Integrable dynamics of Toda type on square and triangular lattices

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In a recent paper we constructed an integrable generalization of the Toda law on the square lattice. We construct other examples of integrable dynamics of Toda type on the square lattice, as well as on the triangular lattice, as nonlinear symmetries of the discrete Laplace equations on square and triangular lattices. We also construct the τ-function formulations and Darboux-Bäcklund transformations of these dynamics.

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I. INTRODUCTION

The Toda lattice [1–3]

$$\frac{d^2 q_m}{dt^2} = \Delta_m e^{\Delta_m q_{m-1}}, \tag{1}$$

where $\Delta_m f_m = f_{m+1} - f_m$ is the difference operator and $q_m(t)$ is a dynamical function on a one-dimensional lattice, is one of the most well-known integrable nonlinear lattice equations. It describes the dynamics of a one-dimensional physical lattice, the masses of which are subjected to an interaction potential of exponential type. The infinite, finite and periodic Toda lattice (1), as well as its numerous extensions [4–12], have applications in various other physical and mathematical contexts [13–19].

Its integrability properties follow from the basic fact that Eq. (1) is the compatibility condition for the following "Lax pair" [20,21]:

$$A_m \Psi_{m+1} + A_{m-1}\Psi_{m-1} = F_m \Psi_m + \lambda \Psi_m, \tag{2}$$

$$\frac{d\Psi_m}{dt} = \frac{1}{2}(A_m \Psi_{m+1} - A_{m-1}\Psi_{m-1}), \tag{3}$$

where λ is the constant eigenvalue of the self-adjoint three-point scheme (2), the eigenfunction $\Psi_m(t,\lambda)$ solves simultaneously the Lax pair (2) and (3), and the dynamical fields $A_m(t)$ and $F_m(t)$, solutions of the nonlinear equations

$$\frac{dF_m}{dt} + \Delta_m A_{m-1}^2 = 0, \quad \frac{1}{A_m}\frac{dA_m}{dt} + \frac{1}{2}\Delta_m F_m = 0, \tag{4}$$

are related to the Toda field $q_m(t)$ in the following way:

$$F_m = -\frac{dq_m}{dt}, \quad A_m = e^{(1/2)\Delta_m q_m}. \tag{5}$$

Indeed, from such a Lax pair, using a standard procedure common to all soliton equations in $(1+1)$ dimensions (see, for instance, [1,22–24]), (i) one constructs the Darboux-Bäcklund transformations (DBTs), discrete symmetries of Eqs. (1)–(3), allowing one to generate recursively explicit analytic solutions of (1)–(3), from simpler analytic solutions; (ii) one solves the Cauchy problem for the infinite and periodic Toda chains, through, respectively, the inverse spectral transform (IST) [22–24] and the finite-gap method [25–27] applied to the eigenvalue problem (2).

Motivated by the numerous applications of the Toda lattice (1) and by its powerful integration scheme, we find it important to construct integrable generalizations of the Toda law (1) to regular planar lattices—i.e., to the square, triangular, and honeycomb lattices. To achieve this goal, one needs to identify proper two-dimensional generalizations of the one-dimensional self-adjoint spectral problem (2) associated with (1). Since (2) is an "integrable" discretization of the one-dimensional stationary Schrödinger spectral problem [where by integrable we now mean that the corresponding operator admits, as its continuous counterpart, a large set of continuous and discrete symmetries, like the Laplace and Darboux transformations (DTs)], such a project requires the identification of proper integrable discretizations of self-adjoint second-order operators on the plane first. A key progress in this direction was made in [28], where it was established that the self-adjoint scheme on the star of the triangular lattice admits Laplace transformations, and in [29,30], where it was established that the self-adjoint schemes on the stars of the square, triangular, and honeycomb lattices admit DTs as their natural continuous counterparts. In addition, in [30], a novel discrete time dynamics on the triangular lattice was introduced, in connection with its Laplace transformation. To construct integrable nonlinear dynamics associated with these self-adjoint operators, gauge equivalent to the discrete Laplace equations on weighted graphs, is the main goal of the paper.

It is necessary to mention that these three planar schemes (on the square, triangular, and honeycomb lattices) are directly connected (see [31] and [30]), via the sublattice approach [31], to the so-called discrete Moutard [32,33] (or B-quadrilateral [34]) lattice in $\mathbb{Z}^N$, and therefore they are all reductions of the multidimensional (planar) quadrilateral lattice [35–38]. We also remark that the above three linear schemes are distinguished examples of Laplace equations on graphs, obtainable from the discrete Moutard equations on bipartite planar quad-graphs [39–42].

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Using the self-adjoint scheme on the star of the square lattice as spectral problem, we have recently constructed in [43] an integrable generalization of the Toda law on the square lattice. In this paper we construct other examples of integrable dynamics of Toda type on the square lattice, as well as on the triangular lattice. More precisely, we first introduce, using the self-adjoint scheme on the star of the square lattice as spectral problem, the following nonlinear dynamical system on the regular square lattice:

$$\frac{dF_{m,n}}{dt}+\frac{1}{\xi_{m,n}}\Delta_m(\xi_{m,n}\xi_{m-1,n}A^2_{m-1,n})+\frac{1}{\eta_{m,n}}\Delta_n(\eta_{m,n}\eta_{m,n-1}B^2_{m,n-1})=0,$$

$$\frac{1}{A_{m,n}}\frac{dA_{m,n}}{dt}+\frac{1}{2}\Delta_1(\xi_{m,n}F_{m,n})=0,$$

$$\frac{1}{B_{m,n}}\frac{dB_{m,n}}{dt}+\frac{1}{2}\Delta_2(\eta_{m,n}F_{m,n})=0,$$

$$A_{m,n}B_{m,n}(\xi_{m,n}+\eta_{m,n})=A_{m,n+1}B_{m+1,n}(\xi_{m+1,n+1}+\eta_{m+1,n+1}),$$

$$A_{m,n+1}B_{m,n}(\xi_{m,n+1}-\eta_{m,n+1})=A_{m,n}B_{m+1,n}(\xi_{m+1,n}-\eta_{m+1,n}), \tag{6}$$

corresponding to the Lax pair

$$A_{m,n}\Psi_{m+1,n}+A_{m-1,n}\Psi_{m-1,n}+B_{m,n}\Psi_{m,n+1}+B_{m,n-1}\Psi_{m,n-1}=F_{m,n}\Psi_{m,n}, \tag{7a}$$

$$\frac{d\Psi_{m,n}}{dt}=\frac{\xi_{m,n}}{2}(A_{m,n}\Psi_{m+1,n}-A_{m-1,n}\Psi_{m-1,n})+\frac{\eta_{m,n}}{2}(B_{m,n}\Psi_{m,n+1}-B_{m,n-1}\Psi_{m,n-1}), \tag{7b}$$

where $A_{m,n}(t)$, $B_{m,n}(t)$, $F_{m,n}(t)$, $\xi_{m,n}(t)$, and $\eta_{m,n}(t)$ are dynamical functions on the square lattice and $\Psi_{m,n}(\lambda,t)$ is the eigenfunction defined on its vertices.

Using instead the self-adjoint scheme on the star of the triangular lattice as spectral problem, we introduce the following nonlinear dynamical system on the regular triangular lattice:

$$\frac{dF}{dt}+\frac{1}{\xi}\Delta_1(\xi\xi_{-1}A^2_{-1})+\frac{1}{\eta}\Delta_2(\eta\eta_{-2}B^2_{-2})+\frac{1}{\zeta}\Delta_3(\zeta\zeta_{-3}C^2_{-3})=0, \tag{8a}$$

$$\frac{1}{A}\frac{dA}{dt}+\frac{1}{2}\Delta_1(\xi F)-\frac{1}{2}\frac{BC_1}{A}(\eta+\zeta)_2+\frac{1}{2}\frac{B_{-3}C_{-3}}{A}(\eta+\zeta)_{-3}=0, \tag{8b}$$

$$\frac{1}{B}\frac{dB}{dt}+\frac{1}{2}\Delta_2(\eta F)+\frac{1}{2}\frac{A_3C}{B}(\xi-\zeta)_3-\frac{1}{2}\frac{AC_1}{B}(\xi-\zeta)_1=0, \tag{8c}$$

$$\frac{1}{C}\frac{dC}{dt}+\frac{1}{2}\Delta_3(\zeta F)+\frac{1}{2}\frac{B_{-1}A_{-1}}{C}(\xi+\eta)_{-1}-\frac{1}{2}\frac{A_3B}{C}(\xi+\eta)_2=0, \tag{8d}$$

$$AB_1(\xi-\eta)_1=A_2B(\xi-\eta)_2, \tag{8e}$$

$$AC(\xi+\zeta)=A_3C_1(\xi+\zeta)_2, \tag{8f}$$

$$B_{-3}C_1(\eta-\zeta)_1=BC_{-3}(\eta-\zeta), \tag{8g}$$

corresponding to the Lax pair

$$A\Psi_1+A_{-1}\Psi_{-1}+B\Psi_2+B_{-2}\Psi_{-2}+C\Psi_3+C_{-3}\Psi_{-3}=F\Psi,$$

$$\frac{d\Psi}{dt}=\frac{\xi}{2}(A\Psi_1-A_{-1}\Psi_{-1})+\frac{\eta}{2}(B\Psi_2-B_{-2}\Psi_{-2})+\frac{\zeta}{2}(C\Psi_3-C_{-3}\Psi_{-3}), \tag{9}$$

where A, B, C, F, ξ, η, and ζ are dynamical functions on the regular triangular lattice and $\Psi(\lambda,t)$ is the eigenfunction defined on its vertices. For Eqs. (8a)–(8g) and (9) we find it convenient to use the concise notation $f_i=T_if$, where T_i, $i=1,2,3$, are the three elementary forward translation operators on the triangular lattice (such that $T_1T_3=T_2$), and $f_{-i}=T_i^{-1}f$.

In addition, we construct the natural reductions of the systems (6) and (8a)–(8g), we present their τ-function formulations, in which the τ function of the BKP hierarchy [44] plays a central role due to the already mentioned common origin of the associated spectral problems, and we derive their DBTs.

As most of the integrable multidimensional generalizations of soliton equation in (1+1) dimensions [45,46], the (2+1)-dimensional generalizations (6) and (8a)–(8g) of the Toda law (1) are nonlocal, introducing the auxiliary fields ξ, η, and ζ coupled to the main fields A, B, and C through relations not involving t derivatives. In analogy with the theory first developed in [47,48] for the Davey-Stewartson 1 (DS1) equation [49], an integrable (2+1)-dimensional generalization of the celebrated nonlinear Schrödinger equation [50], well-posed initial-boundary value (IBV) problems for (6) and (8a)–(8g) on the whole lattices, solvable by the IST associated with the self-adjoint spectral problems (7a) and (8a), can be constructed. For instance, for the system (6) in which the auxiliary fields $(\xi+\eta)$ and $(\xi-\eta)$ are constructed along the two main diagonals of the square lattice from the main fields A and B, one can assign arbitrarily (i) the main fields A, B, and F at $t=0$ on the whole lattice, going to constant values as $m^2+n^2\to\infty$; (ii) the auxiliary field $(\xi+\eta)$, say, at the end $\mu=m+n=-\infty$ of the main diagonal, as an arbitrary function f^+ of $\nu=n-m$ and t; and (iii) the auxiliary field $(\xi-\eta)$, say, at the end $\nu=-\infty$ of the second diagonal, as an arbitrary function f^- of $\mu=m+n$ and t:

$$\xi_{m,n}(t) \pm \eta_{m,n}(t) \to f^{\pm}_{n\mp m}(t), \quad n \pm m \to -\infty, \tag{10}$$

where both fields $f^{\pm}$ are localized in the space variable. With these prescriptions, one constructs well-defined functions $\xi_{m,n}(t)$ and $\eta_{m,n}(t)$ in terms of A and B:

$$\xi_{m,n} + \eta_{m,n} = \prod_{j=0}^{\infty} \frac{A_{m-1-j,n-1-j}B_{m-1-j,n-1-j}}{A_{m-1-j,n-j}B_{m-j,n-1-j}} f^{+}_{n-m}(t),$$

$$\xi_{m,n} - \eta_{m,n} = \prod_{j=0}^{\infty} \frac{A_{m+j,n-1-j}B_{m+1+j,n-1-j}}{A_{m+j,n-j}B_{m+j,n-1-j}} f^{-}_{n+m}(t). \tag{11}$$

Applying instead the finite-gap theory to the self-adjoint spectral problems (7a) and (8a), the periodic problem for the systems (6) and (8a)–(8g) can be, in principle, investigated.

The paper is organized as follows. In Sec. II we construct an integrable dynamics of Toda type on the square lattice, invariant under $\pi/2$ rotation, its τ-function formulation, and its two natural reductions transforming into each other under a $\pi/2$ rotation. One of these two reductions coincides with the two-dimensional Toda system introduced in [43]. In Sec. III we construct an integrable dynamics of Toda type on the triangular lattice, invariant under a $\pi/3$ rotation, its τ-function formulation, and its natural reductions. The DBTs for all the above systems are presented in Sec. IV.

II. DYNAMICS OF THE SQUARE LATTICE

In this section we construct examples of integrable dynamics of Toda type on the square lattice. To simplify the form of the equations, we will be using the following notation: f instead of $f_{m,n}$, $f_{\pm 1}$ instead of $f_{m\pm 1,n}$, $f_{\pm 2}$ instead of $f_{m,n\pm 1}$, $f_{\pm 1\pm 2}$ instead of $f_{m\pm 1,n\pm 1}$, $f_{\pm 1\pm 1}$ instead of $f_{m\pm 2,n}$, and $f_{\pm 2\pm 2}$ instead of $f_{m,n\pm 2}$. Moreover, we denote by T_1 and T_2 the basic translation operators acting on the lattice—i.e., $T_i f = f_i$, $i=1,2$.

The dynamics wanted are associated with the linear self-adjoint five-point scheme

$$A\Psi_1 + A_{-1}\Psi_{-1} + B\Psi_2 + B_{-2}\Psi_{-2} = F\Psi \tag{12}$$

on the star of the square lattice, involving its black center (●) and the four vertices of the star, denoted by the symbol □ in Fig. 1. In the spectral problem (12), the eigenfunction Ψ is defined at the vertices of the graph, while the fields A,B are defined on the nonoriented edges of the lattice. Equation (12), a natural discretization of the self-adjoint second-order equation

$$(a\psi_x)_x + (b\psi_y)_y = f\psi, \tag{13}$$

admits, like its continuous counterpart, DTs [29].

In analogy with Eq. (3), we restrict our investigation to evolution equations for the eigenfunction Ψ involving only the four vertices □ of the five-point scheme:

$$\frac{d\Psi}{dt} = \alpha\Psi_1 + \beta\Psi_{-1} + \gamma\Psi_2 + \delta\Psi_{-2}, \tag{14}$$

where the fields α, β, γ, and δ, defined on the oriented edges of the lattice, will be specified in the following. A term proportional to Ψ in (14) can always be expressed, using (12), in terms of the values of Ψ at the four vertices □ of the star; therefore, it is omitted.

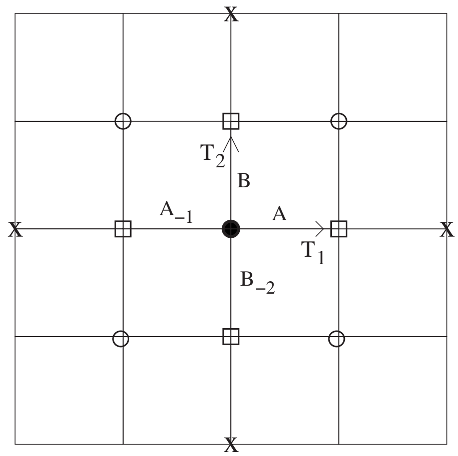

FIG. 1. The square lattice and the points involved in the commutation.

We remark that, due to the $\pi/2$-rotation symmetry of the square lattice, under which the two basic translations T_1 and T_2 transform as

$$T_1 \to \tilde{T}_1 = T_2, \quad T_2 \to \tilde{T}_2 = T_1^{-1}, \tag{15}$$

the coefficients of the five-point scheme and of the evolution equation (14) are subjected to the following transformations:

$$A \to \tilde{A} = B, \quad B \to \tilde{B} = A_{-1}, \quad F \to \tilde{F} = F,$$

$$\alpha \to \tilde{\alpha} = \gamma, \quad \beta \to \tilde{\beta} = \delta, \quad \gamma \to \tilde{\gamma} = \beta, \quad \delta \to \tilde{\delta} = \alpha. \tag{16}$$

The compatibility between Eqs. (12) and (14) leads to an equation involving the values of Ψ at all the marked points ○, ●, □, and x in Fig. 1. Using the scheme (12) centered at the origin and at the points □, one expresses the values of Ψ at the origin and at the points x in terms of the eight independent values of Ψ at the points □ and ○. As a result of this procedure, the compatibility condition becomes a linear equation for the eight independent values of Ψ at the points □ and ○. Equating to zero their eight coefficients, one obtains a determined system of eight nonlinear equations for the eight coefficients A, B, C, F, α, β, γ, and δ.

In the rest of this section we report the results of the analysis of such system leading to Toda-type dynamics.

A. Rotationally invariant dynamics

Setting

$$\alpha = \frac{\xi}{2}A, \quad \beta = -\frac{\xi}{2}A_{-1}, \quad \gamma = \frac{\eta}{2}B, \quad \delta = -\frac{\eta}{2}B_{-2}, \tag{17}$$

where ξ and η are lattice fields to be specified, the corresponding evolution for Ψ,

$$\frac{d\Psi}{dt}=\frac{\xi}{2}(A\Psi_1-A_{-1}\Psi_{-1})+\frac{\eta}{2}(B\Psi_2-B_{-2}\Psi_{-2}), \qquad (18)$$

is compatible with the five-point scheme (12) iff (up to a trivial gauge transformation) the coefficients A, B, F, ξ, and η satisfy the following determined system of five nonlinear equations:

$$\frac{dF}{dt}+\frac{1}{\xi}\Delta_1(\xi\xi_{-1}A_{-1}^2)+\frac{1}{\eta}\Delta_2(\eta\eta_{-2}B_{-2}^2)=0, \qquad (19)$$

$$\frac{1}{A}\frac{dA}{dt}+\frac{1}{2}\Delta_1(\xi F)=0, \quad \frac{1}{B}\frac{dB}{dt}+\frac{1}{2}\Delta_2(\eta F)=0, \qquad (20)$$

$$AB(\xi+\eta)=A_2B_1(\xi+\eta)_{12}, \quad A_2B(\xi-\eta)_2=AB_1(\xi-\eta)_1. \qquad (21)$$

Equations (20) suggest the introduction of the new fields q,r defined by

$$\frac{dq}{dt}=-\xi F, \quad \frac{dr}{dt}=-\eta F. \qquad (22)$$

With this choice:

$$A=ae^{(1/2)\Delta_1 q}, \quad B=be^{(1/2)\Delta_2 r}, \qquad (23)$$

where a and b are arbitrary constants. Choosing, without loss of generality, $a,b=1$, the system (19)–(21) takes the form of the following novel integrable generalization of the Toda exponential law of interaction to the square lattice:

$$\xi\eta\frac{d}{dt}\left(\frac{1}{\xi}\frac{dq}{dt}\right)=\eta\Delta_1(\xi\xi_{-1}e^{\Delta_1 q_{-1}})+\xi\Delta_2(\eta\eta_{-2}e^{-\Delta_2 r_{-2}}), \qquad (24a)$$

$$\xi\frac{dr}{dt}=\eta\frac{dq}{dt}, \qquad (24b)$$

$$\frac{(\xi+\eta)}{(\xi+\eta)_{12}}=e^{\Delta_1\Delta_2(q+r)/2}, \qquad (24c)$$

$$\frac{(\xi-\eta)_1}{(\xi-\eta)_2}=e^{\Delta_1\Delta_2(q-r)/2}. \qquad (24d)$$

Remark 1. In the natural one-dimensional limit in which all the fields are invariant under the T_2 translation, Eqs. (24c) and (24d) imply that ξ and η are constant, and Eq. (24a) reduces to the one-dimensional Toda lattice (1).

Remark 2. Using (23) with $a=b=1$, the five-point scheme (12) takes the following form:

$$\frac{\Gamma}{\Gamma_1}\Psi_1+\frac{\Gamma_{-1}}{\Gamma}\Psi_{-1}+\frac{\hat{\Gamma}}{\hat{\Gamma}_2}\Psi_2+\frac{\hat{\Gamma}_{-2}}{\hat{\Gamma}}\Psi_{-2}=F\Psi,$$

$$\Gamma=e^{-q/2}, \quad \hat{\Gamma}=e^{-r/2}. \qquad (25)$$

It is easy to verify that the spectral problem (25) reduces, in the natural continuous limit, to the stationary Schrödinger equation in the plane: $\Psi_{xx}+\Psi_{yy}+u\Psi=0$. It is therefore a natural integrable discretization of the Schrödinger operator, more general than that introduced in [29].

Remark 3. Using (15)–(17) and (22), it is easy to verify that, under a $\pi/2$ rotation,

$$\xi\to\xi'=\eta, \quad \eta\to\eta'=-\xi, \quad q\to q'=r, \quad r\to r'=-q, \qquad (26)$$

from which it follows that the system (19)–(21) [or (24a)–(24d)] is invariant under this transformation.

B. Reductions not invariant under rotation

The system (19)–(21) [or (24a)–(24d)] admits two distinguished reductions for $\xi=\pm\eta$.

(i) *The reduction* $\xi=\eta$. In this case, the Lax pair (12) and (18) reduces to

$$A\Psi_1+A_{-1}\Psi_{-1}+B\Psi_2+B_{-2}\Psi_{-2}=F\Psi,$$

$$\frac{d\Psi}{dt}=\frac{\xi}{2}(A\Psi_1-A_{-1}\Psi_{-1}+B\Psi_2-B_{-2}\Psi_{-2}), \qquad (27)$$

and the nonlinear dynamics (19)–(21) reduces to

$$\frac{dF}{dt}+\frac{1}{\xi}[\Delta_1(\xi\xi_{-1}A_{-1}^2)+\Delta_2(\xi\xi_{-2}B_{-2}^2)]=0, \qquad (28a)$$

$$\frac{1}{A}\frac{dA}{dt}+\frac{1}{2}\Delta_1(\xi F)=0, \quad \frac{1}{B}\frac{dB}{dt}+\frac{1}{2}\Delta_2(\xi F)=0, \qquad (28b)$$

$$AB\xi=A_2B_1\xi_{12}. \qquad (28c)$$

Integrating Eqs. (28b) and (28c) and using (22), which implies that $r=q(\hat{\Gamma}=\Gamma)$, one recovers the Toda-type system

$$\xi\frac{d}{dt}\left(\frac{1}{\xi}\frac{dq}{dt}\right)=\Delta_1(\xi\xi_{-1}e^{\Delta_1 q_{-1}})+\Delta_2(\xi\xi_{-2}e^{\Delta_2 q_{-2}}),$$

$$\frac{\xi}{\xi_{12}}=e^{\Delta_1\Delta_2 q}, \qquad (29)$$

introduced in [43], whose associated five-point scheme is the discrete Schrödinger equation

$$\frac{\Gamma}{\Gamma_1}\Psi_1+\frac{\Gamma_{-1}}{\Gamma}\Psi_{-1}+\frac{\Gamma}{\Gamma_2}\Psi_2+\frac{\Gamma_{-2}}{\Gamma}\Psi_{-2}=F\Psi,$$

$$A=\frac{\Gamma}{\Gamma_1}, \quad B=\frac{\Gamma}{\Gamma_2}, \quad \Gamma=e^{-q/2}, \quad F=-\frac{q}{\xi}, \qquad (30)$$

introduced in [29].

(ii) *The reduction* $\xi=-\eta$. In this case, the time evolution of Ψ reads

$$\frac{d\Psi}{dt}=\frac{\xi}{2}(A\Psi_1-A_{-1}\Psi_{-1}-B\Psi_2+B_{-2}\Psi_{-2}), \qquad (31)$$

and the nonlinear dynamics (19)–(21) reduces to:

$$\frac{dF}{dt}+\xi_1A^2-\xi_{-1}A_{-1}^2-\xi_2B^2+\xi_{-2}B_{-2}^2=0,$$

$$\frac{1}{A}\frac{dA}{dt} + \frac{1}{2}\Delta_1(\xi F) = 0, \quad \frac{1}{B}\frac{dB}{dt} - \frac{1}{2}\Delta_2(\xi F) = 0,$$

$$A_2 B \xi_2 = A B_1 \xi_1. \tag{32}$$

Equivalently, using (22) and noting that, in this case, $r=-q(\hat{\Gamma}=1/\Gamma)$, one obtains the Toda-type system

$$\xi\frac{d}{dt}\left(\frac{1}{\xi}\frac{dq}{dt}\right) = \Delta_1(\xi\xi_{-1}e^{\Delta_1 q_{-1}}) - \Delta_2(\xi\xi_{-2}e^{-\Delta_2 q_{-2}}),$$

$$\frac{\xi_1}{\xi_2} = e^{\Delta_1\Delta_2 q}, \tag{33}$$

whose five-point scheme is another variant of the discrete Schrödinger equation

$$\frac{\Gamma}{\Gamma_1}\Psi_1 + \frac{\Gamma_{-1}}{\Gamma}\Psi_{-1} + \frac{\Gamma_2}{\Gamma}\Psi_2 + \frac{\Gamma}{\Gamma_{-2}}\Psi_{-2} = F\Psi. \tag{34}$$

We end this section remarking that, due to the transformations (16) and (26), the reduced systems (29) and (33) transform into each other under a $\pi/2$ rotation.

C. τ-function formulations

Motivated by the sublattice approach [31] for the self-adjoint five-point scheme (12), we introduce two potentials τ and $\hat{\tau}$ via the equations

$$A = \frac{\tau_1\tau}{\hat{\tau}\hat{\tau}_{-2}}, \quad B = \frac{\tau_2\tau}{\hat{\tau}\hat{\tau}_{-1}}. \tag{35}$$

These allow to resolve the algebraic part (21) of the system (19)–(21), with the fields ξ and η expressed as follows:

$$\xi = \frac{\hat{\tau}_{-1}\hat{\tau}_{-2} + \hat{\tau}_{-1-2}\hat{\tau}}{2\tau^2}, \quad \eta = \frac{\hat{\tau}_{-1}\hat{\tau}_{-2} - \hat{\tau}_{-1-2}\hat{\tau}}{2\tau^2}. \tag{36}$$

Then the remaining equations (19) and (20) form a system of three equations for three fields τ, $\hat{\tau}$, and F :

$$4\frac{d}{dt}\left(\ln\frac{\tau_1\tau}{\hat{\tau}\hat{\tau}_{-2}}\right) + \Delta_1\left[F\left(\frac{\hat{\tau}_{-1}\hat{\tau}_{-2}}{\tau^2} + \frac{\hat{\tau}_{-1-2}\hat{\tau}}{\tau^2}\right)\right] = 0,$$

$$4\frac{d}{dt}\left(\ln\frac{\tau_2\tau}{\hat{\tau}\hat{\tau}_{-1}}\right) + \Delta_2\left[F\left(\frac{\hat{\tau}_{-1}\hat{\tau}_{-2}}{\tau^2} - \frac{\hat{\tau}_{-1-2}\hat{\tau}}{\tau^2}\right)\right] = 0,$$

$$\frac{2}{\tau^2}\frac{dF}{dt} + \frac{1}{\hat{\tau}\hat{\tau}_{-2}}\left(\frac{\hat{\tau}_{1-2}}{\hat{\tau}_{-2}} + \frac{\hat{\tau}_1}{\hat{\tau}}\right) - \frac{1}{\hat{\tau}_{-1}\hat{\tau}_{-1-2}}\left(\frac{\hat{\tau}_{-1-1}}{\hat{\tau}_{-1}} + \frac{\hat{\tau}_{-1-1-2}}{\hat{\tau}_{-1-2}}\right) + \frac{1}{\hat{\tau}\hat{\tau}_{-1}}\left(\frac{\hat{\tau}_{-12}}{\hat{\tau}_{-1}} - \frac{\hat{\tau}_2}{\hat{\tau}}\right) - \frac{1}{\hat{\tau}_{-2}\hat{\tau}_{-1-2}}\left(\frac{\hat{\tau}_{-2-2}}{\hat{\tau}_{-2}} - \frac{\hat{\tau}_{-1-1-2}}{\hat{\tau}_{-1-2}}\right) = 0.$$

Introduction of the fields q and r (or Γ and $\hat{\Gamma}$), which allowed us to simplify Eqs. (20), suggests the introduction of yet other potentials h and $\hat{h}$ such that

$$\tau^2 = \frac{\hat{h}_1\hat{h}_2}{hh_{12}}, \quad \hat{\tau} = \left(\frac{\hat{h}}{h}\right)_{12}. \tag{37}$$

It follows that

$$\Gamma^2 = \frac{h_{12}\hat{h}_2}{h\hat{h}_1}, \quad \hat{\Gamma}^2 = \frac{h_{12}\hat{h}_1}{h\hat{h}_2}, \tag{38}$$

and that Eqs. (21) are identically satisfied, with the fields ξ and η given as follows:

$$\xi = \frac{1}{2}\left(\frac{hh_{12}}{h_1h_2} + \frac{\hat{h}\hat{h}_{12}}{\hat{h}_1\hat{h}_2}\right), \quad \eta = \frac{1}{2}\left(\frac{hh_{12}}{h_1h_2} - \frac{\hat{h}\hat{h}_{12}}{\hat{h}_1\hat{h}_2}\right). \tag{39}$$

Moreover, Eqs. (20) reduce to two equivalent expressions for F,

$$\frac{h_1h_2}{hh_{12}}\frac{d}{dt}\left(\ln\frac{h_{12}}{h}\right) = \frac{\hat{h}_1\hat{h}_2}{\hat{h}\hat{h}_{12}}\frac{d}{dt}\left(\ln\frac{\hat{h}_2}{\hat{h}_1}\right) = \frac{F}{2}, \tag{40}$$

while Eq. (21) reads

$$4\frac{d}{dt}\left[\frac{h_1h_2}{hh_{12}}\frac{d}{dt}\left(\ln\frac{h_{12}}{h}\right)\right] + \left(\frac{h\hat{h}_1}{h_1\hat{h}} + \frac{h_2\hat{h}_{12}}{h_{12}\hat{h}_2}\right)_1\frac{h_1\hat{h}_2}{h\hat{h}_{12}} - \left(\frac{h_1\hat{h}}{h\hat{h}_1} + \frac{h_{12}\hat{h}_2}{h_2\hat{h}_{12}}\right)_{-1}\frac{h_2\hat{h}_1}{h_{12}\hat{h}} + \left(\frac{h\hat{h}_2}{h_2\hat{h}} - \frac{h_1\hat{h}_{12}}{h_{12}\hat{h}_1}\right)_2\frac{h_2\hat{h}_1}{h\hat{h}_{12}} - \left(\frac{h_{12}\hat{h}_1}{h_1\hat{h}_{12}} - \frac{h_2\hat{h}}{h\hat{h}_2}\right)_{-2}\frac{h_1\hat{h}_2}{h_{12}\hat{h}} = 0. \tag{41}$$

Therefore the introduction of the potentials h and $\hat{h}$ allows one to rewrite the Toda-like system (19)–(21) as a coupled nonlinear system of two equations [the first equation of (40) and Eq. (41)].

III. DYNAMICS ON THE TRIANGULAR LATTICE

In this section we construct some examples of integrable dynamics of Toda type on the regular triangular lattice. We recall that, on the triangular lattice, the three main translations T_1, T_2, and T_3 in the directions 1, 2, and 3 are not independent, being connected by the relation

$$T_1T_3 = T_2, \tag{42}$$

and hence $f_3 = f_{-12}$ and $f_{-3} = f_{1-2}$.

The integrable dynamics of Toda type are associated with the linear and self-adjoint seven-point scheme

$$A\Psi_1 + A_{-1}\Psi_{-1} + B\Psi_2 + B_{-2}\Psi_{-2} + C\Psi_3 + C_{-3}\Psi_{-3} = F\Psi \tag{43}$$

on the star of the triangular lattice, involving the black center ● and the six vertices denoted by the symbol □ in Fig. 2. In the spectral problem (43), the eigenfunction Ψ is defined at the vertices of the graph and the fields A, B, and C are defined on the non oriented edges of the lattice. Equation (43), a natural discretization of the most general self-adjoint second-order equation on the plane,

$$(a\psi_x)_x + (b\psi_y)_y + (c\psi_x)_y + (c\psi_y)_x = f\psi, \tag{44}$$

admits, like its continuous counterpart, DTs [29].

As in the previous section, we restrict our investigation to evolution equations for Ψ involving only the six points □ of the seven-point scheme:

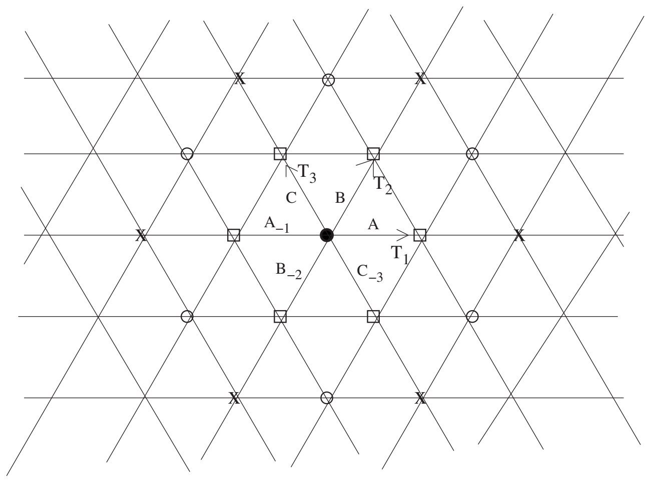

FIG. 2. The triangular lattice and the points involved in the commutation.

$$\frac{d\Psi}{dt} = \alpha\Psi_1 + \beta\Psi_{-1} + \gamma\Psi_2 + \delta\Psi_{-2} + \epsilon\Psi_3 + \nu\Psi_{-3}. \quad (45)$$

We remark that the regular triangular lattice possesses a $\pi/3$-rotation symmetry, under which the three basic translations transform as follows:

$$T_1 \to \tilde{T}_1 = T_2, \quad T_2 \to \tilde{T}_2 = T_3, \quad T_3 \to \tilde{T}_3 = T_1^{-1}. \quad (46)$$

Correspondingly, the coefficients of the seven-point scheme (43) and of the evolution equation (45) transform as follows:

$$A \to B, \quad B \to C, \quad C \to A_{-1},$$

$$\alpha \to \tilde{\alpha} = \gamma, \quad \beta \to \tilde{\beta} = \delta, \quad \gamma \to \tilde{\gamma} = \epsilon, \quad \delta \to \tilde{\delta} = \nu,$$

$$\epsilon \to \tilde{\epsilon} = \beta, \quad \nu \to \tilde{\nu} = \alpha. \quad (47)$$

We proceed adopting the same strategy as in the previous section. The compatibility between Eqs. (43) and (45) leads to an equation involving the values of Ψ at all the 19 marked points in Fig. 2. Using the scheme (43) centered at the origin and at the points □, one expresses the values of Ψ at the origin and at the points x in terms of the 12 values of Ψ at the points □ and ○. As a result of this procedure, the compatibility condition becomes a linear equation for the 12 independent values of Ψ at the points □ and ○. Equating to zero their 12 coefficients, one obtains an overdetermined system of 12 nonlinear equations for the 10 coefficients A, B, C, F, α, β, γ, δ, ϵ, and ν. It turns out that, due to the relation (42) among the three main shifts, such overdeterminacy is resolved and one can construct integrable nontrivial dynamics.

In the rest of this section we report the results of such analysis, leading to the Toda-type dynamics on the triangular lattice.

A. Rotationally invariant dynamics

Setting

$$\alpha = \frac{\xi}{2}A, \quad \beta = -\frac{\xi}{2}A_{-1}, \quad \gamma = \frac{\eta}{2}B, \quad \delta = -\frac{\eta}{2}B_{-2},$$

$$\epsilon = \frac{\zeta}{2}C, \quad \nu = -\frac{\zeta}{2}C_{-3}, \quad (48)$$

where ξ, η, and ζ are lattice fields to be specified, the corresponding evolution for Ψ,

$$\frac{d\Psi}{dt} = \frac{\xi}{2}(A\Psi_1 - A_{-1}\Psi_{-1}) + \frac{\eta}{2}(B\Psi_2 - B_{-2}\Psi_{-2}) + \frac{\zeta}{2}(C\Psi_3 - C_{-3}\Psi_{-3}), \quad (49)$$

is compatible with the seven-point scheme (43) iff the coefficients A, B, C, F, ξ, η, and ζ satisfy the following determined system of seven nonlinear equations:

$$\frac{dF}{dt} + \frac{1}{\xi}\Delta_1(\xi\xi_{-1}A_{-1}^2) + \frac{1}{\eta}\Delta_2(\eta\eta_{-2}B_{-2}^2) + \frac{1}{\zeta}\Delta_3(\zeta\zeta_{-3}C_{-3}^2) = 0, \quad (50)$$

$$\frac{1}{A}\frac{dA}{dt} + \frac{1}{2}\Delta_1(\xi F) - \frac{1}{2}\frac{BC_1}{A}(\eta+\zeta)_2 + \frac{1}{2}\frac{B_{-3}C_{-3}}{A}(\eta+\zeta)_{-3} = 0, \quad (51a)$$

$$\frac{1}{B}\frac{dB}{dt} + \frac{1}{2}\Delta_2(\eta F) + \frac{1}{2}\frac{A_3C}{B}(\xi-\zeta)_3 - \frac{1}{2}\frac{AC_1}{B}(\xi-\zeta)_1 = 0, \quad (51b)$$

$$\frac{1}{C}\frac{dC}{dt} + \frac{1}{2}\Delta_3(\zeta F) + \frac{1}{2}\frac{B_{-1}A_{-1}}{C}(\xi+\eta)_{-1} - \frac{1}{2}\frac{A_3B}{C}(\xi+\eta)_2 = 0, \quad (51c)$$

$$AB_1(\xi-\eta)_1 = A_2B(\xi-\eta)_2, \quad (52a)$$

$$AC(\xi+\zeta) = A_3C_1(\xi+\zeta)_2, \quad (52b)$$

$$B_{-3}C_1(\eta-\zeta)_1 = BC_{-3}(\eta-\zeta). \quad (52c)$$

We remark that, due to three algebraic equations (52a)–(52c), the three equations (51a)–(51c) can be rewritten in the following conservationlike form:

$$\frac{d}{dt}(\ln A^2) + \Delta_1(\xi F) - \Delta_2\left(\frac{B_{-2}C_{-3}}{A_{-2}}(\xi-\eta)\right) - \Delta_3\left(\frac{B_{-3}C_{-3}}{A}(\xi+\zeta)_{-3}\right) = 0,$$

$$\frac{d}{dt}(\ln B^2) + \Delta_2(\eta F) + \Delta_3\left(\frac{AC_{-3}}{B_{-3}}(\eta-\zeta)\right) - \Delta_1\left(\frac{A_{-1}C}{B_{-1}}(\xi-\eta)\right) = 0,$$

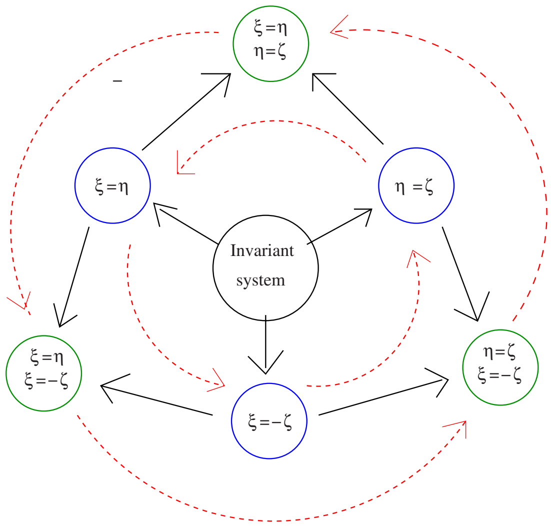

FIG. 3. (Color online) The bold arrows describe the six reductions of the rotationally invariant system. The dashed arrows describe how such reductions transform one into the other, under a $\pi/3$ rotation.

$$\frac{d}{dt}(\ln C^2) + \Delta_3(\zeta F) - \Delta_1\left(\frac{A_{-1}B_{-1}}{C}(\xi+\zeta)_{-1}\right) - \Delta_2\left(\frac{A_{-1}B_{-2}}{C_{-2}}(\eta-\zeta)\right) = 0. \tag{53}$$

Remark 4. Under the transformation (47), the coefficients ξ, η, and ζ transform as follows:

$$\xi \to \tilde{\xi} = \eta, \quad \eta \to \tilde{\eta} = \zeta, \quad \zeta \to \tilde{\zeta} = -\xi, \tag{54}$$

and, as is easy to verify, the nonlinear system (50), (51a)–(51c), and (52a)–(52c) is invariant under a $\pi/3$ rotation.

Remark 5. Like the Toda-system (19)–(21) on the square lattice, also the nonlinear system (50), (51a)–(51c), and (52a)–(52c) is nonlocal, due to the three equations (52a)–(52c), in which the auxiliary fields $(\eta-\zeta)$, $(\xi+\eta)$, and $(\xi-\eta)$ are constructed, from the knowledge of the main fields A, B, and C, along the three main directions 1, 2, and 3 of the triangular lattice. Therefore well-posed IBV problems for the system (50), (51a)–(51c), and (52a)–(52c) of the type discussed in the Introduction can be solved using the IST and finite-gap methods associated with the spectral problem (43).

B. Reductions not invariant under rotation

The nonlinear system (50), (51a)–(51c), and (52a)–(52c) admits the reductions $[\xi=\eta]$, $[\eta=\zeta]$, and $[\zeta=-\xi]$ and the following combinations of them: $[\xi=\eta,\ \eta=\zeta]$, $[\xi=\eta,\ \zeta=-\xi]$, and $[\eta=\zeta,\ \zeta=-\xi]$. They give rise to six integrable dynamics on the triangular lattice. It follows that these dynamics are not rotationally invariant, but they transform one into the other in the way summarized in Fig. 3.

We write down explicitly the two reductions $[\xi=\eta]$ and $[\xi=\eta,\ \eta=\zeta]$, since all the others can be generated from them through rotations.

The reduction $\xi=\eta$. In this case, the evolution of Ψ reads

$$\frac{d\Psi}{dt} = \frac{\xi}{2}(A\Psi_1 - A_{-1}\Psi_{-1} + B\Psi_2 - B_{-2}\Psi_{-2}) + \frac{\zeta}{2}(C\Psi_3 - C_{-3}\Psi_{-3}), \tag{55}$$

and the nonlinear system (50), (51a)–(51c), and (52a)–(52c) reduces to the six equations

$$\frac{dF}{dt} + \frac{1}{\xi}[\Delta_1(\xi\xi_{-1}A_{-1}^2) + \Delta_2(\xi\xi_{-2}B_{-2}^2)] + \frac{1}{\zeta}\Delta_3(\zeta\zeta_{-3}C_{-3}^2) = 0,$$

$$\frac{d(\ln A^2)}{dt} + \Delta_1(\xi F) - \Delta_3\left(\frac{B_{-3}C_{-3}}{A}(\xi+\zeta)_{-3}\right) = 0,$$

$$\frac{d(\ln B^2)}{dt} + \Delta_2(\xi F) + \Delta_3\left(\frac{AC_{-3}}{B_{-3}}(\xi-\zeta)\right) = 0,$$

$$\frac{d(\ln C^2)}{dt} + \Delta_3(\zeta F) - \Delta_1\left(\frac{A_{-1}B_{-1}}{C}(\xi+\zeta)_{-1}\right) - \Delta_2\left(\frac{A_{-1}B_{-2}}{C_{-2}}(\xi-\zeta)\right) = 0,$$

$$AC(\xi+\zeta) = A_3C_1(\xi+\zeta)_2,$$

$$B_{-3}C_1(\xi-\zeta)_1 = BC_{-3}(\xi-\zeta). \tag{56}$$

The reduction $(\xi=\eta,\ \ \eta=\zeta)$. In this case, the evolution of Ψ reads

$$\frac{d\Psi}{dt} = \frac{\xi}{2}(A\Psi_1 - A_{-1}\Psi_{-1} + B\Psi_2 - B_{-2}\Psi_{-2} + C\Psi_3 - C_{-3}\Psi_{-3}), \tag{57}$$

and the nonlinear system (50), (51a)–(51c), and (52a)–(52c) reduces to the five equations

$$\xi\frac{dF}{dt} + \Delta_1(\xi\xi_{-1}A_{-1}^2) + \Delta_2(\xi\xi_{-2}B_{-2}^2) + \Delta_3(\xi\xi_{-3}C_{-3}^2) = 0, \tag{58a}$$

$$\frac{d(\ln A^2)}{dt} + \Delta_1(\xi F) - 2\Delta_3\left(\frac{B_{-3}C_{-3}}{A}\xi_{-3}\right) = 0, \tag{58b}$$

$$\frac{d(\ln B^2)}{dt} + \Delta_2(\xi F) = 0, \tag{58c}$$

$$\frac{d(\ln C^2)}{dt} + \Delta_3(\xi F) - 2\Delta_1\left(\frac{A_{-1}B_{-1}}{C}\xi_{-1}\right) = 0, \tag{58d}$$

$$AC\xi = A_3C_1\xi_2. \tag{58e}$$

In analogy with the previous examples on the square lattice, Eqs. (58b)–(58d) suggest the introduction of the new fields q, ρ, and σ defined by

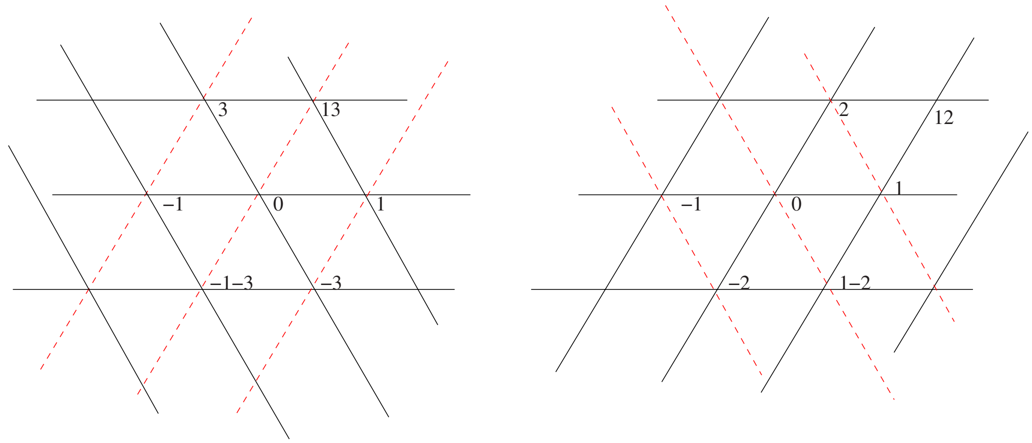

FIG. 4. (Color online) (a) The rhombic lattice for $B=0$. (b) The rhombic lattice for $C=0$.

$$\frac{dq}{dt}=-\xi F,\quad \frac{d\rho}{dt}=-2\frac{BC}{A_3}\xi,\quad \frac{d\sigma}{dt}=-2\frac{AB}{C_1}\xi. \tag{59}$$

With this choice,

$$A=ae^{(1/2)(\Delta_1 q-\Delta_3\rho_{-3})},\quad B=be^{(1/2)\Delta_2 q},\quad C=ce^{(1/2)(\Delta_3 q-\Delta_1\sigma_{-1})}, \tag{60}$$

where a, b, and c are arbitrary constants. Choosing, without lost of generality, $a,b,c=1$, the system (58a)–(58e) can be rewritten in the following Toda-like form:

$$\xi\frac{d}{dt}\left(\frac{1}{\xi}\frac{dq}{dt}\right)+\Delta_1(\xi\xi_{-1}e^{\Delta_1 q_{-1}+\rho_{-2}-r_{-1}})+\Delta_2(\xi\xi_{-2}e^{\Delta_2 q_{-2}})+\Delta_3(\xi\xi_{-3}e^{\Delta_3 q_{-3}+\rho_{-2}-\sigma_{-3}})=0,$$

$$\frac{d\rho}{dt}=-2\xi e^{\Delta_3(q+\rho/2)-(1/2)\Delta_1\sigma_{-1}},$$

$$\frac{d\sigma}{dt}=-2\xi e^{\Delta_1(q+\sigma/2)-(1/2)\Delta_3\rho_{-3}},$$

$$\xi_2=\xi e^{q_3-q_2+q_1-q+(1/2)(\Delta_1^2\sigma_{-1}+\Delta_3^2\rho_{-3})}. \tag{61}$$

C. Reductions to dynamics on the $\mathbb{Z}^2$ graph

We remark that the reduction $\xi=\eta$, $\eta=\zeta$ is compatible with the condition $B=0$, for which all the connections in direction 2 are broken and the triangular lattice reduces to the rhombic lattice in Fig. 4(a). Then direction 2 should be renamed 13 and the system (58a)–(58e) becomes the integrable system (28a)–(28c) on the rhombic lattice of Fig. 4(a) (on the $\mathbb{Z}^2$ graph).

Analogously, it would be possible to show, for example, that the reduction $\xi=-\eta=-\zeta$ is compatible with the condition $C=0$, for which all the connections in direction 3 are broken and the triangular lattice reduces to the rhombic lattice in Fig. 4(b). The system obtained in this case is the integrable system (32) on such a rhombic lattice.

D. τ-function formulations

Motivated by the sublattice approach [30] for the self-adjoint seven-point scheme (43), we introduce three potentials τ, $\hat{\tau}$, and $\check{\tau}$ such that

$$A=\frac{\tau\tau_1}{\hat{\tau}\check{\tau}_2},\quad B=\frac{\tau\tau_2}{\hat{\tau}_3\check{\tau}_2},\quad C=\frac{\tau\tau_3}{\hat{\tau}_3\check{\tau}_3}. \tag{62}$$

The algebraic part (52a)–(52c) of the nonlinear system (50), (51a)–(51c), and (52a)–(52c) is then resolved by the parametrization

$$\xi=\frac{1}{2\tau^2}(\check{\tau}_3\hat{\tau}+\check{\tau}_2\hat{\tau}_{-1}+\check{\tau}\hat{\tau}_3),$$

$$\eta=\frac{1}{2\tau^2}(\check{\tau}_3\hat{\tau}-\check{\tau}_2\hat{\tau}_{-1}+\check{\tau}\hat{\tau}_3),$$

$$\zeta=\frac{1}{2\tau^2}(\check{\tau}_3\hat{\tau}-\check{\tau}_2\hat{\tau}_{-1}-\check{\tau}\hat{\tau}_3). \tag{63}$$

The remaining part of the system gives a system of four equations for the four fields F, τ, $\check{\tau}$, and $\hat{\tau}$.

IV. DARBOUX-BÄCKLUND TRANSFORMATIONS

In this section we present the DBTs for the above Toda-type systems. It is well-known that they allow one to construct, through an elementary procedure, explicit analytic solutions of the above Toda-type systems from simpler solutions.

A. DBTs for the Toda-type system on the square lattice

The Lax pair (12) and (18) is covariant under the Darboux transformation

$$(\Psi,A,\ldots,\eta)\mapsto(\tilde{\Psi},\tilde{A},\ldots,\tilde{\eta}) \tag{64}$$

given by the linear system

$$\Delta_2(K\tilde{\Psi})=-A_{-1}\theta\theta_{-1}\Delta_{-1}\left(\frac{\Psi}{\theta}\right),$$

$$\Delta_1(K\tilde{\Psi}) = B_{-2}\theta\theta_{-2}\Delta_{-2}\left(\frac{\Psi}{\theta}\right),$$

$$\frac{d}{dt}(K\tilde{\Psi}) = \theta_{-1}\theta_{-2}A_{-1}B_{-2}\frac{(\xi+\eta)}{2}\left[\left(\frac{\Psi}{\theta}\right)_{-2} - \left(\frac{\Psi}{\theta}\right)_{-1}\right] + \theta\theta_{-1-2}A_{-1}B_{-1-2}\frac{(\xi_{-1}-\eta_{-1})}{2}\left[\left(\frac{\Psi}{\theta}\right)_{-1-2} - \frac{\Psi}{\theta}\right], \tag{65}$$

where θ is a particular solution of (12); the gauge function K must obey

$$\frac{1}{K}\frac{dK}{dt} + (\theta_{-1}A_{-1} + \theta_{-2}B_{-2})\frac{\xi+\eta}{4\theta} - (\theta_{-2}A_{-1-2} + \theta_{-1}B_{-1-2}) \times \frac{(\xi+\eta)_{-1-2}}{4\theta_{-1-2}} + (\theta_{-1-2}A_{-1-2} + \theta B_{-2})\frac{(\xi-\eta)_{-2}}{4\theta_{-2}} - (\theta A_{-1} + \theta_{-1-2}B_{-1-2})\frac{(\xi-\eta)_{-1}}{4\theta_{-1}} = 0, \tag{66}$$

and the transformation of the other fields is given by

$$\tilde{A} = \frac{KK_1}{B_{-2}\theta\theta_{-2}},$$

$$\tilde{B} = \frac{KK_2}{A_{-1}\theta\theta_{-1}},$$

$$\tilde{F} = K^2\left(\frac{1}{A_{-1}\theta_{-1}\theta} + \frac{1}{A_{-1-2}\theta_{-1-2}\theta_{-2}} + \frac{1}{B_{-2}\theta_{-2}\theta} + \frac{1}{B_{-1-2}\theta_{-1-2}\theta_{-1}}\right),$$

$$\tilde{\xi} - \tilde{\eta} = \frac{\theta_{-1-2}\theta}{K^2}A_{-1}B_{-1-2}(\xi-\eta)_{-1},$$

$$\tilde{\xi} + \tilde{\eta} = \frac{\theta_{-1}\theta_{-2}}{K^2}A_{-1}B_{-2}(\xi+\eta). \tag{67}$$

So the formulas (67) are the Bäcklund transformations (BTs) for the Toda-type system (19)–(21) on the square lattice; i.e., $\tilde{A}$, $\tilde{B}$, $\tilde{F}$, $\tilde{\xi}$, and $\tilde{\eta}$ compose a new solution of (19)–(21). On the level of the τ functions the transformation is given as

$$\tau \mapsto K\hat{\tau}_{-1-2}, \quad \hat{\tau} \mapsto \theta\tau. \tag{68}$$

We remark that, for $\xi = \pm\eta$, the above transformations become the DBTs for the reduced systems (28a)–(28c) and (32).

The spatial parts of the above DBTs were already written in [29]; the temporal parts, describing the time dependence of the transformed solution $\tilde{\Psi}$ and the transformation law for the coefficient ξ, η, and ζ, are new ingredients of this paper.

B. DBTs for the Toda-type system on the triangular lattice

First, for aesthetical reasons, we introduce function

$$S := C_{-2}.$$

The Lax pair (43) and (49) is covariant under the Darboux transformation

$$(\Psi, A, \ldots, \zeta) \mapsto (\tilde{\Psi}, \tilde{A}, \ldots, \tilde{\zeta}) \tag{69}$$

given by the linear system

$$\Delta_1(K\tilde{\Psi}) = -B_{-2}\theta_{-2}\Psi - S\theta_{-2}\Psi_{-1} + (B_{-2}\theta + S\theta_{-1})\Psi_{-2}, \tag{70a}$$

$$\Delta_2(K\tilde{\Psi}) = A_{-1}\theta_{-1}\Psi - (A_{-1}\theta + S\theta_{-2})\Psi_{-1} + S\theta_{-1}\Psi_{-2}, \tag{70b}$$

$$\begin{aligned}\frac{d}{dt}(K\tilde{\Psi}) = -\frac{1}{2}\Bigg\{ & \theta_{-1-2}A_{-1}B_{-1-2}(\xi-\eta)_{-1}\Psi + \theta_{-2}A_{-1-1}S(\xi+\zeta)_{-1}\Psi_{-1-1} - \theta_{-1}B_{-2-2}S(\eta-\zeta)_{-2}\Psi_{-2-2} \\ & + \left[\frac{P_{-2}B_{-1-2}}{B_{-2-2}}(\eta-\zeta)_{-1-2} + \theta_{-1-2}SA_{-1-2}(\xi-\eta)_{-2} + \theta_{-2}A_{-1-2}B_{-1-2}(\xi+\zeta)_{-1-2}\right]\Psi_{-1} \\ & - \left[\frac{P_{-1}A_{-1-2}}{A_{-1-1}}(\xi+\zeta)_{-1-2} - \theta_{-1-2}SB_{-1-2}(\xi-\eta)_{-1} + \theta_{-1}A_{-1-2}B_{-1-2}(\eta-\zeta)_{-1-2}\right]\Psi_{-2} \\ & - \left[\frac{PB_{-1-2}}{B_{-2}}(\xi-\eta)_{-1} - \theta_{-2}SB_{-1-2}(\xi+\zeta)_{-1} + \theta_{-1}SA_{-1-2}(\eta-\zeta)_{-2}\right]\Psi_{-1-2}\Bigg\},\end{aligned} \tag{70c}$$

where θ is a particular solution of the Lax pair (43) and (49), P is given by

$$P := \theta A_{-1}B_{-2} + \theta_{-1}A_{-1}S + \theta_{-2}B_{-2}S,$$

and K is given by the quadrature

$$\frac{1}{K}\frac{dK}{dt}=\frac{-1}{4}\Bigg\{P\theta_{-1-2}\frac{B_{-1-2}}{B_{-2}}(\xi-\eta)_{-1}\left[\frac{A_{-1}}{\theta_{-2}P}-\frac{A_{-1-1}}{\theta_{-1-2}P_{-1}}-\frac{B_{-2}}{\theta_{-1}P}+\frac{B_{-2-2}}{\theta_{-1-2}P_{-2}}-\frac{S_{-1}}{\theta_{-1}P_{-1}}+\frac{S_{-2}}{\theta_{-2}P_{-2}}\right]$$
$$+P_{-1}\theta_{-2}\frac{A_{-1-2}}{A_{-1-1}}(\xi+\zeta)_{-1-2}\left[\frac{A_{-1}}{\theta_{-2}P}-\frac{A_{-1-1}}{\theta_{-1-2}P_{-1}}+\frac{B_{-2}}{\theta_{-1}P}-\frac{B_{-2-2}}{\theta_{-1-2}P_{-2}}+\frac{S_{-1}}{\theta_{-1}P_{-1}}-\frac{S_{-2}}{\theta_{-2}P_{-2}}\right]$$
$$+P_{-2}\theta_{-1}\frac{B_{-1-2}}{B_{-2-2}}(\eta-\zeta)_{-1-2}\left[\frac{A_{-1}}{\theta_{-2}P}-\frac{A_{-1-1}}{\theta_{-1-2}P_{-1}}+\frac{B_{-2}}{\theta_{-1}P}-\frac{B_{-2-2}}{\theta_{-1-2}P_{-2}}-\frac{S_{-1}}{\theta_{-1}P_{-1}}+\frac{S_{-2}}{\theta_{-2}P_{-2}}\right]\Bigg\}. \tag{71}$$

The new eigenfunction $\tilde{\Psi}$ is a solution of the Lax pair (43) and (49) with the new coefficients

$$\tilde{A}=\frac{KK_1}{\theta_{-2}P}A_{-1}, \tag{72a}$$

$$\tilde{B}=\frac{KK_2}{\theta_{-1}P}B_{-2}, \tag{72b}$$

$$\tilde{S}=\frac{K_{-1}K_{-2}}{\theta_{-1-2}P_{-1-2}}S_{-1-2}, \tag{72c}$$

$$\tilde{F}=K^2\left(\frac{A_{-1}}{\theta_{-2}P}+\frac{A_{-1-1}}{\theta_{-1-2}P_{-1}}+\frac{B_{-2}}{\theta_{-1}P}+\frac{B_{-2-2}}{\theta_{-1-2}P_{-2}}+\frac{S_{-1}}{\theta_{-1}P_{-1}}+\frac{S_{-2}}{\theta_{-2}P_{-2}}\right), \tag{72d}$$

$$\tilde{\xi}-\tilde{\eta}=\frac{PB_{-1-2}\theta_{-1-2}}{K^2B_{-2}}(\xi-\eta)_{-1}, \tag{72e}$$

$$\tilde{\xi}+\tilde{\zeta}=\frac{P_{-1}S\theta_{-2}}{K^2S_{-1}}(\xi+\zeta)_{-1}, \tag{72f}$$

$$\tilde{\eta}-\tilde{\zeta}=\frac{P_{-2}S\theta_{-1}}{K^2S_{-2}}(\eta-\zeta)_{-2}. \tag{72g}$$

Therefore formulas (72a)–(72g) constitute the BTs for the Toda type system (50), (51a)–(51c), and (52a)–(52c) on the triangular lattice.

We would like to mention that Eqs. (70a) and (70b) can be easily inverted:

$$\Delta_{-1}\frac{\Psi}{\theta}=\tilde{B}\frac{1}{K_2}\tilde{\Psi}+\tilde{S}_{12}\frac{1}{K_2}\tilde{\Psi}_1-\left(\tilde{B}\frac{1}{K}+\tilde{S}_{12}\frac{1}{K_1}\right)\Psi_2,$$

$$\Delta_{-2}\frac{\Psi}{\theta}=-\tilde{A}\frac{1}{K_1}\tilde{\Psi}-\tilde{S}_{12}\frac{1}{K_1}\tilde{\Psi}_2+\left(\tilde{A}\frac{1}{K}+\tilde{S}_{12}\frac{1}{K_2}\right)\Psi_1. \tag{73}$$

In addition, Eqs. (71) and (72d) can be rewritten by means of “new” solutions as follows

$$\frac{d}{dt}\left(\frac{1}{K}\right)=\frac{1}{2}\left[\tilde{\xi}\left(\frac{\tilde{A}}{K_1}-\frac{\tilde{A}_{-1}}{K_{-1}}\right)+\tilde{\eta}\left(\frac{\tilde{B}}{K_2}-\frac{\tilde{B}_{-2}}{K_{-2}}\right)+\tilde{\zeta}\left(\frac{\tilde{S}_2}{K_{-12}}-\frac{\tilde{S}_1}{K_{1-2}}\right)\right]\Bigg/\left[\frac{\tilde{A}}{K_1}+\frac{\tilde{A}_{-1}}{K_{-1}}+\frac{\tilde{B}}{K_2}+\frac{\tilde{B}_{-2}}{K_{-2}}+\frac{\tilde{S}_2}{K_{-12}}+\frac{\tilde{S}_1}{K_{1-2}}\right]=\frac{\tilde{F}}{K};$$

i.e., $\frac{1}{K}$ is eigenfunction of the tilded Lax pair.

On the level of the τ functions, the transformation is given as follows:

$$\tau\mapsto K\check{\tau}_{-1},\quad \hat{\tau}\mapsto\theta_{-2}\tau_{-2},$$

$$\check{\tau}\mapsto\left(\frac{\theta\tau\check{\tau}_{-1}+\theta_{-1}\tau_{-1}\check{\tau}+\theta_{-2}\tau_{-2}\check{\tau}_3}{\hat{\tau}_{-1}}\right)_{-2}. \tag{74}$$

As before, the DBTs (72a)–(72g) are consistent with all the reductions of the Toda-type system (50), (51a)–(51c), and (52a)–(52c).

The spatial parts of the above DBTs were already written in [29]; the temporal parts, describing the time dependence of the transformed solution $\tilde{\Psi}$, and the transformation law for the coefficient ξ, η, and ζ are new ingredients of this paper.

V. CONCLUSIONS AND FUTURE PERSPECTIVES

In this paper we have constructed integrable dynamics of Toda type on the square and triangular lattices, as nonlinear symmetries of the discrete Laplace equations on the square and triangular lattices, together with their τ-function formulations and their DBTs.

The integrability of these dynamics manifests in this paper in the construction of their basic integrability schemes: the Lax pair and the corresponding DBTs. A systematic use of these DBTs to construct recursively analytic solutions, together with the use of the IST and finite-gap methods to solve IBV problems of the type discussed in the Introduction, are presently under investigation. The interested reader can already see how the DBTs of the type derived in this paper allow one to construct recursively analytic solutions, on the illustrative example presented in [43]. No transparent physical application for these new systems is known, at the moment.

Due to the intimate connections between the self-adjoint schemes on the triangular and honeycomb lattices [30], it is possible, in principle, to construct integrable Toda-type dynamics on the honeycomb lattice from those on the triangular lattice. This project will be developed elsewhere. Another interesting problem for future research is to establish connections between these Toda-like systems and the corresponding Lotka-Volterra systems (see, e.g., [51,52]), as well as the connection, via the sublattice approach, between these Toda-like systems and the integrable dynamics on the discrete Moutard lattice introduced in [53].

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