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Dynamics, correlation scaling, and synchronization behavior in rings of delay-coupled oscillators

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We study the dynamics of unidirectionally delay-coupled nonlinear oscillators. Cascading them within a ring of fixed total propagation delay, we demonstrate simple scaling behavior of correlation properties. In fact, the correlation properties of a ring with N elements can be deduced from the autocorrelation of the single delayed feedback system. Coupling a ring element to a chain of unidirectionally coupled identical oscillators, we achieve complete synchronization between elements of chain and ring, evidencing generalized synchronization among the other elements, even if uncorrelated.

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Coupled nonlinear oscillators are a paradigm in science. The study of their collective dynamical behavior has proven to be successful to describe and understand properties of complex systems in general, including synchronization and resonance phenomena [1–4]. In this paper, we focus on the chaotic dynamics induced by coupling several dynamical elements with a fixed time delay. Delayed coupling is of particular interest as it is common in nature and technology. It was found that delayed coupling can induce both instabilities and synchronization in, e.g., physiological and biological systems [5] and in laser systems [6].

To understand the properties of complex networks of delay-coupled oscillators, it is important to understand the dynamics of basic building blocks such as rings and chains. Rings of self-sustained chaotic oscillators have been studied in the context of death by delay [7] and synchronization [8,9]. Complementary, it has been recently shown that delay and coupling induce complex chaotic behavior in otherwise stable oscillators [6,10–13]. Our aim in this paper is to understand the complex dynamics generated in a ring of unidirectionally coupled oscillators that are stable when uncoupled. Specifically, we address the possibility of explaining this emergent dynamics of many elements in terms of a few elements. To this end, we look for scaling laws of correlation and spectral properties. We anticipate that the spectral and synchronization properties of N elements can be understood and predicted by those of a single element subject to self-feedback.

As depicted in Fig. 1(c), we consider a unidirectional ring configuration consisting of an arbitrary number N of identical nonlinear oscillators (semiconductor lasers or Ikeda oscillators) evenly spaced with a coupling time delay τ/N. In the special case of $N=1$ and $N=2$, the ring configuration is reduced to an oscillator subject to delayed self-feedback and two mutually delay-coupled identical oscillators, respectively, as depicted in Figs. 1(a) and 1(b). Both semiconductor lasers with delayed self-feedback [14] and mutually delay-coupled lasers [6] have been widely studied. In the following, we address how the dynamics of these two systems compares and how the complex dynamics emerges as N is further increased. Finally, we will couple the chaotic output of the ring to a chain of oscillators and demonstrate that identical synchronization between elements of both systems can be achieved.

For the first part of this study, we have chosen semiconductor lasers (SLs) as nonlinear oscillators. SLs have proven to be attractive for studying delay-coupled nonlinear oscillators. They can be well controlled and established models exist. SLs subject to delayed feedback generate chaotic dynamics with intensity pulsations on subnanosecond time scales [15]. Delays do not only occur generically by optical feedback, they also occur when coupling SLs to each other even for short propagation distances.

The model equations for the complex slowly varying envelope of the electric field and carrier number inside the cavity describing a single mode SL with optical injection from another SL in the ring are

$$\frac{dE_j(t)}{dt} = \frac{1+i\alpha}{2}\left[G_j - \frac{1}{\tau_{ph}}\right]E_j + \kappa E_{j-1}\left(t-\frac{\tau}{N}\right)e^{i\Omega(\tau/N)},$$

$$\frac{d\eta_j(t)}{dt} = \frac{I}{e} - \frac{\eta_j}{\tau_n} - G_j|E_j|^2,$$

with $j=0,1,\ldots,N-1$. $G_j \equiv g(\eta_j-\eta_0)/(1+s|E|^2)$ is the optical gain of laser j, Ω is the frequency of the free-running

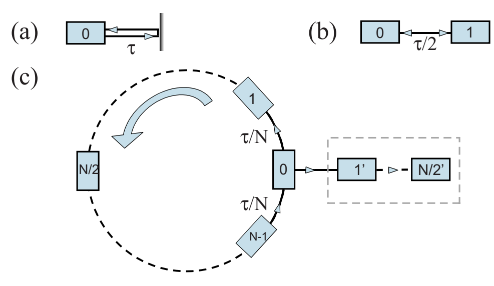

FIG. 1. (Color online) (a) Oscillator with delayed feedback. (b) Two bidirectionally delay-coupled oscillators. (c) A ring configuration of N unidirectionally delay-coupled oscillators which can optionally be coupled to a chain of oscillators.

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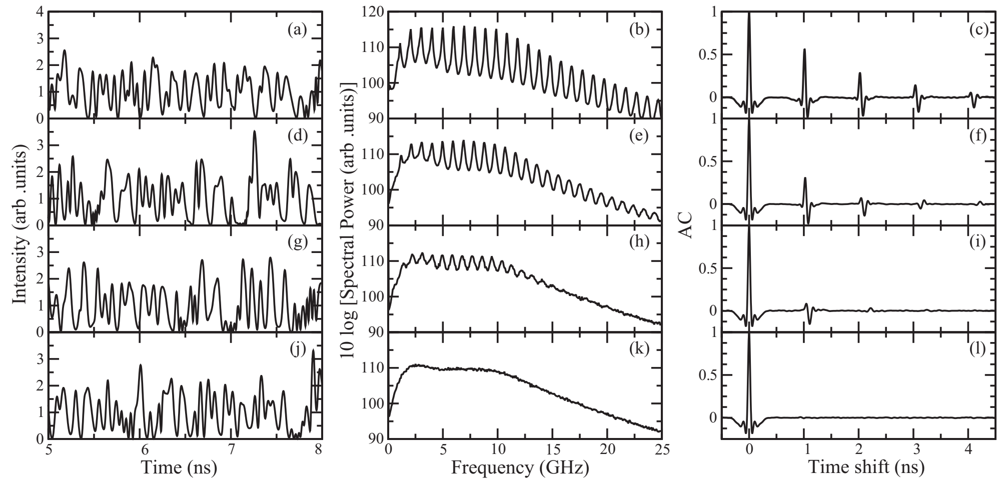

FIG. 2. Time traces (left panel), power spectra (middle panel), and normalized intensity autocorrelation functions (AC, right panel) of the emission of one laser for different values of N. (a)–(c) correspond to a SL with delayed optical feedback, (d)–(f) to two bidirectionally coupled SLs, (g)–(i) to 4 SLs in a ring configuration, and (j)–(l) to 100 SL in a ring configuration.

laser, $\kappa=40\ \mathrm{ns}^{-1}$ is the coupling coefficient, and $\tau=1$ ns is the roundtrip time in the ring, $\alpha=5$ is the linewidth enhancement factor, representing the major nonlinearity, $g=1.5\times10^{-8}\ \mathrm{ps}^{-1}$ is the differential gain parameter, $s=5\times10^{-7}$ is the gain saturation coefficient, $\tau_{ph}=2$ ps is the photon lifetime, $\tau_n=2$ ns is the carrier lifetime, and $\eta_0=1.5\times10^8$ is the carrier number at transparency. We fix the pump current to $I=1.5I_{th}$, where the lasers are stable without coupling, exhibiting a relaxation oscillation frequency of 4.2 GHz. For moderate coupling strengths, they operate in the chaotic coherence collapse regime. The indexing of the lasers is as defined in Fig. 1, while to close the ring $E_0\equiv E_N$.

As an example, we present in Fig. 2 numerical results for one SL with delayed optical feedback ($N=1$), two bidirectionally coupled SLs ($N=2$), $N=4$ and $N=100$. While the time traces (left panel) show no apparent change in the dynamical behavior when the number of SLs is increased, the power spectra (middle panel) and normalized intensity autocorrelation functions (right panel) show evidence of clear changes.

Figure 2(b) shows the power spectrum of one SL with delayed optical feedback. Besides a broadband component the spectrum exhibits discrete frequency peaks related to external cavity modes. Comparing the spectrum in Fig. 2(b) with the power spectrum of bidirectionally coupled SLs depicted in Fig. 2(e), it is striking that the peaks become less defined, while the broadband envelope appears unaffected. In Fig. 2(h), we find that this trend continues for $N=4$, with completely disappearing peaks for high values of N, as shown in Fig. 2(k). The corresponding normalized intensity autocorrelation functions (ACs) shed more light on this apparent dampening of the compound cavity peaks. The AC for $N=1$ in Fig. 2(c) exhibits, besides the peak around $t=0$, several echoes separated by about the total delay time τ. These longer time correlations decrease when N increases. Also note that the shape and amplitude of the peak around $t=0$ does not appear to change when N is increased. In accordance with the symmetry of the system, all elements in the ring show identical power spectra and ACs regardless of the fact that these elements can be uncorrelated among each other. The relation between the AC and the power spectra is given via the Wiener-Khintchine theorem. The underlying spectral shape apparent in Fig. 2(k) is the Fourier transform of the correlation peak around $t=0$ present in the AC for a ring of any N. For N sufficiently large and when the lasers operate in the coherence collapse regime, the injected field acts as a spontaneous emission noise source yielding the broadband spectrum as the corresponding nonlinear response of the SL [16].

To gain more insight into the AC properties, we now focus on the AC around $t=\tau$. This region compares the correlation between the emitted signal of a laser and the returned signal to this element after it has traveled along the ring. The correlation structure consists of a clearly defined peak and a modulation around it. We have studied the dependence of the peak height on the number of lasers. This is shown in Fig. 3(a) (dashed gray line and squares), where an almost exponential decay is observed except for the change in slope (kink) around $N=5$. The kink can be explained as follows. When the light passes through one of the lasers a latency time appears which slightly increases the total effective roundtrip time in the ring. Since an integer number of relaxation oscillation periods shows locking to the roundtrip time, when the latter increases the number of locked relaxation oscillation periods can also increase leading to the kink. This kink disappears when plotting the corresponding local maxima of the envelope of the AC ($\mathrm{AC}_{\max}$) in Fig. 3(a) (in black crosses). The scaling of the envelope is well described by an exponential decay. A fit (in solid black line) yields a scaling factor $\beta=0.4$. The exponential scaling suggests that the effect of the number of elements on the AC can be described by a simple cascading of the effect of one nonlinear element, meaning that the first AC peak for a ring of N elements is simply given by $\exp(-\beta)$ times the AC for $N-1$ elements. Whether it is possible to associate the exponential scaling to the Kolmogorov-Sinai entropy of the delay dynamics is the subject of future studies.

In the following, we compare the correlation properties as

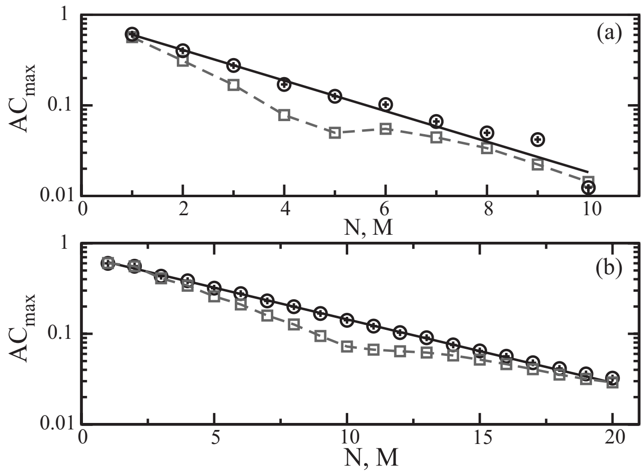

FIG. 3. AC peak (gray squares) and its envelope (black crosses) around the delay time vs the number of elements in the ring for (a) semiconductor lasers and (b) Ikeda oscillators. An exponential fit, $\alpha\exp[-\beta(N-1)]$, has been calculated of the envelope (solid black lines). Black circles indicate the maximum of the envelope of the AC peak at $t=M\tau$ ($t=MT$) of one laser (Ikeda) with delayed feedback. The gray dashed lines are guides to the eyes.

shown in Fig. 2 for different N. We have analyzed the heights of the correlation peaks and find that the peak around $t=2\tau$ in the case of one SL with delayed feedback [Fig. 2(c)] is exactly reproduced around $t=\tau$ in the AC of two bidirectionally coupled lasers [Fig. 2(f)]. In both cases, the signals have passed twice through a nonlinear element. Also, the peak around $t=2\tau$ in Fig. 2(f) is reproduced in Fig. 2(i) at $t=\tau$ and can also be found in Fig. 2(c) at $t=4\tau$. We have verified that any correlation peak of one SL with delayed feedback at $t=M\tau$ is reproduced in the AC of a ring with $N=M$ lasers at $t=\tau$. In Fig. 3(a), we also show the Mth maxima of the envelope of the AC peak of the delayed-feedback system (black circles). We find that they perfectly coincide with the maxima of the envelope of the AC peak at $t=\tau$ of a ring with $N=M$ SLs (black crosses). In general, we find that the shape and position of a correlation peak is defined by the number of passes through a nonlinear element. In fact, we can exactly reconstruct the AC of a ring of N elements by selecting the corresponding peaks in the AC of one SL with delayed feedback.

Our results indicate that cascading nonlinear elements in a ring decreases the correlation properties of the output in a predictable way, but does not change certain properties of the chaotic dynamics. We assume that in accordance with the conjecture of LeBerre the role of the delay accumulated in the ring is to determine the maximum dimension of the chaotic attractor [17]. As the total delay in our system is not (or only slightly) changed, the chaotic dynamics seems not to change its dimension.

To verify whether these results have more general validity and similarly hold for other types of nonlinear oscillator elements, we have extended our calculations to Ikeda oscillators. We consider the following model equation:

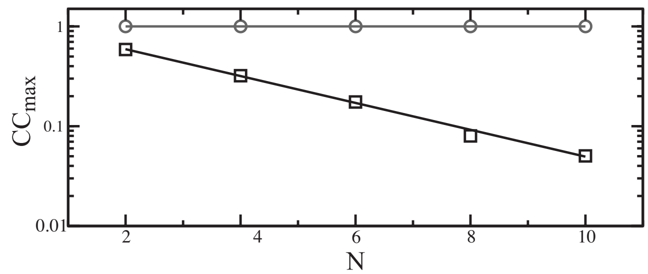

FIG. 4. Maximum value of the cross correlation (CC_{max}) between the sender (SL $N/2$) and the transmitter (SL 0) (black squares) and between the sender and receiver (SL $N/2'$) (gray circles) vs the number N. The solid black line is an exponential fit using $\alpha\exp[-\beta(N-1)]$. The gray line is a guide to the eyes.

$$\frac{dx_j}{dt'} = -x_j + \eta\sin[x_{j-1}(t'-T/N)+\theta],$$

with t' being a dimensionless time, T the round trip time of the ring, η the coupling strength, and θ the coupling phase. We choose $\eta=5$ and $T=50$. We find a similar reduction of the peaks in the ACs and in the power spectra for a ring of N Ikeda oscillators as for the case with SLs. Figure 3(b) shows the envelope of the AC peak around $t'=T$ (black crosses). The scaling follows an exponential decay with a scaling factor $\beta=0.16$. Again, we have compared the Mth peaks of the delayed feedback AC with the AC peak at $t=\tau$ for a ring of N Ikedas. In correspondence to the SL case, we find them to perfectly coincide with each other when $N=M$, as depicted in Fig. 3(b) (black circles and crosses). This indicates that the observed exponential correlation scaling and the changes in the spectra and AC might be a general property of delay-coupled oscillators, at least if we are working in a regime where the AC has decayed within one delay time and no multistability of chaotic attractors exists.

We can still infer more from the dynamics of this complex system. When N is an even number, the system under study can also be understood as two dynamical elements (oscillator 0 and $N/2$) which are delay coupled to each other in a nonlinear way through an array of $N/2-1$ oscillators. When computing the cross-correlation function of the elements 0 and $N/2$, we find that it is strongly affected by the number of nonlinear elements in between. We show in Fig. 4 (with squares), the maximum value of the cross correlation (CC) between laser 0 and $N/2$ as the number of elements in the ring of SLs is increased. When the nonlinearity of the coupling becomes stronger by increasing N, the CC decreases exponentially with a scaling factor $\beta=0.2$. For $N>10$, the two oscillators do not exhibit linear correlations within the statistical accuracy of 10^{-2}. Furthermore, performing a mutual information analysis we do not find indications of such nonlinear correlations within the same statistical accuracy.

In the following, we extend the ring configuration by attaching a chain of $N/2$ oscillators to one of the ring elements [dashed box in Fig. 1(c)]. One may notice that—by exploiting the symmetry properties of this configuration—oscillator 1 and 1′, 2 and 2′ and $N/2$ and $N/2'$ possess an identically synchronized solution, respectively [11]. The cross correlation between the emitted signal of oscillator $N/2$ and the signal emitted by oscillator $N/2'$ is found to be 1 for any N, as shown in Fig. 4 (with circles), indicating that they identically synchronize. Thus, the synchronized solution is stable as in Ref. [18]. This synchronization is mediated through the coupling signal emitted by oscillator 0. The coupling signal is uncorrelated to the two other signals, nevertheless they share a similar power spectrum. The identical synchronization of the elements $N/2$ and $N/2'$ necessarily implies that the coupling signal is generally synchronized to the original one, although an explicit relation cannot be identified. Generalized synchronization means that a function H must exist such that $\lim_{t\to\infty} H[x_i(t), x_j(t)]=0$. Here, the relation is defined by the cascade of nonlinear elements.

In conclusion, we have shown that the delay in a ring configuration of dynamical elements induces chaotic dynamics. When the number of elements is increased, we find that the individual correlation and spectral properties of the elements lose the fingerprint of the round trip time in a predictable way by demonstrating that the correlation properties of a ring with any number of elements can be deduced from the correlation of one single oscillator with delayed feedback, where the autocorrelation decays exponentially. The dynamics generated by the ring of many elements is particularly suited as a chaotic carrier in chaos communication schemes, because it offers both a broadband chaotic spectrum and the possibility that the delay cannot be recovered from its spectral and correlation properties. Finally, we have also shown that we can identically synchronize this chaotic dynamics through an uncorrelated mediating signal, promising interesting options for encrypted information exchange [19,20].

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[1] S. Boccaletti *et al.*, Phys. Rep. **366**, 1 (2002).

[2] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences* (Cambridge University Press, Cambridge, UK, 2001).

[3] A. Pikovsky *et al.*, Phys. Today **56**, 47 (2003).

[4] L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **64**, 821 (1990).

[5] A. Takamatsu, T. Fujii, and I. Endo, Phys. Rev. Lett. **85**, 2026 (2000).

[6] T. Heil, I. Fischer, W. Elsasser, J. Mulet, and C. Mirasso, Phys. Rev. Lett. **86**, 795 (2001).

[7] K. Konishi, Phys. Rev. E **70**, 066201 (2004).

[8] X. L. Deng and H. B. Huang, Phys. Rev. E **65**, 055202(R) (2002).

[9] H. J. Wang, H. B. Huang, and G. X. Qi, Phys. Rev. E **71**, 015202(R) (2005).

[10] J. M. Buldú, M. C. Torrent, and J. García-Ojalvo, J. Lightwave Technol. **25**, 1549 (2007).

[11] I. Fischer *et al.*, Phys. Rev. Lett. **97**, 123902 (2006).

[12] J. Ohtsubo, Opt. Rev. **6**, 1 (1999).

[13] M.-Y. Kim, R. Roy, J. L. Aron, T. W. Carr, and I. B. Schwartz, Phys. Rev. Lett. **94**, 088101 (2005).

[14] D. Lenstra, B. Verbeek, and A. Den Boef, IEEE J. Quantum Electron. **21**, 674 (1985), and references therein.

[15] I. Fischer *et al.*, Phys. Rev. Lett. **76**, 220 (1996).

[16] J. Wang and K. Petermann, IEEE J. Quantum Electron. **27**, 3 (1991).

[17] M. LeBerre, E. Ressayre, A. Tallet, H. M. Gibbs, D. L. Kaplan, and M. H. Rose, Phys. Rev. A **35**, 4020 (1987).

[18] S. Sivaprakasam *et al.*, Opt. Lett. **28**, 1397 (2003).

[19] R. Vicente, C. R. Mirasso, and I. Fischer, Opt. Lett. **32**, 403 (2007).

[20] R. Mislovaty, E. Klein, I. Kanter, and W. Kinzel, Phys. Rev. Lett. **91**, 118701 (2003).