Existence of an upper critical dimension in the majority voter model

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We study the critical properties of the majority voter model on d-dimensional hypercubic lattices. In two dimensions, the majority voter model belongs to the same universality class as that of the Ising model. However, the critical behaviors of the majority voter model on four dimensions do not exhibit mean-field behavior. Using the Monte Carlo simulation on d-dimensional hypercubic lattices, we obtain the critical exponents up to d=7, and find that the upper critical dimension is 6 for the majority voter model. We also confirm our results using mean-field calculation.

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I. INTRODUCTION

The opinion dynamics based on a stochastic spin model using physical notions is one of the active fields in interdisciplinary research [1–6]. The majority voter model is one of the widely studied opinion dynamics [6–19]. In the majority voter model, each agent on a lattice gathers the opinions (spin values) from its neighborhood, and changes its opinion by a majority of votes. When the *effective temperature T* is introduced to put the stochastic process into a majority rule, minority opinions can be accepted due to the nonzero effective temperature and an order-disorder phase transition occurs at the critical effective temperature T_c [7]. Also, critical exponents are obtained at the critical effective temperature. These exponents for the majority voter model in two dimensions are in excellent agreement with those of the Ising model [6–9]. Though the majority voter model is a nonequilibrium model [6,7,9–11], it belongs to the same universality class as the equilibrium Ising model because it is a nonequilibrium model with up-down symmetry [20,21]. However, the majority voter model on the random networks, the small-world lattices or networks, or the scale-free networks does not belong to the Ising universality class [12–15,17,22].

On a one-dimensional lattice, the Ising model does not have an order-disorder phase transition, while it has a phase transition above two dimensions. The upper critical dimension in the Ising model is d_c=4 and thus it exhibits the standard mean-field behavior for $d \geq d_c$. On a one-dimensional lattice, both the Ising model and the majority voter model have exactly the same transition rate. On a two-dimensional lattice, the two models have the same set of critical exponents, although their transition rates are a little bit different [6]. Therefore, it would be of interest to examine the critical behaviors of the majority voter model on the above three-dimensional lattices.

In this paper, we have carried out Monte Carlo simulation for the majority voter model on hypercubic lattices ranging from three to seven dimensions to find out whether there is an upper critical dimension in the majority voter model.

II. MAJORITY VOTER MODEL ON HYPERCUBIC LATTICES

The original majority voter model [6] is a spin model in which each site i is occupied by a spin value either $\sigma_i = +1$ or -1. Kwak *et al.* [7] studied the majority voter model with a *local configuration energy E*, which is defined as follows:

$$E[\sigma_i] = -\sigma_i S\left(\sum_{\langle j\rangle} \sigma_j\right), \tag{1}$$

where the sum runs over the nearest-neighbors of sites, and $S(x)$ is the signum function for $x \neq 0$ and is zero if $x=0$, and $\langle j\rangle$ denotes the nearest neighbors of i. The minus sign in Eq. (1) denotes that the sign of site i follows the majority of the nearest-neighbor spins. Since the energy difference takes on values of +2, 0, −2 with the energy defined in Eq. (1) in the majority voter model, the transition rate is given by

$$w_i(\{\sigma\} \rightarrow \{\sigma'\}) = \frac{1}{2}\left[1 - \sigma_i S\left(\sum_{\langle j\rangle} \sigma_j\right)\tanh \beta_T\right], \tag{2}$$

where $\{\sigma\}$ is the original configuration, $\{\sigma'\}$ is the configuration after spin-flipping, and β_T is an inverse effective temperature. Equation (2) is exactly identical to the transition rate of the original majority voter model with noise parameter q [6] under the relation $(1-2q)=\tanh \beta_T$.

The order parameter m, the susceptibility χ, and Binder's fourth-order cumulant U are defined as follows:

$$m = \frac{1}{N}\sum_{i=1}^{N} \sigma_i, \tag{3}$$

$$\chi = \frac{N}{T}(\langle m^2\rangle - \langle |m|\rangle^2), \tag{4}$$

$$U = 1 - \frac{\langle m^4\rangle}{3\langle m^2\rangle^2}. \tag{5}$$

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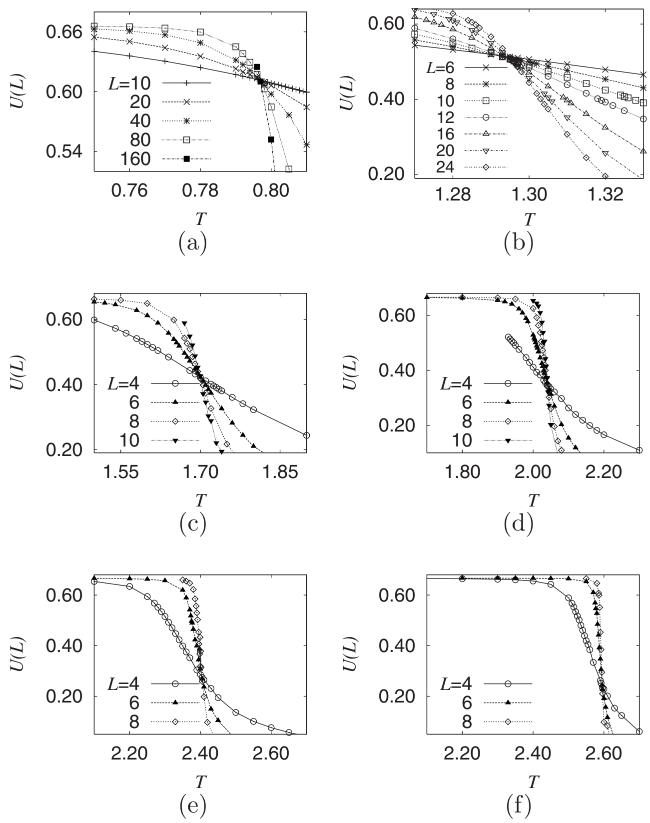

FIG. 1. Plots of Binder's fourth-order cumulant U as a function of effective temperature on d-dimensional hypercubic lattices on (a) 2D [7], (b) 3D, (c) 4D, (d) 5D, (e) 6D, and (f) 7D.

The finite-size scaling forms of the order parameter, the susceptibility, and Binder's fourth-order cumulant for the linear dimension L are given by

$$m(L) = L^{-\beta/\nu}\tilde{m}(L^{1/\nu}t) \quad (t<0), \tag{6}$$

$$\chi(L) = L^{\gamma/\nu}\tilde{\chi}(L^{1/\nu}t), \tag{7}$$

$$U'(L) = L^{1/\nu}\tilde{U}'(L^{1/\nu}t), \tag{8}$$

where the reduced effective temperature is given by $t=(T-T_c)/T_c$ and $U'(L)$ is the derivative of Binder's fourth-order cumulant with respect to effective temperature. According to the finite-size scaling theory [23], the scaling functions $\tilde{m}$, $\tilde{\chi}$, and $\tilde{U}'$ are constant and $m(L)$, $\chi(L)$, and $U'(L)$ are smooth and analytic functions in the vicinity of the critical effective temperature T_c.

III. RESULTS

Simulations are carried out on d-dimensional hypercubic lattices with periodic boundary conditions, where the range of d is from 3 to 7 and L is from 4 to 24. The location of the critical effective temperature for the model is estimated from Binder's fourth-order cumulant Eq. (5).

Figure 1 shows Binder's fourth-order cumulant U as a function of effective temperature for the majority voter model. The critical effective temperature T_c can be found from an intersection point of U for different values of L. We get $T_c=0.796\pm0.005$ for 2D [7], 1.30 ± 0.01 for 3D, 1.70 ± 0.05 for 4D, 2.04 ± 0.05 for 5D, 2.40 ± 0.06 for 6D, and 2.59 ± 0.05 for 7D, respectively. As the dimension of the lattice increases, T_c is also monotonically increased. The relations between T_c and dimensions can be inferred as $T_c \sim \log d$ from Fig. 2.

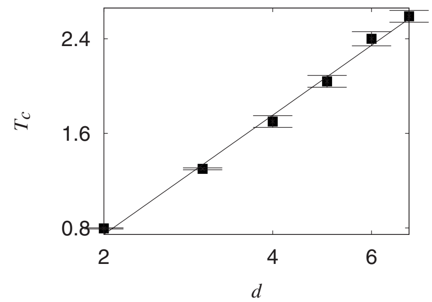

FIG. 2. Semilogarithmic plot of T_c as a function of dimension of a hypercubic lattice.

Figure 3 is the plot of the derivative of Binder's fourth-order cumulant with respect to effective temperature as a function of linear dimension L on d-dimensional hypercubic lattices. The scaling form of $U'(L)$ is given in Eq. (8). When we draw the log-log plot of the maximum values of the finite size $U'(L)$ versus L, the slope of the log-log plot is $1/\nu$ at the finite-size critical effective temperature $T_c(L)$. The estimated values of ν obtained from fitting lines in Fig. 3 are 1.02 ± 0.03 for 2D [7], 0.63 ± 0.01 for 3D, 0.51 ± 0.02 for 4D, 0.40 ± 0.01 for 5D, 0.32 ± 0.02 for 6D, and 0.29 ± 0.02 for 7D, respectively.

The critical exponent β of the order parameter is defined below the critical effective temperature and the scaling relation is given by Eq. (6). From Fig. 4, which shows the plots of scaling function $mN^{\beta/\nu}$ as a function of scaling variable $tL^{1/\nu}$, β/ν is obtained: 0.120 ± 0.005 for 2D, 0.60 ± 0.01 for 3D, 0.77 ± 0.01 for 4D, 1.06 ± 0.04 for 5D, 1.50 ± 0.05 for 6D, and 1.78 ± 0.05 for 7D, respectively. Our simulation result for 3D is different from that of the Ising model, and results for 4D and 5D are far away from the standard mean-field values.

Using the maximum values of the finite-size susceptibility $\chi(L)$ at $T_c(L)$ and the finite-size scaling form in Eq. (7), we are able to find γ/ν. In Fig. 5, the slope of a straight line

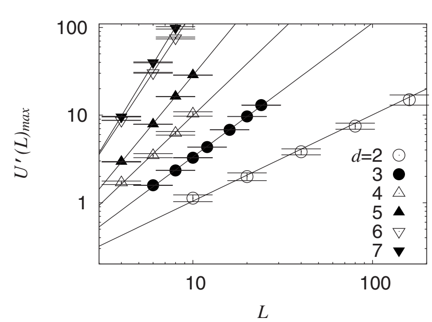

FIG. 3. Log-log plot of the maximum values of U' as a function of linear dimension L on d-dimensional hypercubic lattices.

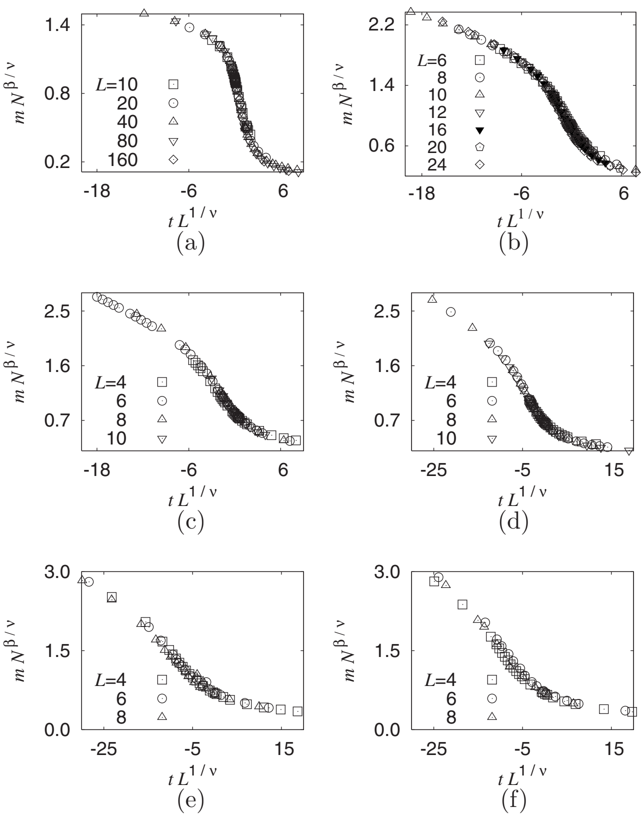

FIG. 4. Plots of $mL^{\beta/\nu}$ as a function of $tL^{1/\nu}$ on hypercubic lattices on (a) 2D [7], (b) 3D, (c) 4D, (d) 5D, (e) 6D, and (f) 7D.

gives $\gamma/\nu=1.78\pm0.05$ for 2D [7], 2.10 ± 0.01 for 3D, 2.46 ± 0.06 for 4D, 2.89 ± 0.08 for 5D, 3.01 ± 0.09 for 6D, and 3.44 ± 0.09 for 7D, respectively. We also obtain the same tendency as β; our simulation result for 3D is different from that of the Ising model, and results for 4D and 5D are far away from the standard mean-field values.

We summarize the obtained values of critical exponents for the majority voter model and the known values of the Ising model in Table I. The critical exponents for the majority voter model on three dimensions are quite different from those of the Ising model, and also the values of β and γ for four and five dimensions are not equal to the standard mean-field values, $\beta=1/2$ and $\gamma=1$, respectively. However, the obtained values of β and γ for six and seven dimensions are well fitted to $\beta=1/2$ and $\gamma=1$, respectively. Our results for

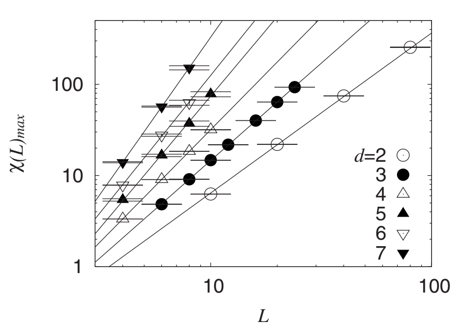

FIG. 5. Log-log plot of the maximum values of $\chi(L)$ as a function of L on a d-dimensional hypercubic lattice.

TABLE I. Estimates of the critical effective temperatures and the critical exponents for the majority voter model (MV) and the Ising model on the hypercubic lattices with different dimensions.

Model	T_c	β	γ	νd
2D MV [7]	0.796(5)	0.122(7)	1.82(10)	2.04(6)
3D MV	1.30(1)	0.38(1)	1.32(3)	1.89(3)
4D MV	1.70(5)	0.39(2)	1.25(8)	2.04(8)
5D MV	2.04(5)	0.42(3)	1.16(6)	2.00(5)
6D MV	2.40(6)	0.48(5)	0.96(9)	1.9(2)
7D MV	2.59(5)	0.52(5)	1.00(9)	2.0(2)
2D Ising [24]		1/8	7/4	2
3D Ising [25]		0.326(5)	1.247(5)	1.887(3)
mean field		1/2	1	2

$d\geq6$ also satisfy a hyperscaling relation from the Rushbrooke and Josephson scaling laws: $2\beta+\gamma=\nu d$, where d is a dimension of hypercubic lattices. We conclude that the upper critical dimension of the majority voter model is $d_c=6$.

IV. THE RATE EQUATION AND THE ORDER PARAMETER

We analytically calculate the magnetization of the majority voter model on the hypercubic lattices. The number of sites N of d-dimensional lattices with a linear dimension L is L^d. N_+ and N_- out of N are the number of up spins and down spins, respectively. The relations between N, N_+, N_-, and m are as follows: $N_++N_-=N$ and $N_+-N_-=mN$. Also the fraction of up (or down) spins is expressed as $P(+)$ [or $P(-)$]. Then, $P(\pm)=(1\pm m)/2$ is obtained easily. If a spin in a site chosen randomly is flipped by the transition rate, Eq. (2), the expectation value of the magnetization change is as follows:

$$\Delta m=\frac{2}{N}P(-\rightarrow+)-\frac{2}{N}P(+\rightarrow-), \tag{9}$$

where $P(\mp\rightarrow\pm)$ is a spin-flipping probability. When a site that has a down (or up) spin is chosen and a spin on a site flips, the expectation value of the magnetization change is represented in the first (or second) term on the right-hand side in Eq. (9). Using the relation $P(\mp\rightarrow\pm)=P(\mp)\cdot(1\pm\tanh\beta_T\mathbf{S})/2$, Eq. (9) can be rewritten as follows:

$$\Delta m=\frac{1}{N}(-m+\tanh\beta_T\mathbf{S}), \tag{10}$$

where $\mathbf{S}=S(\Sigma_{\langle j\rangle}\sigma_j)$.

On a d-dimensional hypercubic lattice, the number of nearest neighbors is $2d$ and the number of possible spin combinations for a neighbor is $2d+1$: $(0,2d)$, $(1,2d-1)$, …, $(2d,0)$, where (a,b) denotes the number of up and down spins. When the number of up (or down) spins is in the range of 0 to $d-1$, the majority spin is down (or up). Therefore, the probabilities that the sign of majority spins for the nearest neighbors is +, 0, or − are as follows:

$$P(\mathbf{S}=\pm)=\sum_{k=0}^{d-1}\binom{2d}{k}P(\pm)^{2d-k}P(\mp)^{k}, \tag{11}$$

$$P(\mathbf{S}=0)=\binom{2d}{d}P(+)^{d}P(-)^{d}, \tag{12}$$

where $\binom{x}{y}=x!/y!(x-y)!$. Then, the expectation value of $\mathbf{S}$ in Eq. (10), $\bar{\mathbf{S}}$, is expressed as $\bar{\mathbf{S}}=1\times P(\mathbf{S}=+)+0\times P(\mathbf{S}=0)-1\times P(\mathbf{S}=-)$. Therefore, by substituting $\bar{\mathbf{S}}$ for $\mathbf{S}$ in Eq. (10), Δm for a d-dimensional lattice is represented as follows:

$$\Delta m=-\frac{m}{N}+\frac{\tanh\beta_T}{N}\frac{1}{2^{2d}}\sum_{k=0}^{d-1}\binom{2d}{k}(1-m^2)^k \times[(1+m)^{2d-2k}-(1-m)^{2d-2k}]. \tag{13}$$

Under the conditions of $m=0$ and $\Delta m=0$, the critical effective temperature is obtained as follows:

$$\frac{1}{\tanh\frac{1}{T_c}}=\frac{1}{4^{d-1}}\sum_{k=0}^{d-1}\binom{2d}{k}(d-k). \tag{14}$$

For $d=1$, $\tanh(1/T_c)=1$ is obtained from Eq. (14). Thus, the system is always disordered phase for $T>0$. This result is the same as the Ising model, because the transition rate is exactly the same.

In general, by expanding Eq. (13) near T_c with the condition $\Delta m=0$, we obtain the following relation:

$$\tanh\frac{1}{T_c}+T_c\left(\frac{d\tanh\frac{1}{T}}{dT}\right)_{T=T_c}t+O(t^2) =\frac{1}{4^{d-1}}\sum_{k=0}^{d-1}\binom{2d}{k}(d-k)+O(m^2). \tag{15}$$

Each of the first terms on the left- and right-hand side is eliminated. Therefore, $m\sim t^{1/2}$ is acquired regardless of dimension, and we thus get the mean-field value $\beta=1/2$ from the simple calculation using the rate equation.

V. CONCLUSION

The majority voter model on two dimensions has the same set of critical exponents as the Ising model, and both models belong to the same universality class [6,7]. However, for hypercubic lattices above three dimensions, the set of critical exponents for the majority voter model differs from that for the Ising model. Our simulation results for six and seven dimensions follow well the values of the standard mean-field values, $\beta=1/2$, $\gamma=1$: $\beta=0.48\pm0.05$ and $\gamma=0.96\pm0.05$ for 6D, and $\beta=0.52\pm0.05$ and $\gamma=1.00\pm0.09$ for 7D, respectively. We thus conclude that the upper critical dimension is $d_c=6$. Our results satisfy well a hyperscaling relation with $2\beta+\gamma=\tilde{\nu}$, where $\tilde{\nu}=\nu d=2$ and d is a dimension of hypercubic lattices.

TABLE II. Comparison of configuration energy difference ΔE between local and global, where d is a dimension of regular lattice and $[n,m]$ denotes a closed interval with integer values.

$\Delta E_{\rm local}$	$\Delta E_{\rm global}$
+2	$[3-d,2(d+1)]$
0	$[-d,d]$
−2	$[-2(d+1),-(3-d)]$

We consider the difference of the local and the global configuration energies between before and after spin-flipping to find out what factor makes the critical exponents of the majority voter model differ from those of the Ising model. The global configuration energy is obtained from summing the local configuration energy all over the sites. From the spin-flipping on a site, $\Delta E_{\rm local}$, the difference of a configuration energy of a selected site always has one of the values +2, 0, or −2 for the majority voter model. However, $\Delta E_{\rm global}$, the difference of a configuration energy of the whole system after flipping a spin in a selected site, can have various values such as shown in Table II.

For the Ising model, the differences of the global and the local configuration energy are exactly identical, regardless of dimension. The energy decreasing spin-flipping is always chosen with high probability, Whereas for the majority voter model, the global and the local configuration energy differences are not identical. For the majority voter model in two dimensions, the global and the local configuration energy differences are not identical, but the sign of both energies is the same. For $d\geq 3$, the sign of the global and local configuration energy is not always the same. Table II shows that, for the local energy difference $\Delta E_{\rm local}<0$ (or $\Delta E_{\rm local}>0$), there are some probabilities to have the global energy difference $\Delta E_{\rm global}>0$ (or $\Delta E_{\rm global}<0$). Therefore, the local energy minimization cannot produce the global energy minimization for the majority voter model above three dimensions. We conjecture that this discordance of the sign of the energy difference between the global and local energy is responsible for the different critical behaviors of the majority voter model from those of the Ising model.

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