Localized states beyond the asymptotic parametrically driven amplitude equation

M. G. Clerc, S. Coulibaly,* and D. Laroze
Departamento de Física, Facultad de Ciencias Físicas y Matemáticas, Universidad de Chile, Casilla 487-3, Santiago, Chile
(Received 16 August 2007; revised manuscript received 8 January 2008; published 19 May 2008)

[1]We study theoretically a family of localized states which asymptotically connect a uniform oscillatory state in the magnetization of an easy-plane ferromagnetic spin chain when an oscillatory magnetic field is applied and in a parametrically driven damped pendula chain. The conventional approach to these systems, the parametrically driven damped nonlinear Schrödinger equation, does not account for these states. Adding higher order terms to this model we were able to obtain these localized structures.

DOI: 10.1103/PhysRevE.77.056209 PACS number(s): 05.45.−a, 45.70.Qj, 47.54.−r

Recently, emerging macroscopic particle-type solutions or localized states in dissipative systems have been observed in different fields, such as domains in magnetic materials, chiral bubbles in liquid crystals, current filaments in gas discharge, spots in chemical reactions, localized states in fluid surface waves, oscillons in granular media, isolated states in thermal convection, solitary waves in nonlinear optics, among others. In one-dimensional systems, localized states can be described as spatial trajectories that connect one steady state with itself, that means, they are homoclinic orbits from the dynamical system point of view (see Ref. [1], and references therein), while domains or walls are seen as spatial trajectories joining two different steady states—heteroclinic curves—of the corresponding spatial dynamical system [2]. For quasireversible systems—time reversible systems perturbed with injection and dissipation of energy [3]—the prototypical model that exhibits localized structures is the parametrically driven damped nonlinear Schrödinger equation [4]. This model has been derived in several contexts to describe pattern and localized structures, such as the vertically oscillating layer of water [5], nonlinear lattices [6], optical fibers [7], Kerr-type optical parametric oscillators [8], magnetization in an easy-plane ferromagnetic exposed to an oscillatory magnetic field [9], and a parametrically driven damped chain of pendula [10]. However, this equation is not sufficient to describe a family of localized states that link asymptotically uniform oscillations. Indeed in parameter space of the parametrically driven damped nonlinear Schrödinger equation, we have observed numerically both in vertically driven damped pendula chain and easy-plane ferromagnetic spin chain exposed to an oscillatory magnetic, families of particle type solutions unexpected inside the Arnold tongue.

The aim of this work is to theoretically study a family of localized states that asymptotically connect a uniform oscillatory state in a magnetic wire forced with a transversal oscillatory magnetic field and in a parametrically driven damped pendula chain. These localized states are not contained in the conventional approach to these systems, the parametrically driven damped nonlinear Schrödinger equation. Adding in this model the higher order term, we are able to explain these localized states, the stable uniform homogeneous oscillations, and the kink solution between them. Hence, using this amended amplitude equations we recover the original dynamical behavior of these systems.

A one-dimensional easy-plane ferromagnetic such as $(CH_3)_4NMnCl_3$ (TMMC) or $Ni_{80}Fe_{20}$ is described by the well-known Landau-Lifshitz-Gilbert equation, which in dimensionless form may be written as [9,11,12]

$$\partial_t \mathbf{M} = \mathbf{M} \times \mathbf{M}_{zz} - \beta(\mathbf{M} \cdot \hat{z})(\mathbf{M} \times \hat{z}) + \mathbf{M} \times \mathbf{H} - \alpha \mathbf{M} \times \mathbf{M}_t, \tag{1}$$

where $\mathbf{M}$ stands for the unit vector of the magnetization, $\beta > 0$ is the easy-plane anisotropy constant, $\hat{z} \equiv (0,0,1)$ denotes the unit vector along the hard axis, and α is the relaxation constant. Let us consider an external magnetic field $\mathbf{H} = (H_0 + h_0 \sin \omega t)\hat{x}$, which has both a constant and a parametric forcing with amplitude h_0 and fixed frequency ω. Notice that, when $\beta \gg H_0$ the above Landau-Lifshitz-Gilbert equation can be reduced to the quasireversible sine-Gordon equation [13]

$$\ddot{\theta}(z,t) = -[\omega_0^2 + \gamma \sin(\omega t)]\sin(\theta) - \mu\dot{\theta} + k\partial_{zz}\theta, \tag{2}$$

where $\theta(z,t)$ is the azimuthal angle in the easy plane, $\omega_0^2 \equiv H_0\beta$, $\gamma \equiv h_0/\beta$, $\mu \equiv \alpha/\beta$, and $k \equiv \beta^{-1}$. The magnetization is related to θ by [13]

$$M = \{\cos\theta \cos(\dot{\theta}/2\beta), \sin\theta \cos(\dot{\theta}/2\beta), \sin(\dot{\theta}/2\beta)\}.$$

The model (2) also describes in the continuum limit a vertically driven damped chain of pendula, where $\theta(z,t)$ is the angle formed by the pendulum and the vertical axis in the z position at time t; ω_0 is pendulum natural frequency, $\{\mu, k, \gamma, \omega\}$ are the damping, elastic coupling, amplitude, and frequency of the parametric forcing, respectively. We remark that, the pendula chain has the trivial reflection symmetry $\theta \to -\theta$ that corresponds to reflection invariance of the Landau-Lifshitz-Gilbert equation given by $\mathbf{M} = (M_x, M_y, M_z) \to (M_x, -M_y, -M_z)$.

A simple homogeneous state of Eq. (2) is $\theta = 0$, which represents an uniform vertical oscillation of pendula and an homogeneous magnetization $\mathbf{M} = \hat{x}$ in the Landau-Lifshitz-Gilbert model (1). When the pendula chain is forced close to the double of the natural frequency $\omega = 2(\omega_0 + \nu)$, where 2ν is

*Permanent address: Laboratoire de Cristallograplie et Physique Moléculaire (LACPM), UFR Sciences de Structures de la Matière et Technologies (UFR SSMT), Université de Cocody, Abidjan, Côte d'Ivoire.

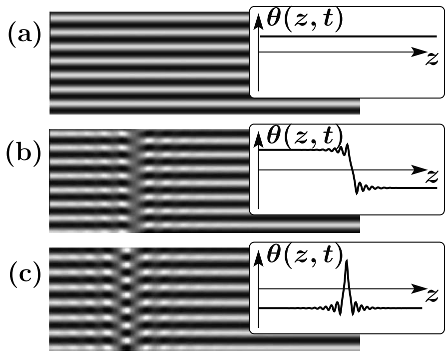

FIG. 1. Spatiotemporal diagram of θ of the parametrically driven damped pendula chain (2) inside Arnold's tongue for $\omega_0=1$, $\gamma=0.28$, $\nu=-0.02$, and $\mu=0.01$. The gray color is proportional to θ. (a) Homogeneous oscillation, (b) kink state between two homogenous oscillation, and (c) localized state. The insets represent an instantaneous θ profile.

the detuning parameter, the vertical solution becomes unstable at $\mu^2+\nu^2=\gamma^2$ for small $\{\mu,\gamma,\nu\}$ —Arnold's tongue. This bifurcation gives rise to a uniform attractive periodic solution parametric resonance [14], so the pendula chain oscillates uniformly $[\theta(t+T)=\theta(t)$, where $T\approx 2\pi/\omega_0]$. The simulation shown in Figure 1(a)[18] shows the characteristic uniform oscillation observed inside Arnold's tongue for Eq. (2). This uniform oscillation gives an account of a uniform or synchronized precession motion of the magnetization around the easy axis $\hat{x}$ in the yz plane with frequency $\sqrt{H_0(\beta+H_0)}$, where this frequency is in units of the gyrofrequency. Figure 2(a) illustrates a uniform precession for the corresponding magnetization field.

Due to the reflection symmetry, there is another uniform oscillation solution out of phase in π. Hence, we can expect to find solutions that link these two uniform oscillations—kink or wall solutions. Figure 1(b) depicts the wall solution obtained by Eq. (2). Thus, the magnetic system has walls that segregates two synchronized precessions shifted in π. Figures 2(b) and 3(b) describe the typical wall for parametrically driven magnetic system and the M_y component observed in the Landau-Lifshitz-Gilbert model (1), respectively. As can be observed, these kink solutions have spatial damping oscillation or oscillatory tails [see the inset Fig. 1(b) or Fig. 3(b)]. When wall solutions have this spatial behavior, it is well known that the nature of the kink and anti-

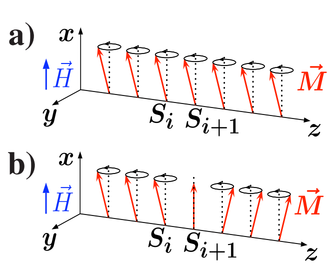

FIG. 2. (Color online) Schematic representation of the different states exhibited by the Landau-Lifshitz-Gilbert Eq. (1). (a) The homogeneous precession and (b) kink state that links two uniform precessions.

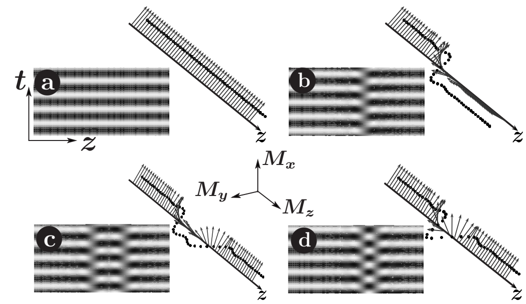

FIG. 3. Spatiotemporal diagram of the M_y component of magnetization obtained from the Landau-Lifshitz-Gilbert Eq. (1) inside Arnold's tongue for $\beta=20$, $H_0=1$, $h_0=0.57$, $\omega=2[\sqrt{H_0(H_0+\beta)}+\nu]$, $\nu=-0.057$, and $\alpha=0.05$. The gray color is proportional to θ. (a) Homogeneous state, (b) kink state between two synchronized precessions shifted by π, (c) and (d) are localized states. The insets represent an instantaneous distributions of spins over the chain and projection of the M_y component.

kink interactions alternates between attractive and repulsive [15]. Therefore, we expect to find a family of localized states with thicknesses that are roughly multiples of the characteristic length of the damped spatial oscillation present in the kink solution. Figures 1(c) and 3(d) show the shortest localized state observed in the pendula chain and those observed in the Landau-Lifshitz-Gilbert equation, respectively. This localized state represents a defect in the synchronized precession, in which the magnetization, at the core of the defect, has a precession shifted in π. Consequently, the magnetic system has a family of novel oscillatory localized states.

To study this intriguing family of localized states exhibited by the above systems, it is necessary to consider the quasi-reversible limit of these systems, corresponding to weak values of both forcing and damping parameters [3]. In the case of this magnetic model, the limit holds when $h_0 \sim \alpha$ are small with respect to the constant field H_0. Thus, the analog pendula chain is described by the perturbed sine-Gordon equation with $\gamma\sim\mu\sim\nu\sim\epsilon$ for the pendula chain, where ϵ is an arbitrary small parameter $\epsilon \ll 1$. We introduce in Eq. (2) for the pendula chain the ansatz

$$\theta(z,t) = 2\sqrt{\frac{\epsilon}{\omega_0}}A(\zeta,\tau)e^{i(\omega_0+\nu)t} - 2\sqrt{\frac{\epsilon}{\omega_0}}\left\{\frac{A^3(\zeta,\tau)}{48} + \frac{i\gamma}{16\omega_0^2}A(\tau,\zeta) - \frac{i\gamma\epsilon}{8\omega_0^3}|A(\zeta,\tau)|^2A(\zeta,\tau)\right\}e^{3i(\omega_0+\nu)t} + \text{c.c.} + \text{h.o.t.}, \quad (3)$$

where $\tau=\epsilon t$, $\zeta=\sqrt{2\epsilon\omega_0/k}z$ are slow variables, in Eq. (2). After straightforward calculation the amplitude A satisfies

$$\partial_\tau A = -i\nu A - i|A|^2A - i\partial_\zeta^2 A - \mu A + \gamma\bar{A} + \text{h.o.t.} \quad (4)$$

The explicit terms of above equation are of order $\epsilon^{3/2}$ and the higher order terms (h.o.t.) are at least of order $\epsilon^{5/2}$. Hence, close to the parametric instability the dynamics, asymptotically, is driven by Eq. (4), a well-known situation as parametrically driven damped nonlinear Schrödinger equation. This model has been used intensively to describe patterns and solitons in several systems such as: Vertically oscillating layer of water [5], localized structures in nonlinear lattices [6], light pulses in optical fibers [7], and the Kerr-type optical parametric oscillator [8].

In the quasireversible limit, the amplitude of small oscillation of the magnetic model (1) is approached by the parametrically driven damped nonlinear Schrödinger equation [9], where the natural frequency for the Landau-Lifshitz-Gilbert model is $\Omega=\sqrt{H_0(H_0+\beta)}$. The parametrically driven damped nonlinear Schrödinger equation has the homogeneous state $A=0$, which represents $\theta(z,t)=0$ and $\mathbf{M}(z,t)=\hat{x}$, respectively. Inside the Arnold's tongue this model also has the uniform state $A_+=\pm[1+i\sqrt{(\gamma-\mu)/(\gamma+\mu)}]x_0$, where $x_0\equiv\sqrt{(\gamma+\mu)(-\nu+\sqrt{\gamma^2-\mu^2})/2\gamma}$. These three states merge together through a pitchfork bifurcation at $\gamma^2=(\nu^2+\mu^2)$, with $\mu>0$. The $|A_\pm|$ represents the amplitude of the homogeneous oscillations for the pendula chain and the magnetic system. However, these uniform states are linear unstable fixed points for the parametrically driven damped nonlinear Schrödinger equation. Also, they are marginal for $\nu=0$, that is, whatever perturbations of the form $A=A_\pm+a_0e^{\lambda t+ikz}$ ($a_0\ll1$) satisfies $\lambda(k)\leq0$ and there are critical wave numbers $k_c=\pm\sqrt{-\nu+2\sqrt{\gamma^2-\mu^2}}$ for which $\lambda(k_c)=0$. At this surface ($\nu=0$), we numerically observe that the uniform state is nonlinearly stable; however, the kink and the localized state which connect these states are unstable. Hence, the model (4) does not account for features of homogenous oscillation and consequently it is unable to describe kinks and families of localized states observed in the original systems. As all these states asymptotically converge to the uniform state, the stability of these particle-type solutions depend of steadiness of these uniform states.

In order to describe the kink solutions and the family of localized states exhibited by the pendula chain and the magnetic system under study, we are required to consider higher order terms in the Eq. (4), since these terms may restore the features of the uniform states and particle type solutions. In the parameter region where the uniform state $|A_\pm|$ is marginal ($\nu=0$), we expect that any small corrections of the amplitude equation can render this state linear stable or unstable. When we consider the higher order terms the amplitude equation reads

$$\partial_\tau A=-i\nu A-iA|A|^2-i\partial_\zeta^2A-\mu A+\gamma\bar{A}+\frac{\epsilon}{\omega_0}\Bigg\{\frac{i}{6}|A|^4A-\frac{3\gamma}{2}|A|^2\bar{A}+\frac{7\gamma}{6}A^3-\mu|A|^2A+\frac{i}{2}(\gamma^2-\mu^2)A-\nu\gamma\bar{A}-2i\left[\frac{1}{4}\partial_\zeta^4A+|A|^2\partial_\zeta^2A+A|\partial_\zeta A|^2+\frac{1}{2}\bar{A}(\partial_\zeta A)^2\right]\Bigg\}, \quad (5)$$

where the terms inside the brackets are order $\epsilon^{5/2}$. For small detuning, we observe numerically that the amended Eq. (5), has stable uniform solutions close to A_+ or A_-, and in this parameter region Eq. (5) exhibits stable solutions connecting these states—kink solutions. Figure 4(a) shows these

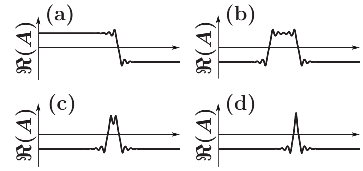

FIG. 4. Localized states exhibited by model (5) for $\mu=0.05$, $\gamma=0.1025$, and $\nu=-0.002$. (a) Kink solution, (b) and (c) localized domains, and (d) smallest localized structure.

particle-type solutions. These walls have spatial damping oscillation, due to the kink and antikink interactions which alternate between attractive and repulsive [15], we find a family of localized states with thickness which are roughly multiples of the characteristic length of the damping spatial oscillation. Figure 4 illustrates the typical localized states observed in model (5). Therefore, the incorporation of higher order terms in the parametrically driven damped nonlinear Schrödinger equation can account for particle-type solutions linking two homogeneous states. We remark that Eq. (4) has been used in several physical contexts [5–9] neglecting the higher order terms and disregarding then the possibility of studying the localized states which connect two uniform oscillations.

The parametrically driven damped nonlinear Schrödinger Eq. (4) includes only cubic nonlinear terms and linear damping and forcing. In order to understand the validity of this approach, we have considered the nonlinear terms of lowest order in Eq. (2), neglecting the fifth order in θ and also ignoring the nonlinear terms proportional to γ. Numerically this extended parametrically driven nonlinear oscillator does not exhibit stable kinks and families of localized states linking two uniform oscillations. When we add the nonlinear terms of the fifth order and the nonlinear parametric forcing up to order three, the system presents the particle-type solutions which connect the uniform oscillations. So, the parametrically driven damped nonlinear Schrödinger Eq. (4) can account for linear parametrically driven extended cubic oscillators.

Kink interactions is the main tool to understand localized states in one-dimensional systems [16], however in two dimensions this interaction is replaced by surface tension [1]. The family of localized structures disappear and only some localized states survive. To analyze the localized states in quasi-reversible systems supported by uniform oscillations, in the 2D situation, we consider the amended amplitude equation

$$\partial_\tau A=-i\nu A-i|A|^2A-i\nabla^2A-\mu A+\gamma\bar{A}+ia|A|^4A+\gamma(b|A|^2\bar{A}+cA^3), \quad (6)$$

where $\nu\sim\gamma\sim\mu\ll1$ and $\{a,b,c\}$ are parameters of order 1. Inside Arnold's tongue and for small detuning, the above model has two uniform states close to $|A_\pm|$ and this model exhibits localized structures. Figure 5 illustrates typical localized states. Hence, we suggest that quasireversible parametrically driven systems in two-dimensions must exhibit localized states.

In conclusion, we have presented a type of localized states

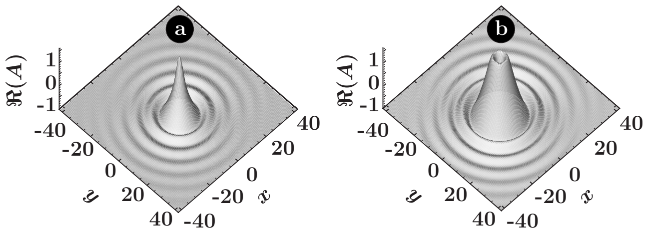

FIG. 5. Localized states observed in model (6) for μ=0.24, γ=0.80, ν=−0.10, a=1.0, and b=1.0.

which link asymptotically homogeneous precession states in a magnetic wire parametrically driven with a magnetic field in the transversal direction. The conventional approach of this system, the parametrically driven damped nonlinear Schrödinger equation in one dimension has been a successful model to explain pattern and localized states which connect uniform states in parametrically driven quasireversible systems. However, this model lacks this family of localized states, which asymptotically connect a uniform oscillatory state with itself. The improvement of this model by the consideration of higher order terms allow us to recover and to account for this localized state. Due to the unified description that we have considered, the same family of localized states is observed in parametrically driven damped pendula chain. The study of the dynamics, in particular the permanent complex dynamics [17], and the interaction of these states in one and two dimensions is in progress.

The authors thanks the support of ring program ACT15 of *Programa Bicentenario* of the Chilean Government. M.G.C. acknowledges the financial support of FONDAP, Grant No. 11980002. D.L. acknowledges the partial support of ring program ACT24 of *Programa Bicentenario* of the Chilean Government. S.C. is thankful for the financial support of FONDECYT, Grant No. 3080041.

[1] P. Coullet, Int. J. Bifurcation Chaos Appl. Sci. Eng. **12**, 2445 (2002).
[2] W. van Saarloos and P. C. Hohenberg, Phys. Rev. Lett. **64**, 749 (1990).
[3] M. Clerc, P. Coullet, and E. Tirapegui, Phys. Rev. Lett. **83**, 3820 (1999); M. Clerc, P. Coullet, and E. Tirapegui, Int. J. Bifurcation Chaos Appl. Sci. Eng. **11**, 591 (2001); M. Clerc, P. Coullet, and E. Tirapegui, Opt. Commun. **167**, 159 (1999); M. Clerc, P. Coullet, and E. Tirapegui, Phys. Lett. A **287**, 198 (2001); M. Clerc, P. Coullet, and E. Tirapegui, Prog. Theor. Phys. Suppl. **139**, 337 (2000).
[4] I. V. Barashenkov and E. V. Zemlyanaya, Phys. Rev. Lett. **83**, 2568 (1999).
[5] J. W. Miles, J. Fluid Mech. **148**, 451 (1984); W. Zhang and J. Viñals, Phys. Rev. Lett. **74**, 690 (1995); X. Wang and R. Wei, Phys. Rev. E **57**, 2405 (1998).
[6] B. Denardo, B. Galvin, A. Greenfield, A. Larraza, S. Putterman, and W. Wright, Phys. Rev. Lett. **68**, 1730 (1992).
[7] J. N. Kutz *et al.*, Opt. Lett. **18**, 802 (1993).
[8] S. Longhi, Phys. Rev. E **53**, 5520 (1996).
[9] I. V. Barashenkov, M. M. Bogdan, and V. I. Korobov, Europhys. Lett. **15**, 113 (1991).
[10] N. V. Alexeeva, I. V. Barashenkov, and G. P. Tsironis, Phys. Rev. Lett. **84**, 3053 (2000).
[11] N. Grønbech-Jensen, Y. S. Kivshar, and M. R. Samuelsen, Phys. Rev. B **47**, 5013 (1993).
[12] M. A. Hoefer, M. J. Ablowitz, B. Ilan, M. R. Pufall, and T. J. Silva, Phys. Rev. Lett. **95**, 267206 (2005).
[13] H. J. Mikeska, J. Phys. C **11**, L29 (1978).
[14] V. I. Arnold, *Geometrical Methods in the Theory of Ordinary Differential Equations* (Springer-Verlag, New York, 1983).
[15] M.G. Clerc, D. Escaff and V.M. Kenkre, Phys. Rev. E **72**, 056217 (2005), and references therein.
[16] M. G. Clerc and C. Falcon, Physica A **356**, 48 (2005).
[17] M.G. Clerc, A. Petrossian, and S. Residori, Phys. Rev. E **71**, 015205(R) (2005).
[18] The simulation software, DimX, was developed at INLN and used for all numerical simulations presented in this paper.