Predicting the synchronization time in coupled-map networks

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An analytical expression for the synchronization time in coupled-map networks is given. By means of the expression, the synchronization time for any given network can be predicted accurately. Furthermore, for networks in which the distributions of nontrivial eigenvalues of coupling matrices have some unique characteristics, analytical results for the minimal synchronization time are given.

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I. INTRODUCTION

Synchronization occurs ubiquitously and performs important roles in natural as well as artificial systems [1]. In the brain, synchronization of neural activity has been shown to exist between different columns and hemispheres and underlies information processing and learning roles [2]. Based on the synchronization of chaotic lasers, secure communications have been realized both in the laboratory and over commercial fiber-optic channels [3]. To understand these diverse synchronization phenomena, theoretical studies have considered various node dynamics (e.g., excitable, periodic, and chaotic) and network topologies (e.g., regular, random, small world, scale-free, and modular) [4]. For completely regular (such as all-to-all, nearest-neighbor, long-range) [5] or completely random (such as Erdös-Rényi) [6] coupled-map networks, some analytical results have been given. However, it is relatively difficult to give analytical results for more complex networks (such as small world, scale-free, and modular) [7].

When studying the synchronization in coupled-map networks, one important point is to find the time needed for the networks to become synchronized: *the synchronization time*. It reflects not only whether the networks can synchronize, but also how fast the synchronization is. Normally, the synchronization time is obtained through direct numerical simulations, while in this paper we give an analytical expression for it. By means of this expression, we can predict the synchronization time for any given network. In Sec. II, the theoretical analysis for a general coupled-map network and some necessary definitions are given. In Sec. III, a directed random network is studied as an illustration to verify the theoretical analysis. For two classes of networks in which the distributions of nontrivial eigenvalues of coupling matrices have some unique characteristics, the analytical and numerical results of the minimal synchronization time are given in Sec. IV. In Sec. V, nonreciprocity effects on the synchronization are studied. Finally, a discussion and conclusions are given in Sec. VI.

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II. THEORETICAL ANALYSIS

The dynamics of a general network of N coupled identical maps is described by

$$\mathbf{x}_i(t+1)=\mathbf{F}[\mathbf{x}_i(t)]+\epsilon\sum_{j=1}^{N}C_{ij}\mathbf{H}\{\mathbf{F}[\mathbf{x}_j(t)]\}, \qquad (1)$$

where $i=1,\ldots,N$, $\mathbf{x}(t+1)=\mathbf{F}[\mathbf{x}(t)]$ is an m-dimensional map, ϵ is the global coupling strength, $\mathbf{C}=(C_{ij})$ is the coupling matrix, $C_{ij}=A_{ij}/k_i$ for $j\neq i$ and $C_{ii}=-1$, where k_i is the in-degree of node i and A_{ij} is an element of the adjacency matrix $\mathbf{A}$, and $\mathbf{H}$ is the coupling function. Since the rows of $\mathbf{C}$ have zero sum, the network permits a synchronization manifold $\mathbf{x}_1(t)=\mathbf{x}_2(t)=\cdots=\mathbf{x}_N(t)=\mathbf{s}(t)$, where $\mathbf{s}(t)$ is the synchronous state with $\mathbf{s}(t+1)=\mathbf{F}[\mathbf{s}(t)]$.

The stability of the synchronous state can be determined by letting $\mathbf{x}_i(t)=\mathbf{s}(t)+\delta\mathbf{x}_i(t)$ and linearizing Eq. (1) around $\mathbf{s}(t)$. This leads to

$$\delta\mathbf{x}_i(t+1)=F'[\mathbf{s}(t)]\delta\mathbf{x}_i(t)+\epsilon\sum_{j=1}^{N}C_{ij}H'\{\mathbf{F}[\mathbf{s}(t)]\}F'[\mathbf{s}(t)]\delta\mathbf{x}_j(t), \qquad (2)$$

where F' and H' are the Jacobian matrices of the corresponding vector functions. Diagonalizing $\mathbf{C}$ yields a set of eigenvalues $\{\lambda_i, i=1,\ldots,N\}$, whose real parts are nonpositive. We sort the eigenvalues as $0=\lambda_1^{\mathrm{real}}\geq\lambda_2^{\mathrm{real}}\geq\cdots\geq\lambda_N^{\mathrm{real}}$ and denote the corresponding normalized eigenvectors by $\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_N$. The denser the distribution of the nontrivial eigenvalues is, the stronger is the synchronizability of the network, i.e., both the eigenratio $\lambda_N^{\mathrm{real}}/\lambda_2^{\mathrm{real}}$ and $\max\{|\lambda_{l,l=2,\ldots,N}^{\mathrm{imaginary}}|\}$ are simultaneously smaller [8]. The transform $\delta\mathbf{y}=\mathbf{O}^{-1}\delta\mathbf{x}$, where $\mathbf{O}$ is a matrix whose columns are the set of normalized eigenvectors, leads to the block-diagonally decoupled form $\delta\mathbf{y}_i(t+1)=(\mathbf{I}+\epsilon\lambda_i H'\{\mathbf{F}[\mathbf{s}(t)]\})F'[\mathbf{s}(t)]\delta\mathbf{y}_i(t)$. The synchronization is stable if $\lim_{t\to\infty}(1/t)\ln[|\delta\mathbf{y}_i(t)|/|\delta\mathbf{y}_i(0)|]<0$ for any $i\in\{2,\ldots,N\}$. Direct calculation gives $\lim_{t\to\infty}(1/t)\ln\prod_{\tau=0}^{t-1}|F'[\mathbf{s}(\tau)]| +\lim_{t\to\infty}(1/t)\ln\prod_{\tau=0}^{t-1}|1+\epsilon\lambda_i H'\{\mathbf{F}[\mathbf{s}(\tau)]\}|<0$, where the first term is the largest Lyapunov exponent of a single map: $\Lambda(0)$. Replacing $\epsilon\lambda_i$ by $\alpha+i\beta$, we get the master stability function (MSF) [9] $\Lambda(\alpha+i\beta) =\Lambda(0)+\lim_{t\to\infty}(1/t)\ln\prod_{\tau=0}^{t-1}|1+(\alpha+i\beta)H'\{\mathbf{F}[\mathbf{s}(\tau)]\}|<0$. To

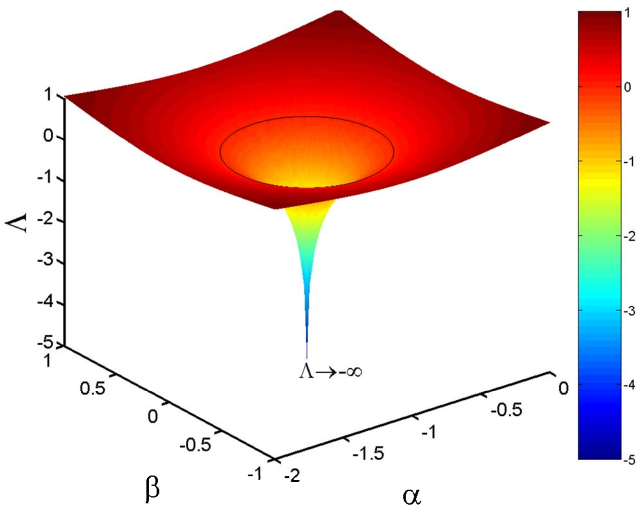

FIG. 1. (Color online) MSF for the logistic map with $\mu=4$ $[\Lambda(0)=\ln 2]$. The black solid line denotes where $\Lambda=0$.

be concrete, we choose $\mathbf{F}(\mathbf{x})=\mu x(1-x)$ (the logistic map) and $\mathbf{H}(\mathbf{x})=x$; then we get

$$\Lambda(\alpha+i\beta)=\Lambda(0)+\ln|1+\alpha+i\beta|<0, \tag{3}$$

which is equivalent to $\sqrt{(1+\alpha)^2+\beta^2}<e^{-\Lambda(0)}$. Here, $e^{-\Lambda(0)}=0.5$ for $\mu=4$ $[\Lambda(0)=\ln 2]$. Figure 1 shows the MSF, where at the point $(\alpha,\beta)=(-1,0)$, $\Lambda\to-\infty$. It should be noted that, if the coupling function $\mathbf{H}$ is nonlinear, $H'\{\mathbf{F}[\mathbf{s}(t)]\}$ will depend on the value of $\mathbf{s}(t)$ through $\mathbf{F}[\mathbf{s}(t)]$ and it is then difficult to obtain an explicit expression for the MSF just like Eq. (3).

In numerical simulations, synchronization can be defined as the synchronization error $\Delta_{\rm sync}(t)=\frac{1}{N}\sum_{j=1}^{N}|\mathbf{x}_j(t)-\langle\mathbf{x}(t)\rangle|\le\delta$, where $\langle\mathbf{x}(t)\rangle=\frac{1}{N}\sum_{j=1}^{N}\mathbf{x}_j(t)$ and δ is the synchronization accuracy. The synchronization time $T_{\rm sync}$, defined as the average time required for the network to become synchronized, i.e., $\Delta_{\rm sync}(t)<\delta$ for $t>T_{\rm sync}$, can be used to characterize the synchronizability of the network. If the network is unsynchronizable, $T_{\rm sync}\to\infty$. Otherwise, the synchronization error can be expressed as

$$\begin{aligned}\Delta_{\rm sync}(t)&=\frac{1}{N-1}[\Delta_{\rm sync}(0)e^{\Lambda(\epsilon\lambda_2)t}+\Delta_{\rm sync}(0)e^{\Lambda(\epsilon\lambda_3)t}\\&\quad+\cdots+\Delta_{\rm sync}(0)e^{\Lambda(\epsilon\lambda_N)t}]\\&=\Delta_{(\rm sync)}(0)(\sum_{j=2}^{N}e^{\Lambda(\epsilon\lambda_j)t})/(N-1)\end{aligned}$$

(exact expression) with $\Lambda(\epsilon\lambda_j)<0$, $\forall\ j\in\{2,\ldots,N\}$. Because the terms in the exact expression of $\Delta_{\rm sync}(t)$ with more negative Λ quickly approach zero as t increases, at last only the term with the least negative Λ dominates. Therefore, the synchronization error can be simplified to $\Delta_{\rm sync}(t)=\Delta_{\rm sync}(0)\frac{e^{\max\{\Lambda(\epsilon\lambda_j)\}t}}{N-1}$ (approximate expression). By setting $\Delta_{\rm sync}(0)\frac{e^{\Lambda^{\max}T_{\rm sync}}}{N-1}=\delta$ [here $\Lambda^{\max}=\max\{\Lambda(\epsilon\lambda_j)\}$], we get

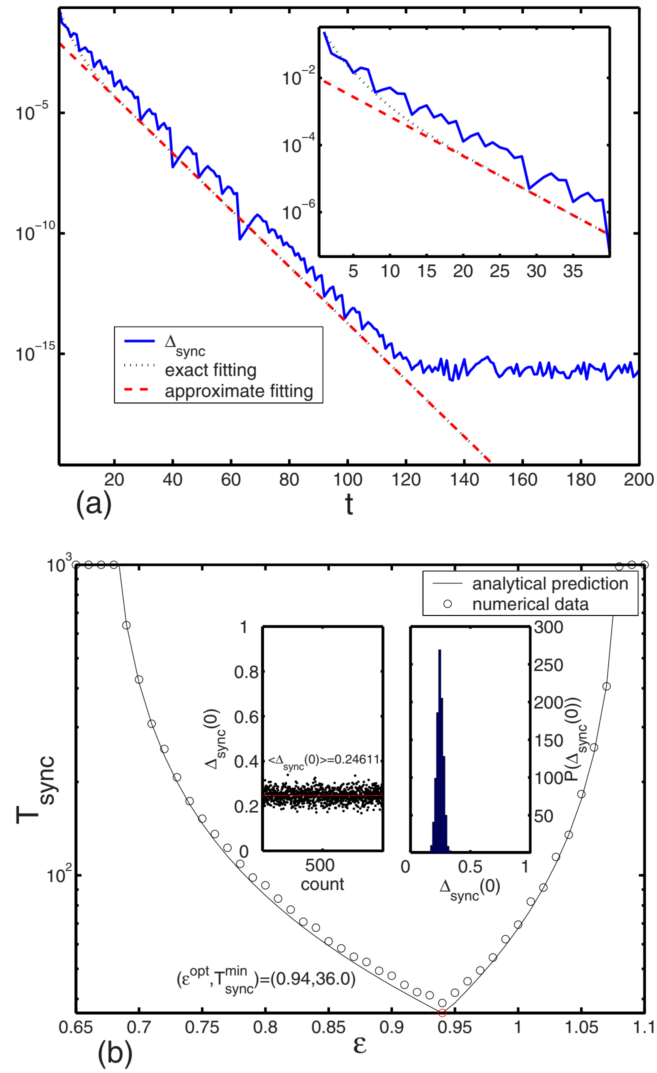

FIG. 2. (Color online) (a) Time series of $\Delta_{\rm sync}$ and two kinds of fitting for a directed random network with $\epsilon=1.0$. (b) Analytical prediction and numerical data for $T_{\rm sync}$ versus ϵ. Each numerical data point is the result of averaging over ten random initial conditions. Left inset shows each value of $\Delta_{\rm sync}(0)$ (black dots) and its average value $\langle\Delta_{\rm sync}(0)\rangle=0.246\,11\approx0.25$ (red solid line) for 1000 realizations. Right inset shows the statistical distribution of $\Delta_{\rm sync}(0)$. Here, $N=30$, $p=0.25$, and $\delta=10^{-10}$.

$$T_{\rm sync}=-\frac{1}{\Lambda^{\max}}[-\ln\delta-\ln(N-1)+\ln\Delta_{\rm sync}(0)]. \tag{4}$$

Defining the speed of synchronization as $v=-\Lambda^{\max}$, we have $T_{\rm sync}\sim1/v$, i.e., the network with higher v synchronizes faster than that with the lower one.

III. ILLUSTRATION

To verify the theoretical analysis given above, we study a directed random network with $N=30$ and connection probability $p=0.25$ as an illustration; each initial value of x is randomly selected in [0,1]. The nontrivial eigenvalues of $\mathbf{C}$ are all in the negative region of the MSF. First, we fix ϵ

$=1.0$ and calculate the time series of $\Delta_{\rm sync}$ numerically; then fit it with the exact and approximate expressions given above, respectively [see Fig. 2(a)]. We find that these two expressions fit $\Delta_{\rm sync}$ very well after some time, while for the first several steps the former fits better than the latter [see the inset of Fig. 2(a)]. Second, for different ϵ, we calculate the corresponding $T_{\rm sync}$ analytically and numerically and then draw them together [see Fig. 2(b)]. We find that they match each other very well. Here, $\Delta_{\rm sync}(0)=0.25$ in Eq. (4), because $\Delta_{\rm sync}(0)$ is around 0.25 with the highest probability, as shown in the right inset of Fig. 2(b) for the uniform distribution in [0,1] of $x(0)$. When N is larger, the statistical distribution of $\Delta_{\rm sync}(0)$ becomes more sharply peaked around 0.25. An interesting phenomenon should be noted: there exists an optimal value of ϵ, $\epsilon^{\rm opt}$, at which the network synchronizes with the minimal value of $T_{\rm sync}$, $T_{\rm sync}^{\rm min}$; here $(\epsilon^{\rm opt},T_{\rm sync}^{\rm min})=(0.94,36.0)$ [see Fig. 2(b)]. A similar phenomenon has also been found in a coupled-map lattice consisting of a chain of chaotic logistic maps exhibiting power law interactions [10]. In conclusion, by means of Eq. (4), we can get $T_{\rm sync}$ analytically without resorting to direct numerical simulations.

IV. TWO CLASSES OF NETWORKS

For some networks, in which the distributions of nontrivial eigenvalues of **C** have some unique characteristics, we can give the analytical results for $T_{\rm sync}^{\rm min}$. (i) For some bidirectional networks, if they are sufficiently random, the spectra of the **C**'s tend to the semicircle law for large N with arbitrary expected degree [11]. This means that $\lambda_2 \approx -1-2/\sqrt{K}$, $\lambda_N \approx -1+2/\sqrt{K}$, where K is the average degree, which is found to provide a good approximation under the condition $\min\{k_i\} \gg 1$, regardless of the degree distribution, e.g., for bidirectional K-regular random networks, bidirectional random networks with connection probability $p=K/(N-1)$, and bidirectional scale-free networks with $d=K/2$ [12]. (ii) For some unidirectional networks, e.g., unidirectional K-regular random networks and unidirectional random networks with connection probability $p=K/(N-1)$, with sufficiently large N, the distribution of nontrivial eigenvalues resembles a disk in the complex plane that is centered at -1 and has a radius $r_{\rm RMT}=(\frac{1}{K}-\frac{1}{N})^{1/2}$, which can be deduced by random matrix theory (RMT) [13]. It follows that $\lambda_j \in \{-1+r_j(\cos\theta_j+i\sin\theta_j)\}$ for $j=2,\ldots,N$; here $r_j\in[0,r_{\rm RMT}]$ and $\theta_j\in[0,2\pi]$. We numerically calculate $\Lambda^{\rm max}$ defined above in the $(\epsilon$-$K)$ parameter space for two classes of networks. From the insets of Figs. 3(a) and 3(b) we find that $\epsilon^{\rm opt}=1$, i.e., where $\Lambda^{\rm max}$ is most negative. Setting $\epsilon=\epsilon^{\rm opt}$, for the bidirectional networks, we get $\Lambda^{\rm max}=\Lambda(0)+\ln\frac{2}{\sqrt{K}}$. For $\mu=4$ [$\Lambda(0)=\ln 2$], only when $K>K_{\rm bidirectional}^{\rm min}=16$ is $\Lambda^{\rm max}<0$. In addition, from Eq. (4) we get $T_{\rm sync}^{\rm min}\approx-\frac{1}{\Lambda(0)+\ln(2/\sqrt{K})}[-\ln\delta-\ln(N-1)+\ln\Delta_{\rm sync}(0)]$, while, for the unidirectional networks, we get $\Lambda^{\rm max}=\Lambda(0)+\ln(\frac{1}{K}-\frac{1}{N})^{1/2}\approx\Lambda(0)+\ln\frac{1}{\sqrt{K}}$ for $N\gg 1$. Here, only when $K>K_{\rm unidirectional}^{\rm min}=4$ is $\Lambda^{\rm max}<0$. Similarly, from Eq. (4) we get $T_{\rm sync}^{\rm min}\approx-\frac{1}{\Lambda(0)+\ln(1/\sqrt{K})}[-\ln\delta-\ln(N-1)+\ln\Delta_{sync}(0)]$. Figures 3(a) and 3(b) show both the analytical predictions (the curves) and the numerical results (the symbols) for $T_{\rm sync}^{\rm min}$ for different values of K; they match each other very well. Based on the results given above, we conclude that, for these two classes of networks, $T_{\rm sync}^{\rm min}$ depends only on the average in-degree K: $T_{\rm sync}^{\rm min}$ decreases with increasing K, while the network size N and detailed network structure have almost no effect on it. This phenomenon is reminiscent of the results of pulse-coupled leaky integrate-and-fire neurons interacting on asymmetric random networks [13] and an array of chaotic logistic maps coupled with random delay times [14].

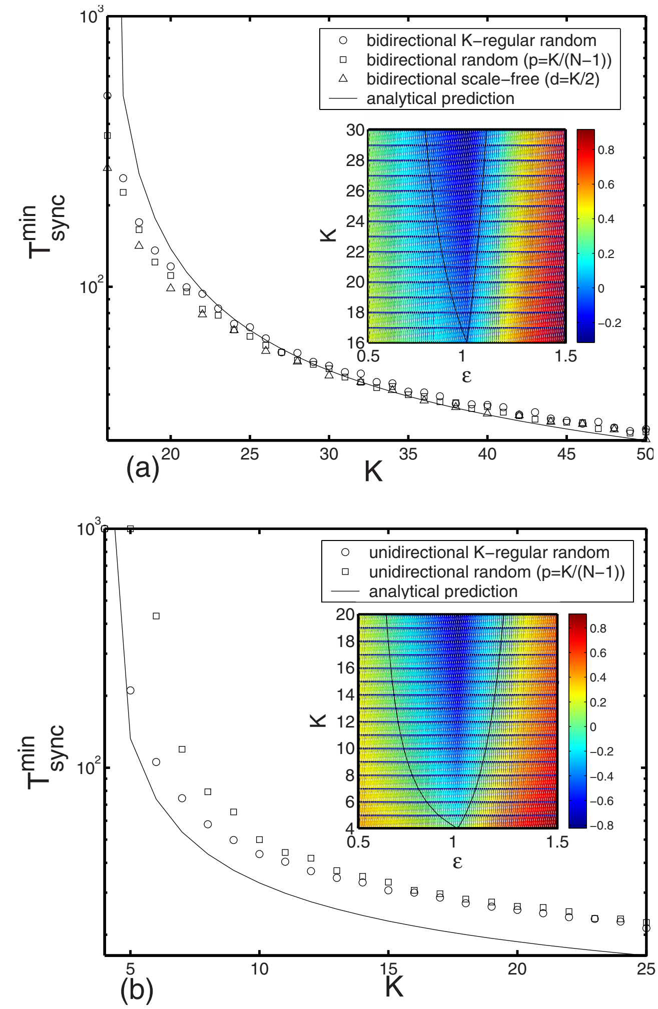

FIG. 3. (Color online) Analytical prediction and numerical data for $T_{\rm sync}^{\rm min}$ versus K for two classes of networks: the bidirectional (a) and unidirectional (b) networks. Each numerical data point is the result of averaging over ten random initial conditions. Insets: $\Lambda^{\rm max}$ in $(\epsilon$-$K)$ parameter space, and the black solid line denotes where $\Lambda^{\rm max}=0$. Here, $N=1000$ and $\delta=10^{-10}$.

V. NONRECIPROCITY EFFECTS

The networks discussed above are either purely bidirectional or purely unidirectional. What will happens if the networks are in between? To answer this question, we put forward a rewiring scheme with which a general network can be transformed from purely bidirectional to purely unidirectional continuously while the in-degree of each node in the

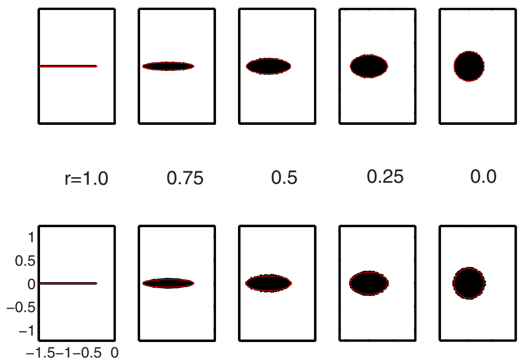

FIG. 4. (Color online) Distributions of eigenvalues of coupling matrices for several values of r. Top: K-regular random networks. Bottom: Random networks with connection probability $p=K/(N-1)$. Here, red solid lines denote the RMT predictions, and $N=1000$, $K=10$.

original network is conserved during the transformation. First, we generate a purely bidirectional network with a given degree distribution. Second, we select one connection in the original network randomly and break it with probability $p^{\rm rewire}$. Third, we select two other nodes randomly, which have no connection to either node of the broken connection, and connect them to the two nodes of the broken connection. Finally, we repeat the aforementioned procedure until each connection in the original network has been chosen just one time. We define $r=1-p^{\rm rewire}$, which is just the so-called reciprocity in network terminology, defined as the ratio of the number of connections pointing in both directions to the total number of connections [15]. Therefore, by increasing the value of $p^{\rm rewire}$ in $[0,1]$ continuously, we can study the nonreciprocity effects on the synchronization systematically.

A. *K*-regular random networks and random networks with connection probability $p=K/(N-1)$

We apply the rewiring scheme to K-regular random networks and calculate the eigenvalues of the coupling matrices numerically. The distribution of nontrivial eigenvalues as shown in the up row of Fig. 4 is reminiscent of the generalized circle law of RMT for Gaussian asymmetric random matrices [16]. Following the spirit of Ref. [13], we extend the generalized circle law to sparse matrices ($K \ll N-1$) and deduce the analytical boundary of nontrivial eigenvalues for different values of r when $N\to\infty$. To directly compare the nontrivial eigenvalues of coupling matrices, which have average eigenvalue $\langle\lambda_i\rangle=\frac{1}{N}\Sigma_{i=1}^{N}\lambda_i=-1$, to those of the Gaussian ensemble, we transform $C'_{ij}=C_{ij}+\delta_{ij}$, shifting the average eigenvalue to $\langle\lambda'_i\rangle=0$. Here δ_{ij} denotes the Kronecker delta, $\delta_{ij}=1$ if $i=j$ and $\delta_{ij}=0$ otherwise. For the variance of C' we obtain $\langle C'^2_{ij}\rangle=\frac{1}{N^2}\Sigma_{i=1}^{N}\Sigma_{j=1}^{N}C'^2_{ij}=\frac{1}{N}\frac{1}{K}=\sigma^2_{C'}$, and $\langle C'_{ij}C'_{ji}\rangle=\frac{1}{N^2}\Sigma_{i=1}^{N}\Sigma_{j=1}^{N}C'_{ij}C'_{ji}=r\frac{1}{N}\frac{1}{K}=r\sigma^2_{C'}$; here $\langle C'_{ij}\rangle^2=(\frac{1}{N^2}\Sigma_{i=1}^{N}\Sigma_{j=1}^{N}C'_{ij})^2=\frac{1}{N^2}\to 0$ when $N\to\infty$. Therefore, we get $r_{\rm RMT}=N^{1/2}\sigma_{C'}=\frac{1}{\sqrt{K}}$, i.e., all the nontrivial eigenvalues are distributed uniformly in an ellipse $\frac{(x+1)^2}{a^2}+\frac{y^2}{b^2}\le r^2_{\rm RMT}$; here $a=1+r$ and $b=1-r$. It should be noted that, even for random networks constructed by choosing every connection with probability $p=K/(N-1)$, we get good predictions if K is replaced by $\langle K\rangle=\frac{1}{N}\Sigma_{i=1}^{N}k_i$ (see the bottom row of Fig. 4). It shows that the RMT predictions match the numerical results very well.

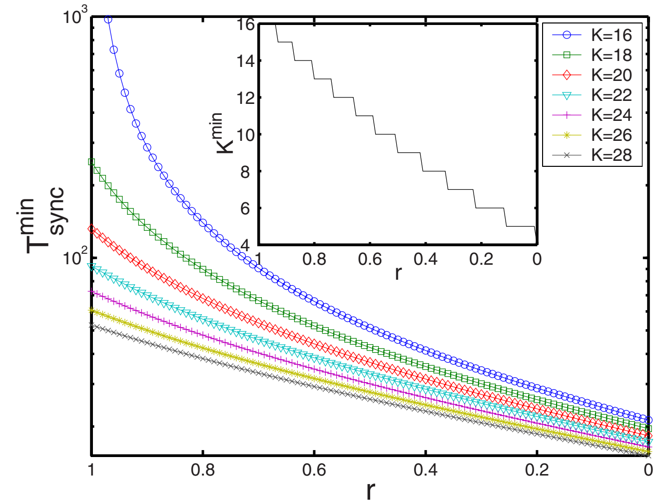

FIG. 5. (Color online) $T^{\rm min}_{\rm sync}$ versus r for several values of K. Inset: $K^{\rm min}$ versus r. Here, $N=1000$ and $\delta=10^{-10}$.

Based on the results given above, when $\epsilon=\epsilon^{\rm opt}=1$, $\Lambda^{\rm max}=\Lambda(0)+\ln[(1+r)\frac{1}{\sqrt{K}}]$. Therefore, from Eq. (4) we get

$$T^{\rm min}_{\rm sync}\approx-\frac{1}{\Lambda(0)+\ln\left[(1+r)\frac{1}{\sqrt{K}}\right]}\times[-\ln\delta-\ln(N-1)+\ln\Delta_{\rm sync}(0)]. \quad (5)$$

Equation (5) comprises the purely bidirectional case ($r=1$) and the purely unidirectional case ($r=0$) discussed above. Figure 5 shows the relation between $T^{\rm min}_{\rm sync}$ and r for several values of K. With decreasing r, $T^{\rm min}_{\rm sync}$ decreases logarithmically. If the network is synchronizable, $\Lambda^{\rm max}$ should be negative, i.e., $\Lambda(0)+\ln[(1+r)\frac{1}{\sqrt{K}}]<0$. We get $K>[4(1+r)^2]$ for $\mu=4$ [$\Lambda(0)=\ln 2$]; here $[\cdot]$ denotes the integral part: $K^{\rm min}_{\rm bidirectional}=16$ for $r=1$; $K^{\rm min}_{\rm unidirectional}=4$ for $r=0$ just as in the results given in Sec. IV. The relation between $K^{\rm min}$ and r is shown in the inset of Fig. 5. $K^{\rm min}$ decreases nearly as the square of r, but noncontinuously, with decreasing r from 16 to 4. Based on the results given above, we conclude that synchronization is enhanced by the nonreciprocity.

B. Small-world and scale-free networks

Up to now we have studied networks in which each node has fully uniform (K-regular random networks) or almost uniform [random networks with connection probability $p=K/(N-1)$] in-degrees. Applying the rewiring scheme to other structured networks, e.g., small-world [17] and scale-free [12] networks, we find that the enhancement effect of the nonreciprocity exists, too. As shown in Fig. 6, the distribution of the nontrivial eigenvalues becomes denser with decreasing r. Furthermore, the enhancement is more obvious for small-world networks than for scale-free networks when

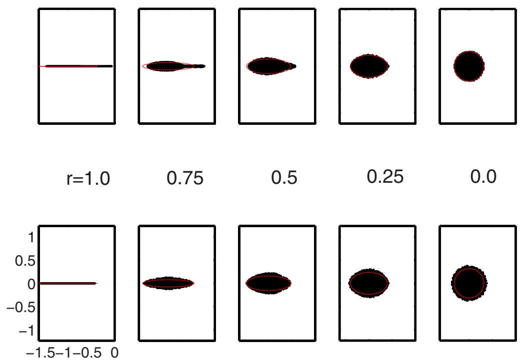

FIG. 6. (Color online) Distributions of eigenvalues of coupling matrices for several values of r. Top: Small-world networks. Bottom: Scale-free networks. Here, red solid lines are identical to those of Fig. 4 just for comparison, and $N=1000$, $K=10$.

r is relatively large, e.g., from 1.0 through 0.75 to 0.5 (see Fig. 6). When $r=0.0$, the nontrivial eigenvalue distribution for small-world networks is almost identical to those of the two random networks discussed above. For scale-free networks, the nontrivial eigenvalues distribute more sparsely than those of the other three networks (see the rightmost column of Fig. 6). We explain it as follows: the in-degree dispersions of the former three networks are almost identical and very small, while for scale-free networks, the dispersion of in-degrees is larger because of the power law characteristic. Using RMT, we have $r_{\mathrm{RMT}}=N^{1/2}\sigma_{C'}$, and $\sigma_{C'}^2=\langle C'^2_{ij}\rangle=\frac{1}{N^2}\sum_{i=1}^N\sum_{j=1}^N C'^2_{ij}=\frac{1}{N^2}\sum_{i=1}^N\frac{1}{k_i}$, where $C'_{ij}=A_{ij}/k_i$, while $\sum_{i=1}^N\frac{1}{\langle K\rangle}=\sum_{i=1}^N 1/(\frac{1}{N}\sum_{i=1}^N k_i)=\frac{N^2}{k_1+k_2+\cdots+k_N}$. Since $(k_1+k_2+\cdots+k_N)(\frac{1}{k_1}+\frac{1}{k_2}+\cdots+\frac{1}{k_N})\geq N^2$, we get $\sum_{i=1}^N\frac{1}{\langle K\rangle}\leq\sum_{i=1}^N\frac{1}{k_i}$. Therefore, if each node has almost the same in-degree as the average in-degree of the whole network, $\sigma_{C'}$ will be small, i.e., r_{RMT} will be small.

VI. DISCUSSION AND CONCLUSION

The method for predicting the synchronization time given above is valid only for diagonalizable networks. Although synchronizability is optimal in some nondiagonalizable networks, as discussed in Ref. [18], the synchronization time increases when these networks become larger [19]. In this case, Eq. (4) is not valid any more. Furthermore, while the rewiring scheme is applied to a network to change the reciprocity, the average path length of the network is also changed: with decreasing value of r, the average path length decreases, too. This may explain the enhancement effect by nonreciprocity found in Sec. V.

In conclusion, we give an analytical expression for the synchronization time in coupled-map networks. By means of this expression, we can accurately predict the synchronization time for any given network. For networks in which the distributions of nontrivial eigenvalues of coupling matrices have some unique characteristics, the analytical results for the minimal synchronization time are given. Our work may provide fresh insight into the control of the speed of synchronization in engineering, as for secure communications [20].

ACKNOWLEDGMENTS

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