active medium, which is considered in Sec. IV). Let us emphasize that expressions (2.1)–(2.3) satisfy Maxwell's equations and material relations irrespective of the determination of the root $s(\omega)=\sqrt{s^2(\omega)}$. This function should be determined in accordance with so-called "radiation principles" [26].

If the dissipation of electromagnetic energy is considered to be negligible, the function $s^2(\omega)$ is real for real values of ω. In this case, in the domain of radiated frequencies [i.e., on those parts of the real axis where $s^2(\omega)>0$], Mandelshtam's principle of radiation should be fulfilled [26]. Conforming to this principle, the group velocity's $\vec{V}_g=d\omega/d\vec{k}$ direction is away from the source. Note that, in dispersive media, Sommerfeld's principle of radiation—i.e., the requirement that the phase velocity $\vec{V}_p$ be directed away from the source—is not always true. According to Mandelshtam's principle, the group velocity must be directed away from the charge motion line, $V_{g\rho}=d\omega/ds>0$. Since $ds^2/d\omega=2sds/d\omega$, this requirement can be written in the form $\frac{1}{s}\frac{ds^2}{d\omega}>0$. Along the parts of the real axis where $s^2(\omega)<0$, the requirement that the "local waves" exponentially decrease should be met, and such a requirement corresponds to the condition $\text{Im}\, s>0$. As a result, we obtain

$$s=\begin{cases}|s|\,\text{sgn}(ds^2/d\omega) & \text{for } s^2>0\\ i|s| & \text{for } s^2<0.\end{cases} \tag{2.4}$$

According to Eq. (2.3), in the domain of radiated frequencies, where $n^2\beta^2-1>0$, we obtain

$$\text{sgn}\left(\frac{ds^2}{d\omega}\right)=\text{sgn}\left\{\omega\left[1+\frac{\beta^2\omega}{2(n^2\beta^2-1)}\frac{dn^2}{d\omega}\right]\right\}. \tag{2.5}$$

In the case of a normal dispersion of $n^2(\omega)=\varepsilon(\omega)\mu(\omega)$, when $\omega dn^2/d\omega>0$, we obtain $\text{sgn}(ds^2/d\omega)=\text{sgn}\,\omega$—i.e., $\text{sgn}\, s=\text{sgn}\,\omega$. This means that Sommerfeld's radiation principle is true. If the dispersion of $n^2(\omega)$ is anomalous, $\text{sgn}(ds^2/d\omega)$ can be equal to both $\text{sgn}\,\omega$ and $-\text{sgn}\,\omega$ (in the latter case, Sommerfeld's principle is untrue). Let us note that the frequency band of anomalous dispersion $n^2(\omega)$ (where $\omega dn^2/d\omega<0$) may be characterized by a relatively small absorption (i.e., a negligible imaginary part of n^2). For example, such a situation is possible in the case of the so-called left-handed medium. (Cherenkov radiation in left-handed media is analyzed in Ref. [27] both for the lossless case and for the lossy one). Furthermore, some active media may have negligible values of $\text{Im}(n^2)$ in the range of anomalous dispersion (see Sec. IV).

The function $s(\omega)$ must be defined on the complex plane of ω in such a way that the requirements (2.4) are met on the real axis. It is expedient to draw the cuts in segments where $\text{Im}\, s=0$, fixing the "physical" sheet of the Riemann surface by the requirement $\text{Im}\, s>0$. Then, on those parts of the real axis that coincide with the cuts, the contour of integration should lie along those banks of the cuts where $V_{g\rho}>0$—i.e., where the condition (2.5) is fulfilled.

For the case where electromagnetic energy absorption in the medium is taken into account, the refractive index for real frequencies is complex. In this case, for the definition of the function $s(\omega)$, it is enough to apply the requirement $\text{Im}\, s>0$ on the real axis. To determine the "physical" sheet of the Riemann surface, it is expedient, as before, to extend this requirement over the whole complex plane of ω.

Let us obtain some properties of the function $s(\omega)$ and functions (2.2) on the "physical" sheet of the Riemann surface. For this purpose, we will use the following relations:

$$\varepsilon(-\bar{\omega})=\bar{\varepsilon}(\omega),\quad \mu(-\bar{\omega})=\bar{\mu}(\omega),\quad n^2(-\bar{\omega})=\bar{n}^2(\omega), \tag{2.6}$$

where the overbar indicates complex conjugation. These formulas are not related to the selection of a medium model; they arise only from a requirement for the reality of the field components in the Maxwell equations. Using Eqs. (2.3) and (2.6), we obtain the following:

$$s^2(-\bar{\omega})=\bar{s}^2(\omega). \tag{2.7}$$

Let us assume that both point ω and point $-\bar{\omega}$ belong to the "physical" sheet of the Riemann surface—i.e., $0\le\arg[s(\omega)]\le\pi$ and $0\le\arg[s(-\bar{\omega})]\le\pi$. Then, from Eq. (2.7) it follows that

$$s(-\bar{\omega})=e^{i\pi}\bar{s}(\omega). \tag{2.8}$$

This property will play a considerable role in subsequent transformations.

Based on the requirement of reality of the field components in the Maxwell equations, it is easy to show that the following relations are valid:

$$e_\rho(-\bar{\omega})=\bar{e}_\rho(\omega),\quad e_z(-\bar{\omega})=\bar{e}_z(\omega),\quad h_\phi(-\bar{\omega})=\bar{h}_\phi(\omega). \tag{2.9}$$

[Let us note that these proportions can be derived directly from Eq. (2.2) using Eqs. (2.6) and (2.8) and certain properties of the cylindrical function.]

Furthermore, the initial contour will be transformed in such a way that the new contour will be symmetrical with respect to the imaginary axis. In this connection, we note the following important property. Let us assume that a contour Γ consists of two parts: one of them (Γ_+) lies in the domain $\text{Re}\,\omega\ge0$, and the other (Γ_-) lies in the domain $\text{Re}\,\omega\le0$. If the total contour Γ is symmetrical with respect to the imaginary axis, then, with the help of Eq. (2.9), we obtain the following identity:

$$\int_\Gamma f(\omega)d\omega=2\int_{\Gamma_+}\text{Re}[f(\omega)d\omega], \tag{2.10}$$

where $f(\omega)$ is any of the functions (2.2).

Let us write down the asymptotic of function $s(\omega)$ on a "physical" sheet. Taking into consideration that, at $|\omega|\to\infty$ for all real media $\varepsilon(\omega)\to1$ and $\mu(\omega)\to1$, we obtain the following:

$$s(\omega)\underset{|\omega|\to\infty}{\to}\frac{\sqrt{1-\beta^2}}{V}i\omega\,\text{sgn}(\text{Re}\,\omega). \tag{2.11}$$

Let us emphasize that the discontinuity of the asymptotic is explained by the "physical" sheet fixation according to the rule $\mathrm{Im}\, s(\omega)>0$. Using asymptotic forms of the Hankel function [28], we obtain that the functions (2.2) exponentially diminish as $|\omega|\to\infty$ in the following domains:

$$\mathrm{Im}\ \omega > -|\mathrm{Re}\ \omega|\rho\sqrt{1-\beta^2}/\zeta \quad \text{at} \quad \zeta>0,$$

$$\mathrm{Im}\ \omega < |\mathrm{Re}\ \omega|\rho\sqrt{1-\beta^2}/|\zeta| \quad \text{at} \quad \zeta<0. \tag{2.12}$$

Furthermore, an essential role will be played by asymptotes of the steepest descent contour (SDC). The contour itself cannot be found without concretization of a medium model. Its behavior at infinity, however, is determined only by the asymptotic (2.11). The SDC asymptotes lie within the fields of exponential diminution (2.12) and are determined by the equation $\mathrm{Re}(s\rho+\omega\zeta V^{-1})=\mathrm{const}$, which can be written in the following form:

$$\rho\sqrt{1-\beta^2}\mathrm{Im}\ \omega = \zeta|\mathrm{Re}\ \omega| + \mathrm{const} \tag{2.13}$$

(the value of the constant included here is not essential for future studies).

III. MOVING CHARGE FIELD IN A PASSIVE MEDIUM WITH SEVERAL RESONANCE FREQUENCIES

In this section, we consider a passive medium with a resonance-type frequency dispersion. It is assumed that there are several resonance frequencies ω_{rm} (for completeness, let us suppose that $\omega_{rm+1}>\omega_{rm}$). We express the refractive index of the medium in the following typical form:

$$n^2(\omega) = \varepsilon(\omega)\mu(\omega) = 1 + \sum_{m=1}^{M}\frac{\omega_{pm}^2}{\omega_{rm}^2 - 2i\omega_{dm}\omega - \omega^2}, \tag{3.1}$$

where the parameters ω_{pm} can be called "plasma frequencies" and the parameters ω_{dm} determine the dissipation of the electromagnetic energy in the medium.

Let us first consider the case when dissipation in the medium can be neglected—i.e., $\omega_{dm}=0$. In such a situation, the function $s^2(\omega)$ can be presented in the following form:

$$s^2(\omega) = -\frac{(1-\beta^2)\omega^2}{V^2}\frac{\prod_{m=1}^{M}(\omega^2-\omega_{cm}^2)}{\prod_{m=1}^{M}(\omega^2-\omega_{rm}^2)}, \tag{3.2}$$

where ω_{cm} are zeros of the numerator of $s^2(\omega)$. For further consideration, it is important to distinguish two situations. To start, let us assume that $n_0\beta<1$, where $n_0=n(0)=\sqrt{1+\sum_{m=1}^{M}\frac{\omega_{pm}^2}{\omega_{rm}^2}}$ is the refractive index for waves with frequencies essentially less than the lower resonance frequency. This condition means that the phase velocity of the low-frequency waves exceeds the charge movement velocity ($c/n_0>V$). Figure 1(a), part I, shows the dependence $n^2(\omega)$, and the dashed straight line shows level β^{-2} corresponding to the case being considered. The real axis segments, where $n^2(\omega)>\beta^{-2}$, represent the ranges of radiation frequency. Resonance frequencies ω_{rm} are their upper limits, while frequencies ω_{cm} are the lower limits. As we see, in this case, there are M real positive zeros of the function $s(\omega)$. Fixing the "physical" sheet of the Riemann surface with the requirement of $\mathrm{Im}\, s>0$ let us draw cuts in such segments, where $\mathrm{Im}\, s=0$. The system of such cuts is shown in Fig. 1(a), part II. To define the function $s(\omega)=\sqrt{s^2(\omega)}$, it is required to define the radical $\sqrt{\omega^2}$ with the rule of $\sqrt{\omega^2}=\lim_{\delta\to+0}\sqrt{\omega^2+\delta^2}$.

So we have the cuts going from point $i\delta$ to $i\infty$ and from point $-i\delta$ to $-i\infty$ [Fig. 1(a), part II]. It is easy to verify that, within those parts of the contour of integration, where $s^2<0$, the "local waves" are diminished (i.e., $s=i|s|$). At the same time, on the upper banks of the cuts located on the real axis, $\mathrm{sgn}\, s=\mathrm{sgn}\, \omega$. Thus, if we draw the contour of integration along the upper banks [Fig. 1(a), part II], the requirement of a group velocity direction away from the charge movement direction will be fulfilled due to the normal dispersion (see Sec. II).

In addition to the mentioned singularities, integrands of E_ρ and E_z can contain poles—zeros of permittivity $\varepsilon(\omega)$ [in the case when frequency dependence (3.1) is determined by the function $\varepsilon(\omega)$]. Such singularities determine the so-called "plasma trace" of the source. It represents plasma oscillations excited close to the way of the source movement, and it does not transfer electromagnetic energy. This part of the electromagnetic field can exist only in the domain behind the source; i.e., the poles must be bypassed from above.

In the case of $n_0\beta>1$, when the phase velocity of low-frequency waves is less than the charge movement velocity, the Vavilov-Cherenkov radiation spectrum starts from zero frequency [Fig. 1(b), part I]. At the branch points, $\pm\omega_{c1}$ are purely imaginary, but the values of $\pm\omega_{cm}$ at $m\neq 1$ are real. The cuts determined by the equation $\mathrm{Im}\, s=0$ and the contour of integration, which satisfies the radiation principle, are shown in Fig. 1(b), part II. Let us note that, in this case, the radical $\sqrt{\omega^2}$ being a part of $s(\omega)$ must be determined by the rule $\sqrt{\omega^2}=\omega$. The cuts lying on the imaginary axis go out from the points $\pm\omega_{c1}$ [Fig. 1(b), part II].

Further analytical transformations for both cases consist of the following steps. Let us form a closed contour of integration by complementing the initial contour (the real axis) with an infinite semicircle located within the domain of $\mathrm{Im}\ \omega\geq 0$ at $\zeta>0$ and within the domain of $\mathrm{Im}\ \omega\leq 0$ at $\zeta<0$. At $\zeta>0$, only the cut going along the imaginary positive semiaxis contributes to the integrals. It gives a "quasi-Coulomb" field, which represents a forerunner moving together with the charge in front of it. At $\zeta<0$, both the contour embracing the cut going along the imaginary negative semiaxis ("quasi-Coulomb" field) and the contours embracing the cuts located on the real axis (the wave field—i.e., the VCR field) make inputs into the integrals. Moreover, if the permittivity $\varepsilon(\omega)$ is a function of frequency (for instance, in the case when $\mu=1$ and $n^2=\varepsilon$), then the poles $\pm\omega_{sm}$, which represent zeros of the function $\varepsilon(\omega)$, make their contributions to the components E_ρ and E_z (they determine the so-called "plasma trace"). As a result of a series of simple transformations, the following expressions can be obtained:

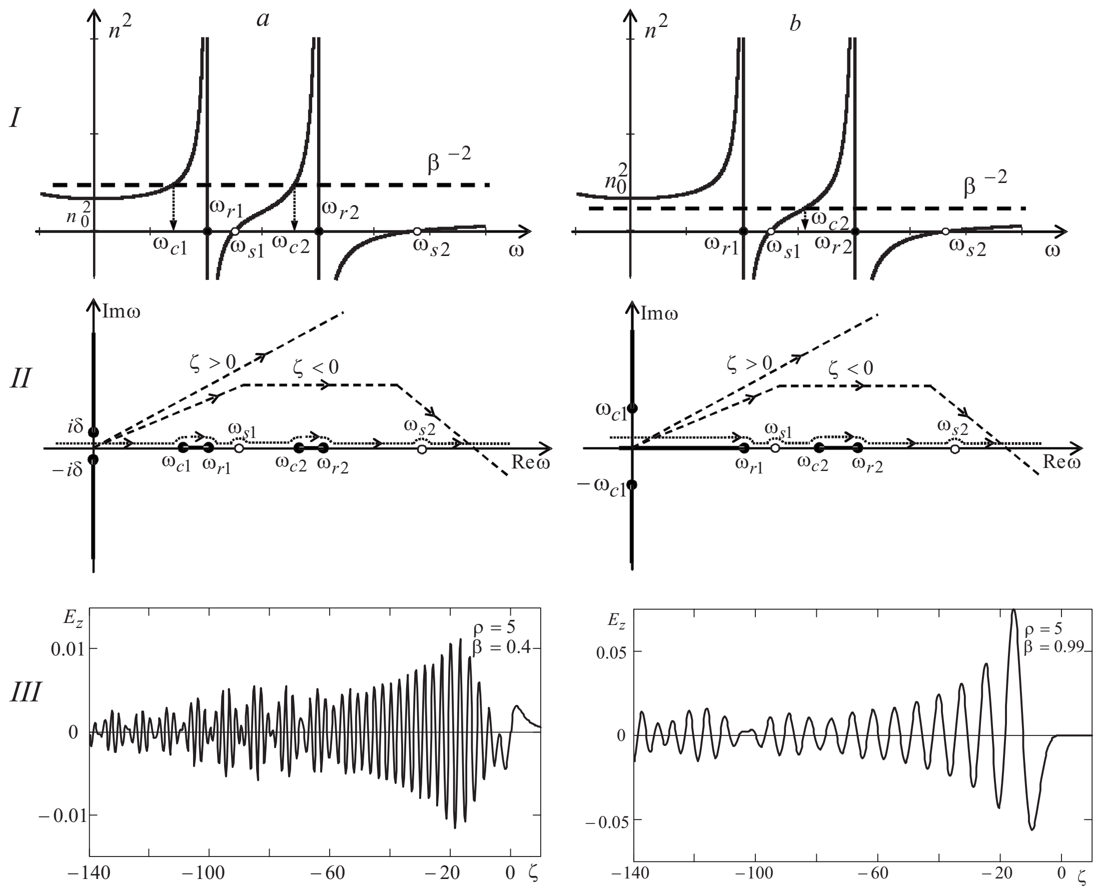

FIG. 1. Part I: dependence of n^2 on ω for (a) $n_0\beta<1$ and (b) $n_0\beta>1$ in the case of a medium with two resonances ($M=2$). The dashed line shows the level of β^{-2}. Part II: view of the cuts (bold lines) and contours of integrations on the complex plane ω for (a) $n_0^2\beta^2<1$ and (b) $n_0^2\beta^2>1$ in the case of a medium with two resonances ($M=2$). The dotted line is the initial contour of integration, and the dashed lines are contours used for computation. The picture for the half-plane Re $\omega<0$ is symmetrical with respect to the imaginary axis. Part III: dependence of the longitudinal component of the electrical field E_z (in units of $q\omega_{r1}^2c^{-2}$) on $\zeta=z-Vt$ (in units of $c\omega_{r1}^{-1}$) for (a) $\beta=0.4$ and (b) $\beta=0.99$ in case of ruby: $M=2$, $\omega_{r1}=2\pi\times 11\times 10^{12}$ s^{-1}, $\omega_{r2}=2\pi\times 14\times 10^{12}$ s^{-1}, $\omega_{p1}=2\pi\times 13\times 10^{12}$ s^{-1}, $\omega_{p2}=2\pi\times 21\times 10^{12}$ s^{-1}, $\omega_{d1}=2\pi\times 0.02\times 10^{12}$ s^{-1}, $\omega_{d2}=2\pi\times 0.06\times 10^{12}$ s^{-1}, and $\rho=5c\omega_{r1}^{-1}$.

$$E_\rho = E_{\rho C} + E_{\rho W} + E_{\rho P}, \quad E_z = E_{zC} + E_{zW} + E_{zP}, \quad H_\phi = H_{\phi C} + H_{\phi W}, \tag{3.3}$$

$$\begin{Bmatrix} E_{\rho C} \\ E_{zC} \\ H_{\phi C} \end{Bmatrix} = \frac{q}{c}\int_{\Omega_0}^{\infty} \begin{Bmatrix} \dfrac{s(i\omega)}{\beta\varepsilon(i\omega)} J_1(\rho s(i\omega)) \\ \dfrac{\omega[1-n^2(i\omega)\beta^2]}{c\beta^2\varepsilon(i\omega)} J_0(\rho s(i\omega))\,\mathrm{sgn}\,\zeta \\ s(i\omega)J_1(\rho s(i\omega)) \end{Bmatrix} \exp\left(-\omega\frac{|\zeta|}{V}\right) d\omega, \tag{3.4}$$

$$\begin{Bmatrix} E_{\rho W} \\ E_{zW} \\ H_{\phi W} \end{Bmatrix} = -\frac{2q}{c}\sum_{m=1}^{M}\int_{\Omega_m}^{\omega_{rm}} \begin{Bmatrix} \dfrac{s(\omega)}{\beta\varepsilon(\omega)} J_1(\rho s(\omega))\sin\left(\omega\dfrac{\zeta}{V}\right) \\ \dfrac{\mu}{c}\left(1-\dfrac{1}{n^2(\omega)\beta^2}\right)\omega J_0(\rho s(\omega))\cos\left(\omega\dfrac{\zeta}{V}\right) \\ s(\omega)J_1(\rho s(\omega))\sin\left(\omega\dfrac{\zeta}{V}\right) \end{Bmatrix} d\omega\Theta(-\zeta), \tag{3.5}$$

$$\begin{Bmatrix} E_{\rho P} \\ E_{zP} \end{Bmatrix} = \frac{4q}{c^2\beta^2}\Theta(-\zeta)\sum_{m=1}^{M}\frac{\omega}{d\varepsilon/d\omega}\left.\begin{Bmatrix} K_1\left(\frac{\omega\rho}{V}\right)\sin\left(\frac{\omega\zeta}{V}\right) \\ -K_0\left(\frac{\omega\rho}{V}\right)\cos\left(\frac{\omega\zeta}{V}\right) \end{Bmatrix}\right|_{\omega=\omega_{sm}}, \tag{3.6}$$

where

$$\Omega_0 = \begin{cases} 0 & \text{for } \beta < \beta_0 \\ |\omega_{c1}| & \text{for } \beta > \beta_0, \end{cases} \qquad \Omega_1 = \begin{cases} \omega_{c1} & \text{for } \beta < \beta_0 \\ 0 & \text{for } \beta > \beta_0, \end{cases}$$

$\Omega_m = \omega_{cm}$ at $m \geq 2$, $\beta_0 = n_0^{-1}$, $J_k(\xi)$ and $K_k(\xi)$ are, respectively, the Bessel function and the modified Hankel function of k order, and $\Theta(\xi)$ is the Heaviside step function: $\Theta(\xi)=0$ at $\xi<0$ and $\Theta(\xi)=1$ at $\xi>0$. Here, the index C is assigned to "quasi-Coulomb" parts, the index W is assigned to wave components, and the index P is assigned to "plasma trace" components. The "quasi-Coulomb" field exists both behind and in front of the moving charge and quickly decreases with distance from it. The wave field (radiation field) exists only behind the charge and oscillates with distance from it. Unlike the radiation field, the "plasma trace" is concentrated close to the charge movement trajectory and exponentially decreases with distance from it.

From formulas (3.3)–(3.6), it is easy to obtain the results for the case of a medium with one resonance frequency. For this purpose, it is sufficient to assume that $\omega_{pm}=0$ at $m \geq 2$ (there is the single integral on the interval of $[\Omega_1, \omega_{r1}]$ in Eq. (3.5)). Let us note that the problem of a field of a charge moving in a medium with one resonance frequency was considered in detail in Refs. [10,11]. In these works, expressions for components of the field are represented in the form of two integrals over segments of the real axis. One of them determines the radiation field, while the other one is the sum of the "quasi-Coulomb" field and the "plasma trace." The integrands, however, contain the Neumann function $N_0(\rho s)$, which has a singularity at $\rho s \to 0$. One advantage of expressions (3.4) and (3.5) is that they do not include the Neumann functions, thereby simplifying the numerical calculation of wave components of the field according to these formulas. Moreover, in formulas (3.4) and (3.5), all three components of the field (the wave and "quasi-Coulomb" components and the "plasma trace") are separated, which simplifies the physical interpretation of the obtained results.

The advantages of the approach stated above are important, for instance, for the analysis of the interaction of two charges moving one after another. It requires one to calculate the field on the charge movement axis, so the absence of a singularity at $\rho \to 0$ in Eqs. (3.4) and (3.5) represents an important advantage of these expressions. In Ref. [29], such a problem is studied for the case of a medium with one resonance frequency. A range of essential physical effects was discovered.

It is noted, however, that there is a more effective algorithm of computation. We can transform the contour of integration in the initial formulas (2.1) using the above analysis of properties of integrands. For relatively small values of ω, it is convenient to transform a contour such that it bypasses all branch points and poles. For the best convergence of an integral at great values of ω, it is necessary to transform a contour so that it is parallel to the asymptote of SDC (2.13). For $\zeta>0$, it is possible to use the contour consisting of two infinite rays parallel to the lines (2.13). For $\zeta<0$, it is possible to use the contour consisting of two trapezoidal dashed lines with half-infinite parts parallel to the asymptotes (2.13) [Figs. 1(a) and 1(b), part II, show the picture for the area Re $\omega>0$]. This transformation allows us to avoid the intersection of the integrands' singularities during the contour transformation. It is noted that the essential advantage of such an approach is that it is possible to choose the most convenient parameters of the dashed contour for the concrete parameters of the problem. We would remind the reader that an arbitrary integral over a symmetric (relative to the imaginary axis) contour is reduced to an integral over the contour part located only in the right half-plane.

Some examples of a calculation of the longitudinal component of an electrical field E_z in the case of a biresonance medium are given in Fig. 1, part III. As an example of the medium, we took a ruby that has two relatively close resonances in the terahertz frequency range. Other resonant frequencies are located far from this domain, so we did not consider them in the calculations. Let us remark that, in such a medium, a charge can move at approximately constant velocity only in a vacuum channel. It is known, however, that the presence of a channel does not affect the VCR if the lengths of the waves exceed the thickness of the channel [7]. Due to this, we can use the results obtained for the homogeneous medium. The model parameters $\omega_{p1,2}$, $\omega_{r1,2}$, and $\omega_{d1,2}$ were determined on the basis of known data [30] by means of interpolation.

Figures 1(a) and 1(b), part III, show the dependence of the component E_z (in units of $q\omega_{r1}^2 c^{-2}$) on distance $\zeta = z - Vt$ (in units of $c\omega_{r1}^{-1}$) at constant distance from the charge movement axis. One can see that the "quasi-Coulomb" component of the field is dominant close to the point $\zeta=0$. In the area $\zeta<0$, we see a typical pattern of wave interference. One can see, at considerable distance from the charge, beating harmonics with two relatively close frequencies (radiation is mainly generated at frequencies that are close to either ω_{r1} or ω_{r2}). Let us also mention that, at high velocities, the "quasi-Coulomb" field is insignificant in comparison with the wave field [Fig. 1(b), part III].

IV. EFFECT OF VAVILOV-CHERENKOV RADIATION AMPLIFICATION IN AN ACTIVE BIRESONANCE MEDIUM

In this section, we will consider the case of active media (or media with inverse occupancy). Such media are usually produced by optical pumping and represent the main components of lasers and masers. In recent years, the attention of researchers has been attracted to the so-called PASER (particle acceleration by stimulated emission of radiation). First of all, this trend is related to a problem in the development of new methods to generate powerful electromagnetic fields. These fields can be used in accelerators based on the wakefield technique [20–25]. In this area, the most important goal is achieving the maximum possible values of the electromagnetic fields. By using the PASER method, an active medium provides the energy that transforms into the energy of VCR, and it is then used to accelerate charged particles of the second bunch. The initial theoretical and experimental works concerning PASERs focus on acceleration in gaseous CO_2 and ammonia laser media [20–23]. The first proof-of-principle experiment on direct particle acceleration by stimulated emission of radiation has been published recently [22,23].

It is necessary to emphasize that active media are characterized by resonant-type dispersion. Waveguide structures, which are completely or partially filled with an active dispersive medium having a single resonance frequency, were considered in Refs. [20,21,24]. It was demonstrated that amplification of VCR is possible even if the refractive index of the medium is purely real [20,21]. The current section demonstrates that amplification of VCR in an active medium with a purely real refractive index is possible without a waveguide. This phenomenon can take place if the medium possesses two resonant frequencies. Furthermore, we consider a medium having resonant frequencies ω_{r1} and ω_{r2} ($\omega_{r2}>\omega_{r1}$). Let us write the expression for the refractive index in the usual form:

$$n^2(\omega)=\varepsilon(\omega)\mu(\omega)=1+\frac{\omega_{p1}^2}{\omega_{r1}^2-2i\omega_{d1}\omega-\omega^2}+\frac{\omega_{p2}^2}{\omega_{r2}^2-2i\omega_{d2}\omega-\omega^2}. \quad (4.1)$$

Let us note that the values $\omega_{p1,2}^2$ are not necessarily positive. In active media, at least one of these parameters is negative; i.e., the respective "plasma frequency" $\omega_{p1,2}$ has an imaginary value [20] (the values of $\omega_{r1,2}$ and $\omega_{d1,2}$ are positive and real, the same as in passive media).

Let us first consider the case when $\omega_{d1}=\omega_{d2}=0$. In this case, the function $s^2(\omega)=\omega^2V^{-2}[n^2(\omega)\beta^2-1]$ can be presented in the following form:

$$s^2(\omega)=-\frac{1-\beta^2}{V^2}\frac{\omega^2(\omega^2-\omega_1^2)(\omega^2-\omega_2^2)}{(\omega^2-\omega_{r1}^2)(\omega^2-\omega_{r2}^2)}, \quad (4.2)$$

where

$$\omega_{1,2}^2=\frac{1}{2}[\omega_{r1}^2+\omega_{r2}^2-\alpha(\omega_{p1}^2+\omega_{p2}^2)\mp\sqrt{D}], \quad (4.3)$$

$$D=[\omega_{r1}^2+\omega_{r2}^2-\alpha(\omega_{p1}^2+\omega_{p2}^2)]^2-4\omega_{r1}^2\omega_{r2}^2+4\alpha(\omega_{p1}^2\omega_{r2}^2+\omega_{p2}^2\omega_{r1}^2), \quad (4.4)$$

$$\alpha=\beta^2(1-\beta^2)^{-1}. \quad (4.5)$$

If $D>0$, the values $\omega_{1,2}^2$ are real, and the roots $\omega_1^{\pm}=\pm\sqrt{\omega_1^2}$ and $\omega_2^{\pm}=\pm\sqrt{\omega_2^2}$ are either purely real or purely imaginary [it is assumed that zeros with index (+) have a non-negative real part; if they are purely imaginary, their imaginary part is positive]. If $D<0$, then the zeros have both real and imaginary parts. It is noted that the following relations take place: $\omega_1^+=-\omega_1^-=\overline{\omega_2^+}=-\overline{\omega_2^-}$, where the overbar designates complex conjugation. The determinant (4.4) can also be represented in the following form:

$$D=[\omega_{r2}^2-\omega_{r1}^2-\alpha(\omega_{p1}^2+\omega_{p2}^2)]^2+4\alpha\omega_{p1}^2(\omega_{r2}^2-\omega_{r1}^2)=[\omega_{r2}^2-\omega_{r1}^2+\alpha(\omega_{p1}^2+\omega_{p2}^2)]^2-4\alpha\omega_{p2}^2(\omega_{r2}^2-\omega_{r1}^2). \quad (4.6)$$

Since $\omega_{r2}>\omega_{r1}$, the determinant D is positive for any parameters of the problems if one of the following situations takes place:

$$\omega_{p1}^2>0,\quad \omega_{p2}^2>0,$$

$$\omega_{p1}^2>0,\quad \omega_{p2}^2<0,$$

$$\omega_{p1}^2<0,\quad \omega_{p2}^2<0. \quad (4.7)$$

The value of D can be negative only in the following case:

$$\omega_{p1}^2<0,\quad \omega_{p2}^2>0, \quad (4.8)$$

i.e., in the case when the lower resonance is "active" and the upper one is "passive." This condition is, however, necessary, though not sufficient. By considering D as a quadratic polynomial with respect to α, it is easy to demonstrate that $D<0$ if the conditions (4.8) are supplemented with the following inequalities:

$$\frac{\omega_{r2}^2-\omega_{r1}^2}{(\omega_{p2}+|\omega_{p1}|)^2}<\alpha<\frac{\omega_{r2}^2-\omega_{r1}^2}{(\omega_{p2}-|\omega_{p1}|)^2}. \quad (4.9)$$

They are equivalent to the inequalities

$$\beta_{\min}<\beta<\beta_{\max},\quad \beta_{\substack{\min\\ \max}}=\sqrt{\frac{\omega_{r2}^2-\omega_{r1}^2}{(\omega_{p2}\pm|\omega_{p1}|)^2+\omega_{r2}^2-\omega_{r1}^2}}. \quad (4.10)$$

Furthermore, we will consider the most interesting case when conditions (4.8) and (4.9) are fulfilled. The dependence of the squared refractive index on frequency for such a situation is shown in Fig. 2. As we see, the straight line β^{-2} does not cross the curve $n^2(\omega)$. It is related to the complex character of the zeros $\omega_{1,2}^{\pm}$ of the function $s^2(\omega)$. These zeros, along with the resonant frequencies, are the branch points of the integrands (2.2). Real positive frequencies of the radiated waves lie within the range of $\omega_{r1}<\omega<\omega_{r2}$, where the condition $n^2\beta^2>1$ is fulfilled [(i.e., $s^2(\omega)>0$]. For these waves, the Mandelshtam's radiation condition is fulfilled. In accor-

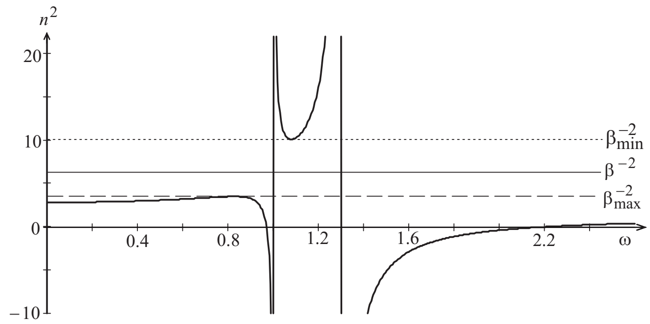

FIG. 2. Dependence of the squared refractive index of an active biresonance medium on frequency ω (in units of ω_{r1}) for the case $|\omega_{p1}^2| < \omega_{p2}^2$.

dance with this condition $V_{g\rho} > 0$, $\operatorname{sgn} s(\omega) = \operatorname{sgn}(ds^2/d\omega)$ (see Sec. II). It is easy to demonstrate that $ds^2/d\omega = 0$ at some point lying on the segment $\omega_{r1} < \omega < \omega_{r2}$ (see Fig. 2), as well as at the symmetrical point on the segment $-\omega_{r2} < \omega < -\omega_{r1}$. Therefore, the function $s(\omega)$ changes its sign at these points; i.e., it has a break. This is possible only in cases when these points lie on the cuts separating the "physical" sheet of the Riemann surface from the "nonphysical" one.

The view of the cuts and the integration contour in the right half-plane are shown in Fig. 3 (in the left half-plane, the picture is symmetrical with respect to the imaginary axis). It is not difficult to demonstrate that the requirement $\operatorname{sgn} s(\omega) = \operatorname{sgn}(ds^2/d\omega)$ is fulfilled if the branch points $\pm\omega_{r1,2}$ are passed from above (Fig. 3). The cuts connecting point ω_1^+ with point ω_2^+ and point ω_1^- with point ω_2^- are passed from above as well. (If these cuts were passed from below, we would obtain a wave field in the domain in front of the charge. This is impossible.) If $n^2 = \varepsilon$, then there are the poles $\pm\omega_{s1,2}$ [zeros of function $\varepsilon(\omega)$]. They determine the "plasma trace" of the source. They are passed from above, as the "plasma trace" can exist only in the domain behind the source [6]. Let us note that the contour passes between the branch points $\pm i\delta(\delta \to +0)$ of function $\sqrt{\omega^2} = \lim_{\delta\to+0}\sqrt{\omega^2+\delta^2}$, which is included in $s(\omega)$. The cuts going out from these points determine the quasistatic field, as in case of passive medium.

The fact that part of the contour is located in the upper half-plane predetermines the effect of the amplification of VCR. It is not difficult to show that, as in the case of a

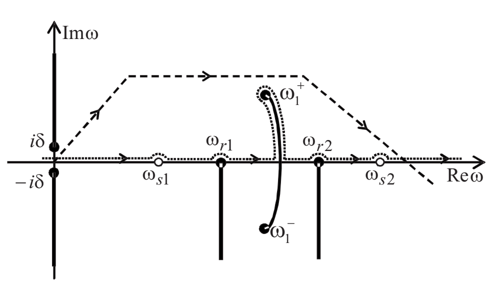

FIG. 3. General view of cuts (bold solid lines) and integration contours for the amplification regime ($\beta_{min} < \beta < \beta_{max}$). The dotted line shows the initial contour of integration, and the dashed line shows the contour for numerical calculation in the domain behind the charge ($\zeta < 0$).

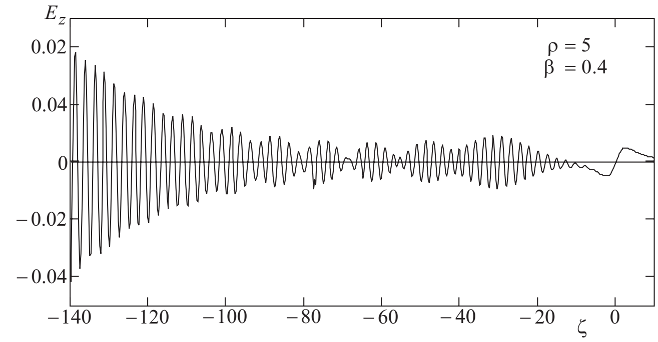

FIG. 4. Dependence of longitudinal component of electrical field E_z (in units of $q\omega_{r1}^2 c^{-2}$) on $\zeta = z - Vt$ (in units of c/ω_{r1}) for ruby in the amplification regime at $\omega_{p1} = 2\pi i \times 1.3 \times 10^{12}$ s^{-1} and $\rho = 5c\omega_{r1}^{-1}$ (other parameters of the medium are the same as in Fig. 1, part III).

passive medium (Sec. II), all the integrals for the field components over the contour Γ, which is symmetrical with respect to the imaginary axis, can be reduced to integrals over its right half Γ_+ [see formula (2.10)]. Therefore, we consider further integrals only over contour Γ_+. Let us note that we do not need to define the precise geometry of the cuts as the contour of integration, Γ_+, can be transformed in such a way that it would be located far from the singularities. In the same way as in the case for a passive medium, for the field in the half-space behind the charge ($\zeta < 0$), it is convenient to replace the initial contour with a dashed line passing from above the poles and branch points and parallel to the asymptote of the steepest descent contour at infinity (Fig. 3). Such a transformation allows us to avoid calculating integrals in the neighborhood of branch points, and it provides good convergence at infinity.

Some results of the calculations are shown in Fig. 4. Let us note that, for the purpose of these calculations, nonzero values of parameters ω_{d1} and ω_{d2} were taken into account (the algorithm described enables us to do this). In the same way as in the case of a passive medium, we will consider a ruby as an example of a medium with two relatively close resonant frequencies. It is assumed that the value $|\omega_{p1}^2|$ in the active regime is 100th of that in the passive regime. In such a situation $\beta_{min} = 0.38$ and $\beta_{max} = 0.42$. As seen in Fig. 4, considerable amplification of VCR takes place at $\beta = 0.4$ despite an negligible inversion of the lower resonance. A similar effect should take place for any active media with two resonant frequencies if the conditions described above have been fulfilled.

It should be noted that the obtained results describe the initial stage of wakefield amplification only. In reality, the field magnitude does not increase permanently with an increase of $|\zeta|$; instead, it reaches some maximum value determined by the energy stored within the medium. The linear approach used in this paper does not describe the saturation process. Such a process can be the object of special research. Nevertheless, it should be emphasized that the analysis carried out in the linear approach is significant because it allowed the discovery of the conditions for wakefield amplification.

V. CONCLUSION

Let us note the main results obtained in this work. First of all, based on the methods of the complex variable functions theory, a new approach to the analysis of a moving charge field in media with arbitrary frequency dispersion has been developed. The general properties of integrands determining field components have been obtained. This provides new opportunities for both an analytical and a numerical analysis of moving charge fields in media with various dispersion behaviors.

As one of the examples of an application of the approach developed, the field of a charge moving in a passive medium with an arbitrary number of resonances has been considered. The field has been presented in the form of the sum of the "quasi-Coulomb" field, the wave field, and the "plasma trace." An efficient method to numerically calculate field components has been developed.

The case of an active medium having two resonant frequencies has been considered as well. It has been demonstrated that radiation can be amplified even with a purely real refractive index. The conditions for realizing such an effect are nontrivial: the "lower" resonance should be active, the "upper" one should be passive, and the charge velocity should lie within a certain range. Numerical analysis of the field has been carried out, taking into consideration a nonzero imaginary part of the refractive index. Preliminary transformation of the integration contour has been used, which has enabled us to develop an efficient algorithm of numerical calculation. The examined example has illustrated a considerable amplification of VCR even at a small inversion of the lower resonance.

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