Gravitational dynamics of an infinite shuffled lattice: Early time evolution and universality of nonlinear correlations

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In two recent papers, a detailed study has been presented of the out-of-equilibrium dynamics of an infinite system of self-gravitating points initially located on a randomly perturbed lattice. In this paper, we extend the treatment of the early time phase during which strong nonlinear correlations first develop, prior to the onset of "self-similar" scaling in the two-point correlation function. We establish more directly, using appropriate modifications of the numerical integration, that the development of these correlations can be well described by an approximation of the evolution in two phases: a first perturbative phase in which particle displacements are small compared to the lattice spacing, and a subsequent phase in which particles interact only with their nearest neighbors. For the range of initial amplitudes considered, we show that the first phase can be well approximated as a transformation of the perturbed lattice configuration into a Poisson distribution at the relevant scales. This appears to explain the universality of the spatial dependence of the asymptotic nonlinear clustering observed from both shuffled lattice and Poisson initial conditions.

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I. INTRODUCTION

Structure formation in the universe is currently addressed primarily using numerical simulations of purely self-gravitating particle systems, with initial configurations generated by displacing the particles slightly from a perfect lattice (see, e.g., [1,2]). The physics of the strongly nonlinear regime of the observed evolution is, in detail, very poorly understood. Progress in understanding would be very useful in providing analytical guidance for numerical simulations, and in particular better control on their precision in representing the relevant continuum limit. In a series of recent papers [3,4], we have studied a reduced version of the full cosmological problem, considering a very simple class of randomly perturbed lattices as initial conditions, and evolution in a static universe.[1]

One of our primary results is that, despite the simplifications, the system we study has qualitative behavior very similar to that observed in the more complex cosmological simulations (correlated perturbations in initial conditions, expanding universe). Notably the evolution is clearly hierarchical (i.e., structures build up at successively larger scales driven by the linearized fluid theory growth of the initial perturbations) and asymptotically self-similar (i.e., the time dependence of the two-point correlation function is given by a simple scaling of the spatial variable which can be inferred from the linearized fluid theory).[2] The *functional form* of the spatial dependence of the nonlinear correlation function is, on the other hand, just as in the cosmological simulations, a fundamental quantity characterizing the results which is determined numerically, but not currently understood (i.e., not predicted analytically or semianalytically). We have noted in [3,4], however, that this form emerges, to a good approximation, in our simulations *prior* to the asymptotic scaling regime, in the preceding transient phase in which strong nonlinear correlations first develop. In this paper we extend and detail our analysis of this phase. We show in greater detail that the emergence of the observed nonlinear two-body correlations can be very well approximated by modeling the evolution as constituted of two subsequent phases, with an abrupt matching from one to the other. During the first phase the particles evolve as described by a perturbative analytical approximation we have introduced and studied in [7,8]; in the second phase the particles evolve under a force coming solely from their nearest neighbors (NNs).

We relate our work here to previous work along similar lines concerning evolution from Poissonian initial conditions [5,6,9]. In this case it has been shown explicitly [5] that the

[1] See also [5,6] for earlier studies of evolution from these initial conditions.

[2] To avoid any possible confusion we note that both these terms are used here with meanings different from those commonly ascribed to them in condensed matter physics. In the latter context both are associated with invariance properties of the spatial correlations under spatial rescalings (see, e.g., [11]). Such properties are not implied by their use in the present context.

TABLE I. Details of the initial conditions studied in this paper, and the numerical parameters used in the simulations. N is the number of particles in the cubic box of side L, and m is the particle mass.

Name	$N^{1/3}$	L	ℓ	Δ	δ	m/m_{64}
SL64	64	1	0.015625	0.015625	1	1
SL32	32	1	0.03125	0.0553	0.177	8
SL24	24	1	0.041667	0.00359	0.0861	18.96
SL16	16	1	0.0625	0.00195	0.03125	64
SL64b	64	1	0.015625	0.0012	0.0768	1
P64	64	1	0.015625	∞	∞	1

emergence of the first strong nonlinear correlations can be very well accounted for by an approximation, at the relevant early times, in which the full gravitational force on each particle is truncated to that due only to its initial NN.[3] In the case of "shuffled lattice" (SL) initial conditions, which we consider in this work, such an approximation is not generically good: when the typical displacement of a particle is small compared to the lattice spacing, the high degree of symmetry means that the force on a typical particle is the sum of comparable contributions from many particles. In this regime, however, we can describe the evolution very well by a simple perturbative approximation, which has been developed fully in [7,8]. This latter approximation breaks down, roughly, when particles start to approach one another, which is precisely when one expects that a NN approximation for the force may become appropriate. We show here that this is indeed the case, and that an abrupt switch between the two phases gives a very good approximation to the evolution. Further, this model allows us to explain the fact that the observed form of the nonlinear correlations in our simulations is independent of the amplitude of the initial shuffling, and the same as that observed from Poisson initial conditions. This is the case because, for the range of amplitude of the initial perturbations we use, the evolution in the first phase brings the system to a distribution with correlation properties which, at the relevant scales, are essentially those of a Poisson distribution.

The main interest of our results here is that they give a semianalytical understanding of the origin of the form of the observed nonlinear two-point correlations for this class of initial conditions, which are qualitatively similar to those used in cosmological-type simulations. As remarked in [3,4], the form of this early time correlation function coincides, to a very good approximation, with that which is also observed in the asymptotic scaling regime attained by the system at longer times. This suggests strongly (but does not prove) that the physical mechanism leading to the former correlations, which we identify here, is also that which gives rise to the latter correlations. In the context of cosmological simulations such a conclusion, if appropriate also for that case, would be very important for the following reason. In this context the results derived from the numerical simulations which are physically relevant are those which are representative of the Vlasov-Poisson (VP) limit of the simulated particle system. The mechanism we describe here for the generation of the nonlinear correlations is, on the other hand, clearly *not* representative of this limit: the effects of interactions with single NN particles are precisely of the kind which are discarded in the VP limit, which is a mean-field approximation. Therefore, if the form of the nonlinear correlations in the long time evolution turns out actually to be determined in such a phase, this form would not be representative, as required, of the VP limit. We discuss this point a little further in our conclusions, and suggest numerical tests which could be performed to determine whether the long time behavior is indeed linked to the early time mechanism we study here.

The paper is organized as follows. In Sec. II we discuss some of the relevant properties of the initial conditions of our simulations. In the next section we give the details of the simulations considered here and summarize briefly the main relevant results of [3,4]. In Sec. III we present in detail the two-phase model which captures the essential elements of the formation of the first nonlinear correlations. We give results here also of numerical simulations. Finally, in Sec. IV we discuss the results and draw our main conclusions.

II. STATISTICAL PROPERTIES OF INITIAL CONDITIONS

As in [3] we study evolution from initial conditions in which the particles are at rest and located at the sites of a perfect simple cubic lattice subjected to random uncorrelated displacements. We adopt the same notation, denoting by $p(\mathbf{u})$ the probability density function (PDF) for the displacements, and by $\mathbf{u}(\mathbf{R})$ the displacement of the particle originally at lattice site $\mathbf{R}$. The variance of the PDF is denoted by Δ^2, and the dimensionless variance $\delta^2 \equiv \Delta^2/\ell^2$, where ℓ is the lattice spacing. The parameter δ we refer to as *the normalized shuffling parameter*. As discussed in [3,4], for the case of purely gravitational interactions, the system is completely characterized, in the infinite volume limit, by the single parameter δ. The Poisson distribution corresponds to the limit in which each particle's position is completely randomized in the infinite volume, i.e., $\delta = \infty$.

The precise details of the different initial conditions of which the evolution is studied numerically below, are summarized in Table I. The PDF $p(\mathbf{u})$ used for generating the displacements is constant in a cube of side 2Δ around the origin (and with sides parallel to the axes of the lattice), and

[3] The importance of NN interactions at early times starting from Poisson initial conditions has also been discussed previously by Saslaw. See [10] and references therein.

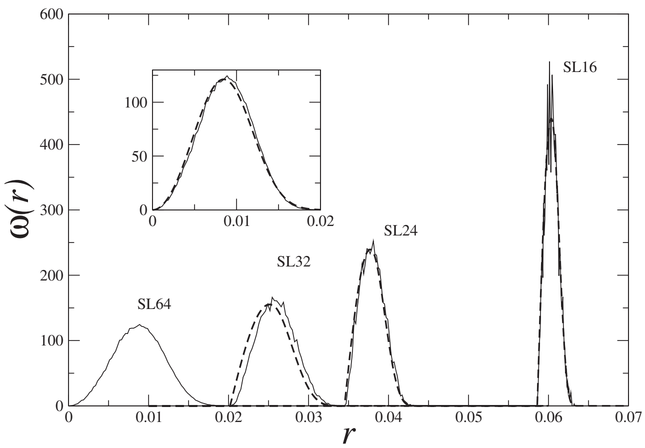

FIG. 1. NN PDF of the SLs considered in this paper (the name of each SL is indicated above the corresponding curve). For SL32, SL24, and SL16, the function (3) is shown for comparison. In the inset panel we show enlarged the SL64 (SL) case together with the behavior of $\omega_p(r)$ for a Poisson distribution (p) with the same number density [i.e., Eq. (2)].

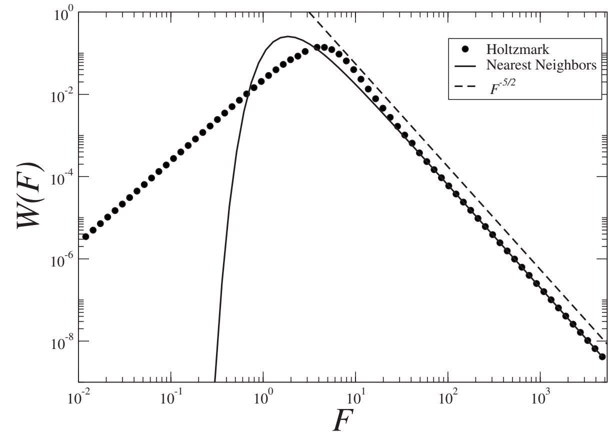

FIG. 2. Holtzmark distribution and the PDF inferred if only NNs contribute, i.e., $W_{NN}(F)dF=\omega(r)dr$. The agreement is very good in the strong field limit where $W(F)\sim F^{-5/2}$. At weak field the PDF due to the NN has a sharp cutoff while the Holtzmark distribution shows a more gentle decrease (see discussion in [11,13]).

zero elsewhere. The (arbitrary) choice of units is as in [3], giving equal dynamical time $\tau_{\rm dyn}\equiv 1/\sqrt{4\pi G\rho_0}$ (where ρ_0 is the mass density).

The first four simulations (SL64 to SL16) are the same ones analyzed in [3]. As explained there (see Sec. III of [3]) the values of δ have been chosen so that, in our units of length, $\delta^2\ell^5$ is constant. This gives (see [3] for details) an amplitude of the power spectrum at small k (i.e., $k\ll\ell^{-1}$ for $\delta<1$) which is equal in all simulations. With time in units of $\tau_{\rm dyn}$ this means that, in the long wavelength fluid limit, the systems are identical initially, and evolve identically. SL64b differs only from SL64 in the value of δ, i.e., they are two simulations with identical values of the parameters characterizing the finite numerical representation of the infinite systems, but with different δ.

To characterize the correlation properties of the distributions we will use the same quantities as in [3,4]: the reduced two-point correlation function $\xi(\mathbf{r})$, the power spectrum $P(\mathbf{k})$ [which is related to $\xi(\mathbf{r})$ by a Fourier transform[4]], and the NN PDF $\omega(r)$. We refer the reader to [3] for the precise definitions of these quantities. We will also consider the stochastic properties of the force, which we characterize using $P(F)$, the PDF for the modulus of the force F.

While $\delta=\infty$ corresponds exactly to the Poisson distribution, one expects any SL with $\delta\gtrsim 1$ to approximate the correlation properties of a Poisson distribution up to a scale of order $\Delta=\delta\ell$. Indeed, if $\delta\gtrsim 1$, the effect of the short distance exclusion of the underlying lattice should disappear and the particles are, to a good approximation, randomly placed in a volume of order Δ^3. This can be seen explicitly for the power spectrum, of which the exact analytical expression may be written [3,11] in the form

$$P(\mathbf{k})=\frac{1}{n_0}+|\tilde{p}(\mathbf{k})|^2A(\mathbf{k}), \tag{1}$$

where n_0 is the mean particle density and $\tilde{p}(\mathbf{k})$ is the characteristic function of $p(\mathbf{u})$ [i.e., its Fourier transform normalized so that $\tilde{p}(0)=1$]. The function $A(\mathbf{k})$ depends only on the initial unperturbed lattice distribution (and not on the shuffling). For any simple form of the PDF for the shuffling (such as the top-hat one considered here and in [3], or, e.g., a Gaussian PDF as used in [4]) $\tilde{p}(\mathbf{k})$ decreases toward zero for $k\Delta>1$, giving that the power spectrum tends to the Poissonian value (given by $1/n_0$). Thus for wave numbers k larger than of order $1/(\delta\ell)$ the power spectrum converges to this Poissonian behavior. We refer the reader notably to Fig. 2 of [3], which shows the power spectrum for the four initial conditions SL16, SL24, SL32, and SL64.

A. Nearest-neighbor distribution

We will consider below often the NN PDF, and it is useful to know its form in the initial conditions just described. For the Poissonian limit $\delta=\infty$ it is straightforward to show analytically [11] that it is

$$\omega_P(r)=4\pi n_0 r^2\exp\left(-\frac{4}{3}\pi n_0 r^3\right), \tag{2}$$

which gives an average distance between NNs $\Lambda_0\approx 0.55 n_0^{-1/3}=0.55\ell$. Since the NN distribution characterizes the small scale properties we expect, following our discussion above, that this expression will be a good approximation for $\delta\gtrsim 1$. For $\delta\ll 1$, on the other hand, one may show[5] that

[4]The power spectrum is the Fourier transform of $\tilde{\xi}(\mathbf{r})$, which differs from what we refer to here as the "correlation function" by a δ function singularity at $r=0$. See [3,4] or [11].

[5]The derivation of the expression given is straightforward, but tedious. For a given shuffling of a particle and its six NNs, one must determine exhaustively the different combinations, and associated probabilities, that lead to a given NN separation. The approximation $\delta\ll 1$ is used in taking the interparticle separations to linear order in δ.

$$\ell\,\omega(r) \approx \frac{1}{(2\delta)^9} f\left(\frac{r}{\ell} - 1\right), \tag{3}$$

where

$$f(x) = \begin{cases} \delta(2\delta + x)(2\delta x^2 + 8\delta^2 x)^2 & \text{if } x \in [-2\delta, -\delta], \\ (\delta^2 - x^2 + \delta x)\left(2\delta x^2 + \frac{4}{3}\delta^3 - 4\delta^2 x + \frac{4}{3}x^3\right)^2 & \text{if } x \in]-\delta, 0], \\ \frac{16}{9}(\delta - x)^8 & \text{if } x \in]\,0, \delta], \\ 0 & \text{otherwise}, \end{cases}$$

corresponding to average distance between NNs $\Lambda_0 = \ell - (86\,827/80\,640)\Delta$.

In Fig. 1 we show the behavior of $\omega(r)$ for most of the SLs studied here. For the SLs with very small shuffling—SL16 or SL24—this PDF is strongly peaked around the average distance between NNs (which is approximately equal to ℓ), and in very close agreement with the analytical approximation given by Eq. (3). For SL32 (with $\delta = 0.177$) a small discrepancy with this approximation is visible, while for SL64 (with $\delta = 1$) the NN PDF is, as expected, in very good agreement with that in the Poisson case given by Eq. (2).

B. Force distribution

The PDF of the modulus of the force $W(F)$ is a useful quantity in our analysis. Notably if the forces on particles are dominated by that coming from their NNs the simple relation

$$W(F)dF = \omega(r)dr \tag{4}$$

must hold. For the case of a Poisson distribution the analytical expression for $W(F)$ was first given by Chandrasekhar [12]. It is proportional to the so-called Holtzmark distribution (see [11] for the explicit result and a simple derivation). In Fig. 2 we show a plot of the full (Holtzmark) distribution, and the PDF inferred if only NNs contribute, $W_{\rm NN}(F)dF = \omega(r)dr$. The domination of the NNs in the force is clearly seen at stronger values of the force. The relation is not valid at weaker values of the force as these correspond to the (rare) particles for which the force picks up comparable (and possibly canceling) contributions from more than one particle. Note that the tail of the PDF at large F decays in proportion to $F^{-5/2}$, which means that the variance (i.e., second moment) of the PDF is infinite.

In [13] we have studied in detail the statistical properties of the force in a SL. As one might expect, one can show that the force PDF is very well approximated by that of a Poisson distribution when the typical displacement is larger than the interparticle spacing, i.e., $\delta > 1$. At small values of the displacements, on the other hand, the force PDF is very different from that in a Poisson distribution, decaying much more rapidly at large values of the force: the strong forces due to NNs are completely absent as the typical particle experiences a comparable effect from its *six* NNs when the configuration is close to a perfect simple cubic lattice. More precisely, for a top-hat PDF of the displacements (as used here), the functional behavior of the PDF at large F changes qualitatively, from exponential decay to a $F^{-5/2}$ power law decay, at $\delta = 0.5$. For $\delta \gtrsim 0.5$ the amplitude of the latter tail is lower than that in the Poisson distribution, with this difference becoming negligible as δ increases to of order unity.

III. NUMERICAL SIMULATIONS

In this section we first report results of numerical simulations in which the initial conditions given in Table I are evolved under the mutual self-gravity of all particles. As such simulations have already been reported in detail in [3,4] (see also [5,9] for Poisson and some SL initial conditions), we restrict ourselves to a very brief summary with an emphasis on the points that are relevant here. We then report results of a set of simulations designed to validate our model of the early time evolution by a direct numerical integration of the appropriate two-phase approximation.

A. Full gravity

As in [3,4] we have used the publicly available code GADGET [14,15] to evolve the system under gravity, modified only by a small scale regularization of the potential below $r = \varepsilon$. This softening parameter ε is taken here in all simulations to be $\varepsilon = 0.001\,75L$ (i.e., in all simulations significantly smaller than the initial average distance between NNs). We have performed the same checks as discussed in [3,4] for the independence of our results to this choice.

In [3] we have found that the SL initial conditions considered here lead, from the time significant nonlinear correlation first develops at small scales, to an evolution in which the correlation function can be approximated by

$$\xi(r,t) \approx \Xi(r/R_s(t)), \tag{5}$$

where $R_s(t)$ is a time-dependent length scale, and a simple functional fit to $\Xi(r)$ is given in [3]. For sufficiently long times—after a transient time of which the duration increases as the value of δ decreases—$R_s(t)$ follows very well the behavior predicted by a simple analysis based on the linearized equations for the system approximated as a pressureless self-

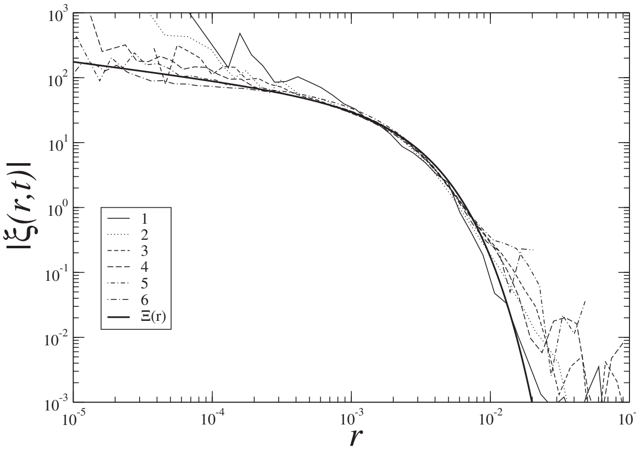

FIG. 3. "Collapse plot" of $\xi(r,t)$ (absolute value) for the P64 initial conditions: for each time $t>1$ we have rescaled r so that $\xi(r,t)=1$ at r_0, where $\xi(r_0,t=1)$. The behavior of the numerical fit given in [3] is also shown for comparison.

gravitating fluid. As described in [3], for such a system with power spectrum of density fluctuations at small wave numbers $P(k)\sim k^n$ one obtains

$$R_s(t) \propto \exp\left(\frac{2}{3+n}\frac{t}{\tau_{\rm dyn}}\right). \tag{6}$$

As reported in [3], the SL initial conditions indeed produce the predicted asymptotic time dependence, corresponding to $n=2$ in this formula. In Fig. 3 we show the same analysis of the evolved Poisson initial conditions P64, using the same numerical fit to $\Xi(r)$ as found in [3] for the SL initial conditions.

In Fig. 4 is shown the associated temporal evolution of $R_s(t)$, and a fit to the theoretical fluid behavior, given by Eq. (6) with $n=0$. The agreement, after a short transient, is as good as that observed in [3] for the SL.

The conclusion which follows is thus that, while the temporal behavior of the scaling depends on the value of δ, the functional form of the spatial dependence in the nonlinear correlation function appears to be the same for all initial conditions. This is what we refer to as "universality" of the nonlinear correlations in this context.

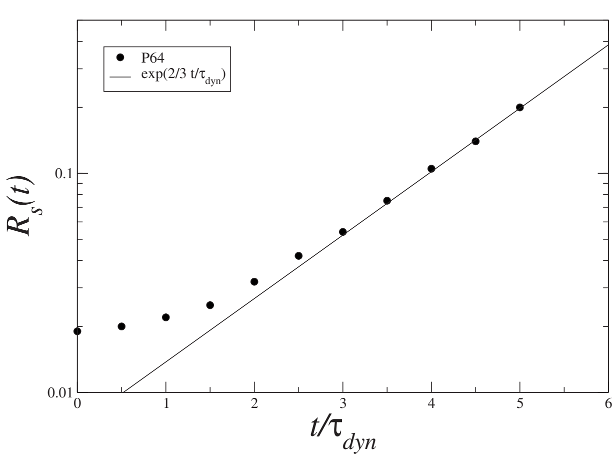

FIG. 4. Evolution of the function $R_s(t)$ in P64 (points) compared with its prediction from linear fluid theory, $R_s(t) \propto \exp[(2/3)t/\tau_{\rm dyn}]$.

We note that, just as underlined for the SL initial conditions in [3,4], it is also true for the Poisson initial conditions that the spatiotemporal scaling of the two-point correlation function, given by Eq. (5), is a good approximation well before the asymptotic scaling behavior, given by Eq. (6), sets in. While the dynamical model we present below is valid only in this first phase (i.e., that prior to the asymptotic regime), it is thus natural to hypothesize that the form of the asymptotic correlation function is in fact determined in this phase. We will discuss this hypothesis further in our conclusions, and, in particular, how it could be tested for.

The second essential result about the development of nonlinear correlations that we recall is the following. In all these simulations (both Poisson [5] and SL [5,3]) we observe that the relation

$$\omega(r)dr = \left(1 - \int_0^r \omega(s)\mathrm{d}s\right)4\pi r^2 n_0[1+\xi(r)]dr \tag{7}$$

holds to a very good approximation, from the time that significant nonlinear correlations first develop until a time of order a dynamical time later. As explained in [3] (see also [11]), it is valid if all but the two-point correlations are trivial. It is thus natural to interpret its observed approximate validity for the correlations which develop in the first phase of nonlinearity to indicate that these correlations develop predominantly *as a result* of the two-body clustering of NN pairs of particles. For the case of the Poisson initial conditions it has been shown explicitly in [5] that this interpretation is correct: by integrating from the initial conditions with only forces between initial NN pairs, the evolution of correlation is well described up to approximately one dynamical time when nonlinear correlations have developed up to a scale of order ℓ.

B. Two-phase model evolution

For SL initial conditions, with small δ, the approximation of forces as NN dominated is, as we have discussed above, not valid at early times. In this limit, however, we have developed in [7,8] an analytical perturbative approach, which at linear order gives a very good approximation to the dynamical evolution. The treatment involves simply a Taylor expansion of the force between particles in their *relative* displacements from their initial lattice positions **R**, and thus breaks down when the latter become equal to the initial separation of the particles. In [8] the precision of the linearized approximation has been explored in detail for the SL initial conditions (and others). For the evolution of the average relative distance between NNs, the approximation turns out to be very good until this quantity becomes quite close to the initial lattice spacing, i.e., until many particles come close to their NNs. We refer to this approximation as *particle linear theory* (PLT) as it is simply a generalization for particles of an analogous standard treatment for the self-gravitating fluid (in the Lagrangian formalism, leading to the so-called Zeldovich approximation). We do not detail further the implementation of this PLT approximation here as a succinct sum-

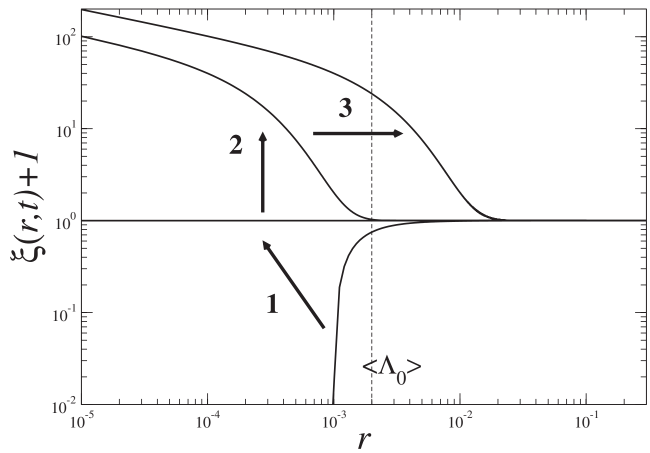

FIG. 5. Schematic representation of the evolution of two-point correlations [specifically $\xi(r)+1$] from SL initial conditions with small δ. During the first phase (1) the initial anticorrelation (i.e., exclusion) at small scales is destroyed, as particles evolve in the regime described by PLT. In the second phase (2) large positive correlations are created at small scales, up to roughly the initial lattice spacing. The forces responsible for these correlations are predominantly those exerted by members of NN pairs on one another. In the subsequent evolution (3), when this approximation is no longer valid, the regime of positive correlations grows in a self-similar way, larger and larger scales becoming nonlinear with time.

mary may be found in [4], and a very complete discussion in [8].

The fact that PLT is observed to work very well up to close to the time of NN domination, and the observation that Eq. (7) is valid when significant nonlinear correlation emerges, leads us to consider the approximation of the early time evolution in which one abruptly matches a PLT phase onto a NN-dominated phase.

Phase 1. From $t=0$ up to a time t_*, particles in the system evolve according to PLT.

Phase 2. For $t \geq t_*$, particles evolve subject only to the gravitational attraction of their NNs at the time t_*.

While the approximation used in the first phase is good for the whole system, and in particular describes well the evolution of correlations at any distance, the second phase will only be valid approximately in describing correlation at some sufficiently small scale, and for sufficiently short times. A schematic representation of the evolution of correlations is given in Fig. 5.

We have implemented the above two phase evolution numerically on the set of initial conditions given in Table I. We have taken the time at which we match the approximations, t_*, as a free parameter and adjusted it to best fit the evolution of the correlations in the full gravity simulations in the phase when strong nonlinear correlations first emerge.[6] We find that for each initial condition there is indeed a choice of the time t_* which gives such a fit, to a very good approximation over a range of amplitudes from $\xi(r) \sim 10^2$ down to considerably less than unity. Results are shown in Fig. 6, for the optimal times t_*:

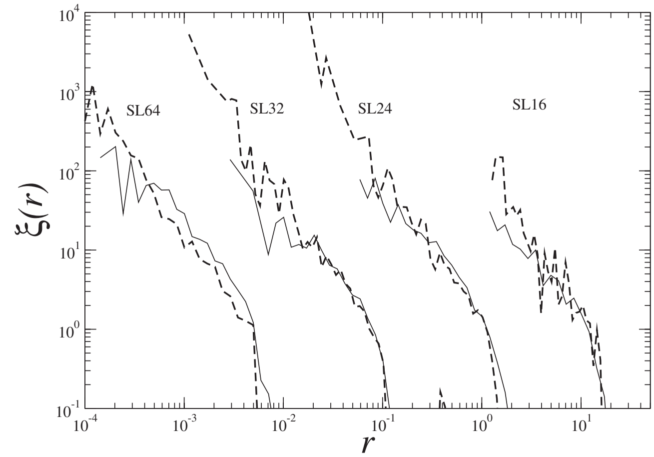

FIG. 6. Two-point correlation function at the times $t_{\max} = (1, 2.5, 3.5, 4.5)\tau_{\rm dyn}$ for the different initial conditions as indicated, in both the full gravity simulations (thick lines) and the simulations of the two-phase model described in the text (thin lines). The corresponding transition times t_* are the optimal ones given in Eq. (8). For clarity the x axis has been rescaled for each initial condition (as otherwise the curves are, to a very good approximation, all superimposed; cf. [3,4]).

$$t_* \approx 0, \ 0.5, \ 1.5 \ \text{and} \ 3.0 \tag{8}$$

for SL64, SL32, SL24, and SL16, respectively (with time in units of $\tau_{\rm dyn}$). The results here are given at the times $t_{\max}$, which are the approximate times at which we observe the evolution under NN interactions to lead to correlations beginning to deviate from (i.e., lag behind) those in the full gravity simulations:

$$t_{\max} = 1, \ 2.5, \ 3.5 \ \text{and} \ 4.5. \tag{9}$$

For times $t > t_{\max}$ the modified simulations stop evolving significantly (as one is then simply seeing the averaged effect of the periodic motion of many NN pairs). In contrast, the full gravity simulations continue to evolve clustering from the collective motion of larger scales which has been completely neglected in the second phase of the approximation.

C. Transition to NN domination

We now consider in more detail the essential time scale t_*, which we determined numerically above. Given our discussion of and motivation for the two-phase model, we might expect it to correspond to the time at which PLT breaks down and the forces on a particle become typically dominated by that due to its NN. As we will now explain, it corresponds in fact to a time somewhat shorter than this.

An approximate characterization of the time $t_{\rm NN}$ at which NN forces dominate can be given numerically by studying the relation between the NN PDF $\omega(r)$ and the force PDF $W(F)$. As explained in Sec. II B, NN forces dominate when the relation Eq. (4) holds, for large values of the modulus of the force F. While in the initial condition SL64 it already holds to a very good approximation, in the distributions ob-

[6]In the second (NN) phase we use the same numerical value of $\varepsilon = 0.00175$ as in the full gravity simulations (and the same functional form of the smoothing as in GADGET).

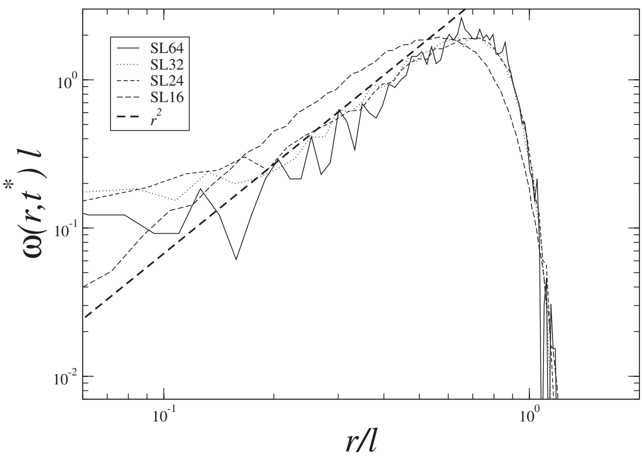

FIG. 7. NN PDF in the distributions evolved to the times $t_{\rm NN}$ given in Eq. (10) for the different SL initial conditions indicated. Also shown is the result for a Poisson distribution [Eq. (2)], and a line indicating its small scale behavior $\omega_P(r)\propto r^2$. Units on the x (y) axis have been divided (multiplied) by the numerical value of the lattice spacing ℓ in each case.

tained by evolution of the SL with smaller δ we find that it becomes good at the times

$$t_{\rm NN}\approx 2.0,\;\; 3.0,\;\; \text{and}\;\; 4.2 \qquad (10)$$

for SL32, SL24, and SL16 respectively. Figure 7 shows the NN PDF in each case at these times, and Fig. 8 the PDF of the force modulus. The quantities have been normalized by the characteristic length or force scale in each case to give the PDF for the corresponding dimensionless quantity: for the NN PDF we have normalized the radial distance to the initial lattice spacing ℓ in each case, and for the force PDF we have normalized to the force $(2m\ell)/\tau_{\rm dyn}^2\propto Gm^2/\ell^2$ (i.e., the force between two particles at the characteristic distance ℓ). We have plotted the two quantities separately to show the

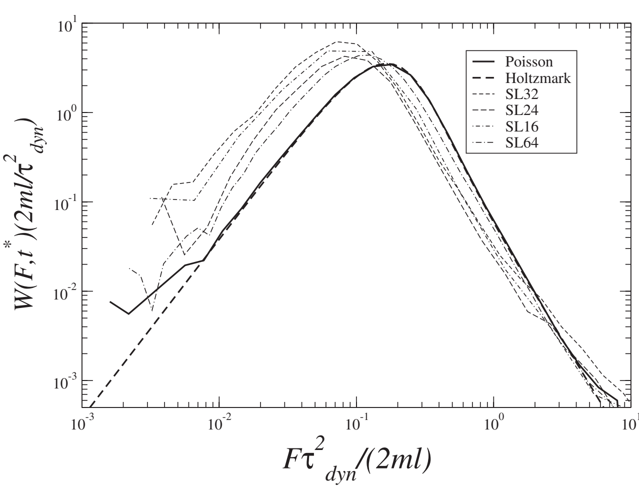

FIG. 8. PDF of the force at the times $t_{\rm NN}$ given in Eq. (10) for the different SL initial conditions as indicated. The force has been rescaled as described in the text. Also shown is the PDF for the P64 initial conditions, as well as the theoretical result for an infinite PDF (i.e., the Holtzmark distribution).

fact that at the times $t_{\rm NN}$ we find in all cases good agreement at small separations and strong forces with the corresponding results for a Poisson distribution. This will be important in our discussion below of the origin of the universality of the observed correlations. As discussed in Sec. II B the relation Eq. (4) is very accurately followed for a Poisson distribution (cf. Fig. 2), and so it follows implicitly from these figures that it is indeed a good approximation here in the regime of strong forces. The very clear differences in the force PDF at smaller values of the force are a reflection of the real difference in the fluctuations at larger scales: the larger amplitude at smaller values of the force, for a force PDF normalized as we have done, is a reflection of the fact that the fluctuations become more suppressed at large scales as δ decreases. In the NN PDF, on the other hand, such differences due to large scales are not present (because it characterizes the small scale properties).

It is clear, from Eqs. (8) and (10), that t_* does not correspond, even approximately, to $t_{\rm NN}$, but rather is shorter by more than a dynamical time. Likewise it is shorter than the estimated time of breakdown of PLT. Indeed, we find that at $t_{\rm NN}$ we have in all cases that the root square variance of the displacements of NN particles, normalized in units of the appropriate ℓ, is $\approx$0.5 which corresponds to the criterion for breakdown of PLT found in [8]. At t_*, on the other hand, we measure the values

$$\delta\approx 0.18,\;\; 0.12,\;\; 0.11 \qquad (11)$$

for SL32, SL24, and SL16 respectively.[7]

This difference, and the approximate value of t_*, can in fact be understood quite easily as follows. An approximation to the evolution that is clearly better than our two-phase approximation at all times is that in which we use the expansion of PLT to approximate all the forces on a particle *except* that due to the particle which becomes its NN in the second phase. Indeed, we would expect such an approximation—let us call it PLT+NN—to be very good for the whole regime we are treating. If we now consider our two-phase treatment (PLT for $t<t_*$, NN for $t\geq t_*$) as an approximation to PLT +NN, it is not difficult to understand why the t_* which makes the approximation optimal is of order a dynamical time smaller than $t_{\rm NN}$. The reason is that the equation of motion for the displacements in PLT reduces, for time scales up to of order a dynamical time, to a simple ballistic approximation.[8] Thus when we turn on the NN interaction at time $t=t_*$ we

[7]These values are, as expected, also in very good agreement with those predicted by PLT. A more appropriate characterization of the breakdown of PLT is in fact given by considering the variance of the *relative* displacements. The difference with respect to the simple (one-point) variance is in fact negligible here for reasons that will be explained below.

[8]It is straightforward to show that one obtains $\mathbf{u}(\mathbf{R},t)=\mathbf{u}(\mathbf{R},0)+\mathbf{v}(\mathbf{R},0)t$, where $\mathbf{v}(\mathbf{R},0)$ is the velocity of the particle associated with the lattice site $\mathbf{R}$ at the (arbitrary) initial time $t=0$, by expanding to linear order the full PLT expression for the evolution of the displacements (see [4] or [8]) in powers of $\epsilon_n(\mathbf{k})t/\tau_{\rm dyn}$, where $\epsilon_n(\mathbf{k})$ are numbers of order unity specifying the eigenvalues of the dynamical matrix for gravity on the lattice [4,8].

follow, for of order a dynamical time, the PLT+NN evolution. If we reduce t_* further we will deviate from PLT+NN more because PLT is poorly approximated; if we increase t_* we will lose precision by excluding the full NN contribution to the force.

Note that PLT "comes for free" in this way for a time after t_* only because we match the velocities at t_*. A simple check on the above explanation is provided by doing a simulation in which we reset the velocities to zero at t_*. We have done this and find indeed that the match to the full gravity evolution is considerably less good, and that, particularly for the SLs with smaller δ, no choice of t_* can produce a good fit. We will return in the next section to discuss further the role of the velocities.

D. Origin of universality of nonlinear correlations

The simple two-phase model thus describes quite well the emergence of the nonlinear correlations at early times for the range of initial conditions studied. We now turn to the fact that these correlations are approximately the same in all cases, i.e., the two point correlation functions, as a function of radial separation, agree quite well (and indeed agree well with these quantities as found in [5] and [3,4]).

1. Small scale correlation properties at t_{NN}

The explanation for this "universality" of the clustering is clearly suggested by the results shown above in Figs. 7 and 8: in all cases the evolution gives at $t \approx t_{\mathrm{NN}}$ a point distribution with correlation properties very similar to those of the Poisson distribution, at the scales relevant to the development of clustering in the following phase. Indeed the force PDF, at stronger values of the force corresponding to the particles which will cluster most rapidly in the subsequent time, follows very closely that in the Poisson distribution. Thus the clustering then develops as in the latter distribution, giving the same correlation properties. These are simply those which emerge, as described in detail in [5], when pairs of particles with initial separations given by the NN PDF $\omega_P(r)$ in the Poisson distribution fall on one another.

Why are the correlation properties so similar at these scales to those of a Poisson distribution at the time t_{NN}? We have emphasized in Sec. II that a Poisson distribution is approximated to increasingly large scales as the amplitude δ of the random *uncorrelated* shuffling increases. It follows that if the evolution described by PLT is to a good approximation simply an amplification of the initial displacements, with only very weak correlation, the transformation to a universal Poisson initial condition at the relevant scales at $t \approx t_{\mathrm{NN}}$ results. To quantify whether this is indeed the case we can consider, e.g.,

$$c_{\mathrm{NN}}(t) \equiv \frac{\frac{1}{6}\sum_{\mathbf{R}_{\mathrm{NN}}} \langle \mathbf{u}(0) \cdot \mathbf{u}(\mathbf{R}_{\mathrm{NN}}) \rangle}{\langle \mathbf{u}^2 \rangle}, \tag{12}$$

where $\mathbf{R}_{\mathrm{NN}}$ are the lattice vectors of the six particles of the simple cubic lattice closest to the origin (where there is a particle). The ensemble average is the average over realizations of the random initial conditions of the SL.[9]

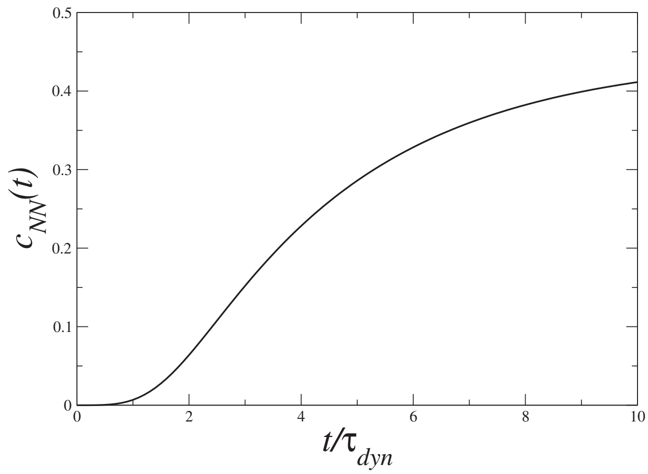

FIG. 9. Temporal evolution of the simple measure $c_{\mathrm{NN}}(t)$ of the correlation in the displacements given by Eq. (12) in PLT.

In Fig. 9 is shown $c_{\mathrm{NN}}(t)$ as calculated exactly in PLT. We see that, for the time scales over which we use PLT (at most about four dynamical times), the correlation which develops between displacements at the relevant (small) scales is indeed weak ($c_{\mathrm{NN}} < 20\%$).

2. The limit $\delta \to 0$

Figure 9 shows that if, instead, our initial conditions had δ sufficiently small so that t_{NN} were greater than a few dynamical times, the approximation of weak correlation of the displacements at small scales would become progressively worse as δ decreases. As a result the basis for the approximate universality in the subsequent evolution would also be expected to become a progressively poorer approximation. The evolution of the displacements in PLT is simply a sum over the appropriately evolved eigenmodes of the displacement fields in the corresponding linear approximation to the interparticle force. The behavior we observe here at long times is a result, as discussed in detail in [4,7,8], of the fact that in this regime the small spread in the eigenvalues of the modes of the displacement field becomes important. The modes with slightly faster growth become arbitrarily dominant, leading to the very specific correlation of displacements described by these modes. For arbitrarily long times, i.e., for arbitrarily small initial δ, one therefore obtains a distribution with correlation properties at all scales very different to that which could form from the Poisson distribution.[10] We thus conclude that the universality we observe in our numerical

[9] When this average is performed, all six approximate NNs are equivalent so that it is in fact sufficient to evaluate $c_{\mathrm{NN}}(t)$ for a single neighbor [and drop the sum and factor of 1/6 in Eq. (12)].

[10] Specifically, as $\delta \to 0$ the evolution will always be dominated at the time when PLT breaks down by the most rapidly growing eigenmode. In a SL lattice (see [7,8]) this eigenmode is one in which adjacent infinite parallel planes fall toward one another.

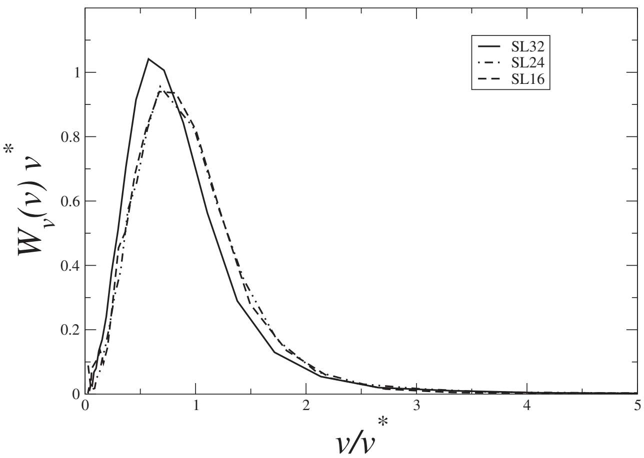

FIG. 10. PDF of the modulus of the velocities at time t_{NN} in the full gravity simulations starting from the initial conditions SL32, SL24, and SL16. The characteristic velocity $\bar{v}$ is defined in Eq. (13).

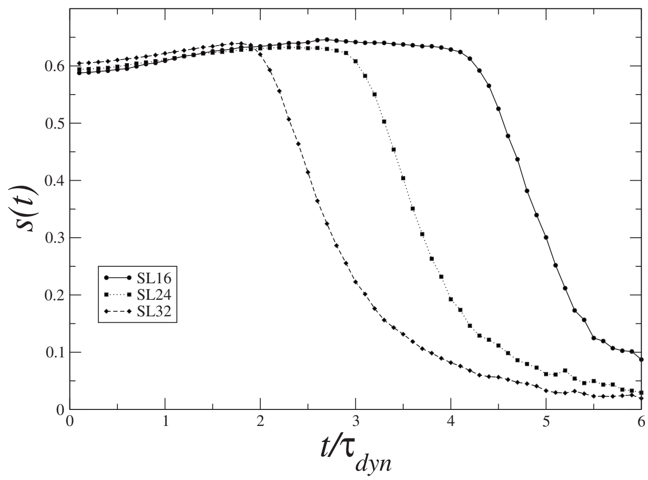

FIG. 11. Behavior of $s(t)$ [as defined in Eq. (14)] in SL32, SL24, and SL16.

simulations is a good approximation in the range of SL initial conditions with small, but not very small,[11] δ.

3. Role of velocities at t_{NN}

In the above discussion we have neglected the role of the velocities: in SL32, SL24, and SL16 nonzero velocities have developed at t_{NN} which make the full initial conditions at this time different from those in SL64 and P64 (with vanishing velocities at $t_{NN}=0$). The PDFs $W_v(v)$ of the modulus of these velocities as measured in the different simulations at this time are shown in Fig. 10. We have normalized for convenience in units of

$$\bar{v} \equiv \sqrt{\frac{Gm}{\ell}}, \tag{13}$$

which is the velocity gained by a particle initially at rest when it reduces its separation by one-half from a particle initially at distance ℓ.

We observe that the different PDF of the velocities agree quite well, which means that the full (space and velocity) initial conditions at t_{NN} are indeed very similar in these simulations. Compared to SL64 and P64, however, the difference in velocities is *a priori* significant: their magnitude is not small, but of order unity, in the units chosen, which is characteristic for the next stage of free fall of NN particles that leads to the correlations. Given that at this time t_{NN} we expect the velocity of particles to be, on average, oriented toward their NNs (since t_{NN} is significantly larger than t_*), and that in the approximated Poisson distribution the average NN distance is 0.55ℓ, the distribution at t_{NN} should be well approximated as one in which pairs of NNs fall on one another, but starting at an earlier time.

This is further illustrated and quantified by Fig. 11, which shows the temporal evolution of the quantity

$$s(t) \equiv \frac{\langle \mathbf{v}\cdot\mathbf{r}_{NN}\rangle}{|\mathbf{v}||\mathbf{r}_{NN}|} \tag{14}$$

for $t>0$, where $\mathbf{v}$ is the velocity of a given particle (at time $t>0$), and $\mathbf{r}_{NN}$ is the vector pointing toward its NN (at the same time t), and the average is over all the particles. The evolution of this quantity is qualitatively very similar in all three simulations: after a slow rise from an initial nonzero value a rapid decrease sets in at a time close to the estimated t_{NN} in each case [cf. Eq. (10)]. The characteristic time for this, roughly exponential, decay of the correlations corresponds well to $\Delta t = t_{max} - t_{NN}$ [where t_{max} is as estimated in Eq. (9)]. We thus see, as anticipated, that there is significant correlation of the direction of velocity with the NN direction at time t_{NN}. While $\Delta t \approx 1$ for SL64 and PL64, we have $\Delta t \approx 0.5$ for the three simulations shown here, with a very similar behavior of the correlation function $s(t)$ in this phase. Thus SL32, SL24, and SL16 all lead to similar nonlinear correlations as those in SL64 and P64, but in a shorter time due to this correlation of velocities with the NN direction acquired before t_{NN}.

The behavior of $s(t)$ observed here can be understood in greater detail in the model we have described. The initial nonzero, approximately constant, value is a result of the fact that at sufficiently small times PLT can be well approximated by its fluid limit. In this case [4,8] the displacement of each particle off its lattice site is simply amplified in time, so that the function $s(t)$ is independent of time and equal to the expression in Eq. (14) with $\mathbf{v}$ replaced by $\mathbf{u}$, the initial displacement of the particle from its lattice site [giving $s(0) \approx 0.6$]. The slow increase of correlation is due to the difference between PLT and this fluid limit. The decrease from about t_{NN} signals that pairs of NN particles have now begun to cross one another, giving a contribution with the opposite sign to $s(t)$. At sufficiently long times a given particle's motion is finally no longer oriented with the direction of its NN, as expected since the gravitational field will become dominated by the collective effect of many particles acting on any given particle.

[11] Such small initial δ is very difficult to simulate numerically because of the precision required.

IV. DISCUSSION

In this paper, we have studied in detail the early time evolution of infinite self-gravitating shuffled lattices, as well as the limiting case given by the Poisson distribution. We have shown that a very good description of the evolution of two point correlations in this phase is given by a simple approximation in which the force on particles abruptly switches from that given by the PLT approximation developed in [7,8] to the force due only to NN particles. Further, in the first phase the system evolves at small scales to always resemble closely the Poisson distribution, explaining the universality of the form of the nonlinear correlation function which emerges. We have noted, however, that this universality will not extend to SL initial conditions with arbitrarily small initial shuffling. In this limit effects come into play in the very long first (PLT) phase leading to a strongly correlated evolution at all scales, which is different from (and unrelated to) that in the Poisson distribution.

We have thus given, for this specific class of initial conditions, an explanation of the nonlinear correlations which emerge at early times. As underlined in [3,4] this nonlinear correlation function coincides with that which is observed at later times, when the system manifests a simple spatiotemporal scaling (or self-similarity). Thus the model appears to explain this asymptotic form of the nonlinear correlations. As described in [4], this can be understood also in the following way: these nonlinear correlations in the system evolving at any later time can be well approximated by those in an evolved coarse-grained "daughter" distribution. In the latter the system may be in the early time phase studied here, while the original distribution is not. The nonlinear correlation functions of the two systems nevertheless coincide.

Our results are of relevance to simulations of structure formation in the universe in cosmology. In this case the goal of numerical simulation is to recover the nonlinear correlations in the Vlasov-Poisson limit of the evolution of a self-gravitating N-body system. These initial conditions are different from those used here—with initially *correlated* displacements and an expanding space—but, as shown in [3,4], the evolution is qualitatively similar to our simpler case. If the same kind of model can be used to explain nonlinear correlations in an early time regime in this case then these correlations clearly are not described by the VP limit: the forces due to NN particles are neglected in this (mean field) limit, while in this model they are the dominant ones. In this context, this would mean that results in this regime would need to be discarded as unphysical (i.e., not representing the required physical limit, but just a numerical effect arising from the method of discretization). Further, if the form of the asymptotic nonlinear correlation function is really determined by this early time evolution, these effects of discreteness are then important at all times and the system never represents the VP limit. This resemblance of the early time and asymptotic correlation function may, however, be a simple coincidence. The evolution at longer times may then indeed be representative of the VP limit while discreteness effects are important only at early times.

We will study these issues further in future work. First the application of our simple two-phase description to cosmological simulations should be investigated. We expect the expansion of space to change nothing qualitatively in our model: the description of the PLT phase is qualitatively unchanged [8], and NN domination during the formation of the first nonlinear structures in such simulations has been explicitly shown in [16]. The fact that the typical initial conditions of such simulations have much more long wavelength power than those considered here may, however, be important [typically one has $P(k)\sim k^m$ with $m\leq -1$, compared to $n=2$ (SL)and $n=0$ (Poisson) considered here]. Second, as we have discussed at some length in the conclusions of [4], it would be instructive to study numerically the relation of the asymptotic regime to the early time regime by completely modifying the former using a large smoothing in the force, i.e., a smoothing scale $\varepsilon \gg \ell$. In this case the dynamics we have described, between NN particles, will not occur. However, at longer times the system should still evolve the same correlations if the VP limit of the system is indeed that represented by the simulations with $\varepsilon \ll \ell$.

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