Thermally enhanced stability in fluctuating bistable potentials

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Overdamped motion of Brownian particles in an asymmetric double-well potential driven by an additive nonequilibrium three-level noise and a thermal noise is considered. In the stationary regime, an exact formula for the mean occupancy of the metastable state is derived, and the phenomenon of enhancement of stability versus temperature is investigated. It is established that in a certain region of the system parameters the mean occupancy can be either multiply enhanced or suppressed by variations of temperature. We show that this effect is due to the involvement of different time scales in the problem. The necessary conditions for several different behaviors of the mean occupancy as a function of temperature are also discussed. The effect is more pronounced when the kurtosis of the three-level noise tends to −2, i.e., in the case of dichotomous noise.

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I. INTRODUCTION

Within the past two decades the behavior of nonequilibrium systems depending on fluctuations (noise) has received considerable attention. Stochastic resonance [1], noise-induced phase transitions in spatially extended systems [2], stochastic transport in ratchets [3], hypersensitive transport [4,5], resonant activation [6], noise-induced multistability, and discontinuous transitions [7,8] are a few phenomena in this field.

One of the objects of special attention has been noise-driven escape processes in a bistable potential. The problem of noise-driven barrier crossing dynamics of a Brownian particle in a double-well potential coupled to a heat bath, represented by an additive white noise, was earlier formulated and solved by Kramers [9]. Since then the Kramers model and many of its variants have been addressed by a large number of studies at various levels of description. The archetypal examples exhibiting bistable (double-well) potentials are nonequilibrium chemical reactions (e.g., the second Schlögl model [10,11]), nonequilibrium Ginzburg-Landau-type bistable stochastic dynamics [10], and optical bistability in laser devices [12,13].

Recent years have witnessed an increasing interest in the dependence of the mean exit time of metastable and unstable systems with fluctuating potentials on noise intensity [14,15]. Related investigations involving noise-induced stability [16] or noise-enhanced stability [17,18] belong to a highly topical interdisciplinary realm of studies, ranging from condensed matter physics to molecular biology, or to cancer growth dynamics [14,19–21]. One should take care not to confuse noise-induced stability (NIS) with noise-enhanced stability (NES): NIS implies complete stabilization in a metastable state, while NES is only a postponement of the system instability [14]. The effect called NES in [15,17,20] is observed in a periodically (or stochastically) driven system with a single metastable minimum. The system remains in the metastable minimum for some time given by the mean first passage time (MFPT) for the barrier, and the MFPT has a maximum at a certain noise intensity. In the case of NIS the potential fluctuates stochastically with a certain correlation time and has two minima. The less stable minimum is the absolute minimum for a certain configuration of the potential, but most of the time this minimum is metastable. Nevertheless, it can be highly occupied [16].

Motivated by investigations into the effect of a periodic electric field on cell membrane proteins [22,23], the author of [16] has considered overdamped motion of a Brownian particle in an asymmetric bistable potential fluctuating according to a dichotomous noise. This biologically motivated model clearly demonstrates the effect of NIS, as for intermediate fluctuation rates the mean occupancy of minima with an energy above the absolute minimum is enhanced. Earlier studies of noise-induced stability as well as related effects, such as NES and resonant activation, have mainly been interested in the dependence of the effects on the potential fluctuation flipping rate. However, the essential role of thermal fluctuations has been recognized by the consideration of NES [17,20]. In spite of the obvious significance of this circumstance, the role of thermal fluctuation intensity has not been much investigated in the context of NIS.

In the present paper we consider a model similar to the one presented in [16], except for some details of the potential profile and for the dichotomous noise being replaced with a trichotomous noise. As the results of [16] show that the phenomenon of NIS is quite universal and manifests itself for arbitrary bistable potential landscapes, we decided to study overdamped motion of Brownian particles in an asymmetric, bistable, piecewise linear potential subjected to both a trichotomous noise and a thermal one. The piecewise linear potential is important for at least two reasons. First, it can be used as a first approximation of the shape of an arbitrary potential and second, it is sufficiently simple to allow an analytic treatment of the relevant quantities, being at the same time physically rich enough to provide most of the effects characteristic of two-well potentials. In order to get the results in exact forms for all values of the noise parameters, the nonequilibrium fluctuations of a potential landscape are modeled as a trichotomous noise. As to the trichotomous Markov process [24], it is most important for our

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purposes that systems driven by trichotomous noises can, in stationary regimes, often be described in full analytic detail. Also, trichotomous noise can, in appropriate limits, be reduced to either white shot noise or Gaussian white noise, and either it can mimic the effects of finite correlation time of real noise or it may directly provide a good representation of an actual physical situation, e.g., thermal transitions between three configurations or states. Finally, it has the advantage that it can easily be implemented as an external noise with finite support. Although both dichotomous and trichotomous noises may be useful in modeling natural colored fluctuations, the latter is more flexible, including all cases of dichotomous noise [24]. Furthermore, it is remarkable that for trichotomous noises the kurtosis φ can be anything from -2 to ∞, unlike the kurtosis for Gaussian colored noise, $\varphi=0$, and symmetric dichotomous noise, $\varphi=-2$. This extra degree of freedom can prove useful in modeling actual fluctuations.

Our purpose is to report some interesting phenomena for Brownian particles, which occur in a simple asymmetric bistable potential by variation of temperature, arising as a consequence of interplay between nonequilibrium noise, thermal noise, and asymmetry of the potential. We restrict ourselves to discussion of the stationary regime of the system investigated, i.e., relaxation processes from initial states to the stationary regime are not considered. For brevity's sake, from now on we shall call the less stable minimum of the average potential (this minimum is metastable most of the time) the metastable state and the absolute minimum of the average potential the stable state of the potential.

The main contribution of this paper is as follows. We provide an exact formula for the analytic treatment of the dependence of the occupancy probability of a metastable state on various system parameters: viz., temperature, potential asymmetry, correlation time, kurtosis, and noise amplitude. We establish a thermal-fluctuation-induced phenomenon: namely, for certain values of the system parameters there exist several ranges of temperature values where the occupancy of the metastable state can be either enhanced or suppressed by variations of temperature. The general highly nonlinear dependence of the occupancy W_0 of a metastable state on the temperature D is characterized by two temperature regions, related to noise-induced stability: for lower temperatures the occupancy of the metastable state is greater than that of the stable state, whereas for higher temperatures the situation is the reverse. In the region of the crossover we have demonstrated the possibility of having, at moderate values of noise correlation times, multiple local extrema and a resonancelike peak in $W_0(D)$. As the noise correlation time τ_c grows, the local extrema in this region disappear, and turn into two wide plateaus at a larger τ_c. We also show that such a behavior of the system is quite robust, and the mentioned phenomenon occurs within a broad range of trichotomous noise parameters (kurtosis, correlation time). Thus, the results indicate the possibility that the stability of the metastable states could be controlled by varying the temperature.

The structure of the paper is as follows. Section II presents the basic model investigated. A master equation description of the model is given and the formula for the occupancy probability of the metastable state is found. Section III analyzes the behavior of the occupancy probability. The phenomenon of multiple enhanced stability of the metastable state versus temperature is established. Section IV contains some brief concluding remarks. Some formulas are delegated to the Appendix.

II. THE MODEL AND THE EXACT SOLUTION

As an archetypical model for systems with a metastable state that are strongly coupled with a noisy environment, we consider one-dimensional overdamped Brownian motion in a fluctuating sawtoothlike asymmetric bistable potential,

$$U(X,Z) = U(X) + XZ(t), \tag{1}$$

where $X(t)$ is the displacement of a Brownian particle at the time t and the variable $Z(t)$ is a Markovian trichotomous noise [24], which consists of jumps between three values $z_1=a$, $z_2=0$, $z_3=-a$, $a>0$. The jumps follow, in time, the pattern of a Poisson process, the values occurring with the stationary probabilities $p_s(a)=p_s(-a)=q$ and $p_s(0)=1-2q$, where $0<q<1/2$. The transition probabilities $T_{ij}:=p(z_i,t+\tau|z_j,t)$ between the states z_n, $n=1,2,3$, can be represented by means of the transition matrix (T_{ij}) of the trichotomous process as follows:

$$(T_{ij}) = (\delta_{i,j}) + (1-e^{-\nu\tau})\begin{pmatrix} q-1 & q & q \\ 1-2q & -2q & 1-2q \\ q & q & q-1 \end{pmatrix},$$

where $\nu>0$, and $\delta_{i,j}$ is the Kronecker symbol. In a stationary state the fluctuation process satisfies $\langle Z(t)\rangle=0$ and $\langle Z(t+\tau)Z(t)\rangle=2qa^2\exp(-\nu\tau)$, where the switching rate ν is the reciprocal of the noise correlation time, $\tau_c=1/\nu$, i.e., $Z(t)$ is a symmetric zero-mean exponentially correlated noise. The trichotomous process is a particular case of the kangaroo process [25] with the kurtosis $\varphi=\langle Z^4(t)\rangle/\langle Z^2(t)\rangle^2-3=1/2q-3$.

We describe the overdamped motion of Brownian particles in dimensionless units, using the Langevin equation

$$\frac{dX}{dt} = h(X) - Z(t) + \xi(t), \quad h(x) = -\frac{dU(x)}{dx}, \tag{2}$$

where the thermal noise $\xi(t)$ satisfies $\langle\xi(t)\rangle=0$ and $\langle\xi(t_1)\xi(t_2)\rangle=2D\delta(t_1-t_2)$. D is the thermal noise intensity, which for the sake of brevity will be called temperature. The piecewise linear asymmetric bistable potential considered has the profile

$$U(x) = \begin{cases} \frac{1}{d}x, & x \in (0,d), \\ 1 + \frac{1+\varepsilon}{1-d}(d-x), & x \in (d,1), \\ U(0)=U(1)=\infty. \end{cases} \tag{3}$$

A schematic representation of the three configurations assumed by the "net potentials" $V_n(x)=U(x)+z_n x$, $n=1,2,3$, associated with the right-hand side of Eq. (2), is shown in Fig. 1. In this work, we restrict ourselves to the system parameter region where the net potentials $V_n(x)$ for all states $n=1,2,3$ of the nonequilibrium noise Z have two minima. More precisely, we assume that

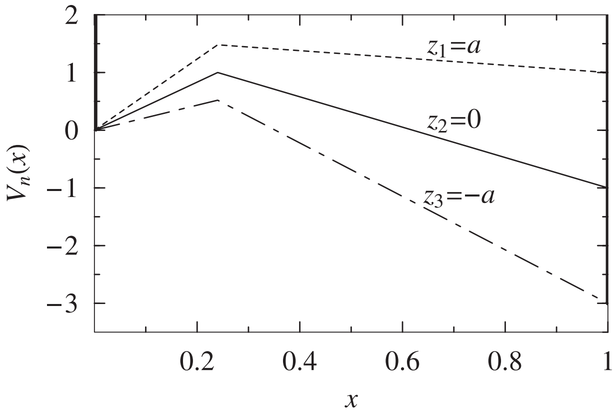

FIG. 1. Representation of different states of the net potentials $V_n(x)=U(x)+z_n x$ with $z_1=a$, $z_2=0$, $z_3=-a$. The potential $U(x)$ is given by Eq. (3) at the parameter values $d=0.24$, $a=2$, and $\varepsilon=1$. All quantities are dimensionless.

$$a < \frac{1+\varepsilon}{1-d}, \quad a < \frac{1}{d}, \quad 0 < d < \frac{1}{2},$$

$$0 < \varepsilon < a(1-2d). \tag{4}$$

The master equation corresponding to Eq. (2) reads

$$\frac{\partial}{\partial t}P_n(x,t) = -\frac{\partial}{\partial x}\left([h(x)-z_n]P_n(x,t) - D\frac{\partial}{\partial x}P_n(x,t)\right) + \sum_{m=1}^{3} S_{nm}P_m(x,t), \tag{5}$$

where $P_n(x,t)$ is the joint probability density for the position variable $x(t)$ and the fluctuation variable $z(t)$, and the transition matrix (S_{nm}) is given by

$$(S_{nm}) = \nu\begin{pmatrix} q-1 & q & q \\ 1-2q & -2q & 1-2q \\ q & q & q-1 \end{pmatrix}.$$

The stationary probability density in the x space, $P^s(x)$, is then evaluated via the stationary probability densities $P^s_n(x)$ for the states (x,z_n):

$$P^s(x) = \sum_{n=1}^{3} P^s_n(x). \tag{6}$$

As the "force" $h(x)=-dU(x)/dx$ is piecewise constant, $h(x)=h_0=-1/d$ for $x\in(0,d)$ and $h(x)=h_1=(1+\varepsilon)/(1-d)$ for $x\in(d,1)$, Eq. (5) splits up into two linear differential equations with constant coefficients for the two vector functions $\boldsymbol{P}^s_i(x)=(P^s_{1i},P^s_{2i},P^s_{3i})$, $i=0,1$, defined on the intervals $(0,d)$ and $(d,1)$, respectively. The solution reads

$$P^s_{ni}(x) = p(z_n)\sum_{j=1}^{5} Y_{ij}A_{nij}e^{-\lambda_{ij}x/D}, \tag{7}$$

where

$$p(z_n) = (1-2q)\delta_{n,2} + q(\delta_{n,1}+\delta_{n,3}),$$

$$A_{nij} = \frac{\nu D}{\nu D - \lambda_{ij}(h_i - z_n + \lambda_{ij})}.$$

Y_{ij} are constants of integration, and $\{\lambda_{ij},\ j=1,\ldots,5\}$ is the set of roots of the algebraic equation

$$\lambda_i^5 + 3\lambda_i^4 h_i + \lambda_i^3(3h_i^2 - a^2 - 2\nu D) + \lambda_i^2 h_i(h_i^2 - a^2 - 4\nu D) + \lambda_i \nu D[\nu D + 2(qa^2 - h_i^2)] + h_i\nu^2D^2 = 0. \tag{8}$$

Nine independent conditions for the ten constants of integration Y_{ij} can be determined at the points of discontinuity, by requiring continuity for the quantities $P^s_{ni}(x)$ and for the stationary current densities $j_{ni}(x) := (h_i - z_n)P^s_{ni}(x) - D\frac{d}{dx}P^s_{ni}(x)$ at the point $x=d$ and the vanishing of the current densities $j_{ni}(x)$ at the boundary points $x=0,1$, i.e.,

$$P^s_{n0}(d) = P^s_{n1}(d), \quad j_{n0}(d) = j_{n1}(d),$$

$$j_{n0}(0) = j_{n1}(1) = 0, \quad n = 1,2,3. \tag{9}$$

It follows from Eq. (5) that the system of linear algebraic equations (9) contains only nine linearly independent equations for Y_{ij}. By including the normalization condition

$$\sum_{n=1}^{3}\int_0^1 P^s_n(x)dx = 1, \tag{10}$$

a complete set of conditions is obtained for ten constants of integration Y_{ij}. Now the constants Y_{ij} can be expressed as quotients of two determinants of the tenth degree:

$$Y_{ij} = \frac{\det[B_{l,r}(1-\delta_{r,j+5i}) + \delta_{l,10}\delta_{r,j+5i}]}{\det(B_{l,r})}, \tag{11}$$

where the matrix $(B_{l,r})$, $l,r=1,\ldots,10$, is defined as follows:

$$B_{6,j+5} = B_{7,j+5} = B_{8,j} = B_{9,j} = 0,$$

$$B_{n,j+5i} = (-1)^i A_{nij}\exp\left(-\frac{d\lambda_{ij}}{D}\right),$$

$$B_{m+3,j+5i} = (h_i - z_{2m-1} + \lambda_{ij})B_{2m-1,j+5i},$$

$$B_{m+5+2i,\ j+5i} = \left[\delta_{0,i}\exp\left(\frac{d\lambda_{ij}}{D}\right) + \delta_{1,i}\exp\left(\frac{(d-1)\lambda_{ij}}{D}\right)\right]B_{m+3,\ j+5i},$$

$$B_{10,j+5i} = \frac{(-1)^i D}{\lambda_{ij}}\left[\exp\left(-\frac{\lambda_{ij}\delta_{1,i}}{D}\right) - \exp\left(-\frac{d\lambda_{ij}}{D}\right)\right], \tag{12}$$

with $n=1,2,3$, $m=1,2$, $j=1,\ldots,5$, and $i=0,1$.

The stationary probability density in the x space, $P^s_i(x)$, with $i=0$ for $x\in(0,d)$ and $i=1$ for $x\in(d,1)$, and the stationary occupancy probabilities W_0 and $W_1=1-W_0$ of the left and right potential wells, respectively, are given by

$$P^s_i(x) = \sum_{j=1}^{5} Y_{ij}\exp\left(-\frac{\lambda_{ij}x}{D}\right), \tag{13}$$

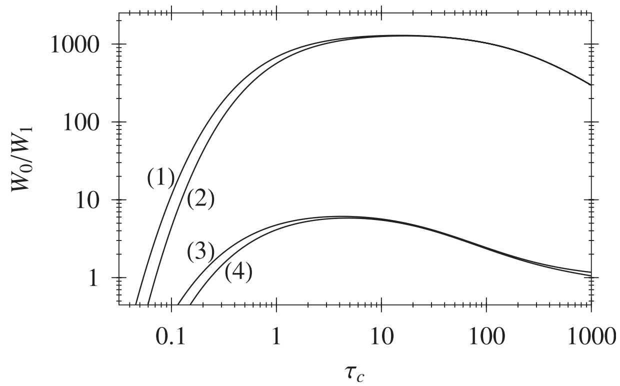

FIG. 2. Ratio W_0/W_1 vs the noise correlation time τ_c at various temperatures D. The occupancy probabilities W_0 and W_1 of the left and right potential wells, respectively, are computed by means of Eq. (14). Parameter values: $a=2$, $\varepsilon=1$, and $d=0.125$. The different curves correspond to the different values of the parameter q and temperature D: (1) $q=0.49$, $D=0.04$; (2) $q=0.35$, $D=0.04$; (3) $q=0.49$, $D=0.07$; (4) $q=0.35$, $D=0.07$.

$$W_0(x) = \int_0^d P_0^s(x)dx = \sum_{j=1}^{5} B_{10,j}Y_{0j}. \qquad (14)$$

We note that there is a profound difference between the situation in which white noise, however weak, is present, and the situation in which there is no white noise, i.e., the limit $D\to 0$ is discontinuous. In the case of $D=0$ the particles are locked in the minimum of an initial potential well and a barrier passage is not possible. For $D>0$, D being arbitrarily small, the crossing of the potential barrier triggered by rare thermal fluctuations is possible and the system relaxes to the stationary regime, which is independent of initial conditions.

The behavior of W_0 in different system parameter regimes will be considered in Sec. III. All numerical calculations are performed by using the software MATHEMATICA 5.0.

III. ENHANCEMENT OF THE STABILITY OF THE METASTABLE STATE

Of central interest to us are the stationary occupancy probability W_0 of the left potential well [see Eq. (14)] and its responses to the switching rate ν and to the temperature $D>0$. Figure 2 exhibits the ratio W_0/W_1 as a function of the switching rate ν at different values of the temperature. It can be seen that the functional dependence of W_0/W_1 on the correlation time $\tau_c=1/\nu$ is of a bell-shaped form. Notably, at low temperatures for intermediate values of ν the mean occupancy of the metastable state (the left potential well) is much larger than the mean occupancy of the stable state, i.e., such fluctuations enhance the occupancy of the left minimum, although most of the time it is not the absolute minimum of the potential. Thus, we observe a noise-correlation-time-induced stability for the metastable state.

The tendency apparent in Fig. 2, namely, an increase of the occupancy probability W_0 as the temperature D decreases, also takes place in the case of lower values of D. Moreover, the decrease of the kurtosis $\varphi=1/2q-3$ of the trichotomous noise Z also enhances the stability of the metastable state (cf. curves 1 and 2 in Fig. 2).

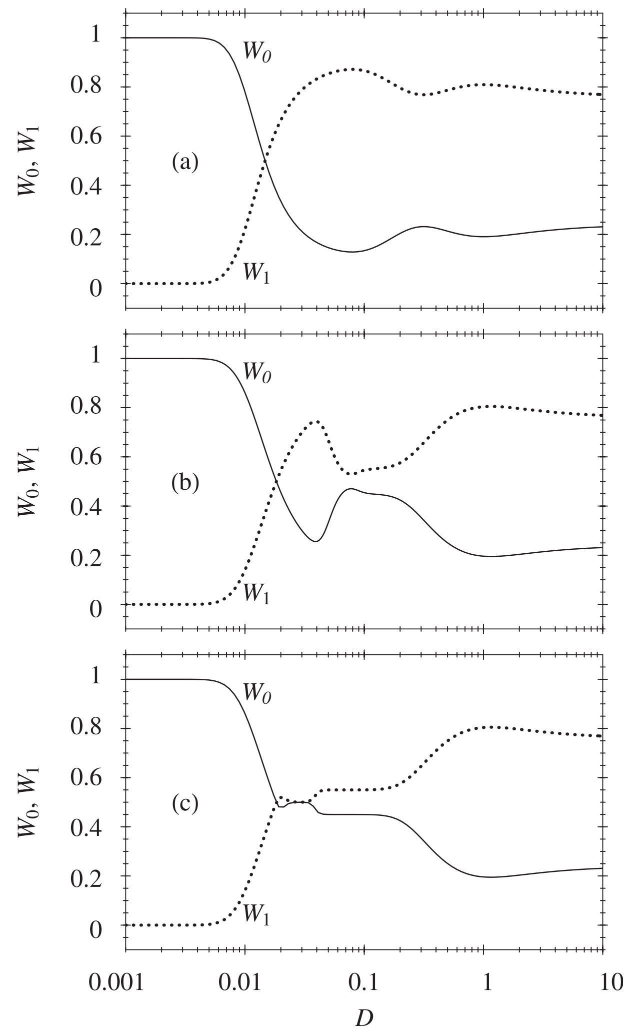

FIG. 3. Occupancy probabilities W_0 (solid line) and W_1 (dotted line) of the left and right potential wells, respectively, versus the temperature D [Eq. (14)]. The parameter values are $a=2$, $\varepsilon=1$, $d=0.24$, and $q=0.45$. At high values of temperature, $D>10$, the probabilities W_0 and W_1 saturate to the values d and $1-d$, respectively. $\nu=$ (a) 1; (b) 10^{-3}; (c) 10^{-9}.

In the case of dichotomous noise, the phenomenon of noise-induced stability in models similar to Eq. (2) has already been examined in [16], where analogous results to Fig. 2 are presented and a comprehensive physical interpretation of the effect is given. So our result exposed in Fig. 2 shows that the phenomenon of the noise-correlation-time-induced stability is robust enough to survive a modification of the noise as well as of the potential profile.

It is of interest to examine the behavior of the exact expression of W_0 [Eq. (14)] versus temperature. In Fig. 3 we have plotted the occupancy probabilities W_0 and W_1 as functions of the dimensionless temperature D for several values of the correlation time τ_c. At intermediate values of τ_c [see Fig. 3(a)] for increasing D, the probability W_0 starts from the value $W_0\approx 1$ and decreases to a minimum. Next it grows to the local maximum and decreases to the other minimum.

Finally, at high temperatures, it grows to the value d.

The interesting peculiarity of Fig. 3 is that there are three temperature regimes where an enhancement of the occupancy of the metastable state can be recognized. At high temperatures the effect is trivial. In this case the Brownian particles "fail to see" the structure of the potential profile and move as in a simple rectangular potential well. For low values of the temperature the effect of enhancement is very pronounced, i.e., nearly all particles are concentrated in the left potential well which has higher energy most of the time. This result is in accordance with the phenomenon of noise-correlation-time-induced stability (see Fig. 2 and [16]). In the case of moderate values of the temperature a resonance-like behavior is observed—enhancement of stability also occurs in a finite interval of the temperature, where the lowest depth of the potential wells is comparable with the thermal energy of the particle.

However, for larger correlation times, the first minimum and the local maximum disappear, and two plateaus occur at moderate temperatures [see Fig. 3(c)]. Notably, in the regions of plateaus, where W_0 is nearly constant over a finite range of temperature values, the occupancy probability of the left potential well is relatively high ($W_0 \approx \frac{1}{2}$ for the first plateau).

To throw some light on the physics of the above-mentioned effects, we shall now study some physical approximations for a simpler description of the dynamics of the system (2) on various time scales.

There are several important time scales in our system: six mean first passage times $T_n^{(i)} = (K_n^{(i)})^{-1}$ for the two minima of $V_n(x)$, $i=0$ and $i=1$ corresponding to the left and right minima, respectively; the intrawell relaxation times for $V_n(x)$; and the correlation time $\tau_c = 1/\nu$ for the fluctuations of the potential. Below we will consider values of temperature that are sufficiently low, $D < \min(\Delta V_n^{(i)})$, to allow the following rate separations:

(i) $\min(\Delta V_n^{(i)}/L_i^2) > \nu > \max(K_n^{(i)})$;
(ii) $K_1^{(0)} < K_2^{(0)} < \nu < K_3^{(0)}$, $K_3^{(1)} < K_2^{(1)} < \nu < K_1^{(1)}$;
(iii) $K_1^{(0)} < \nu < K_2^{(0)} < K_3^{(0)}$, $K_3^{(1)} < K_2^{(1)} < \nu < K_1^{(1)}$;
(iv) $\nu < K_1^{(0)}$. The rates $K_n^{(i)}$ can be approximated to the Kramers rates [26],

$$K_n^{(i)} = \frac{(\Delta V_n^{(i)})^2}{D(L_i)^2} e^{-\Delta V_n^{(i)}/D}, \tag{15}$$

where $L_0 = d$, $L_1 = 1-d$, and $\Delta V_n^{(i)}$ are the depths of the net potential wells $V_n^{(i)}$. Note that, for a fixed ν, all four cases can be subsequently approached by varying the thermal noise intensity D only. If τ_c is long enough compared to the intrawell deterministic relaxation times of the net potential $V_n(x)$, i.e., $\tau_c \gg \max(L_i^2/\Delta V_n^{(i)})$, the condition $D < \min(\Delta V_n^{(i)})$ also guarantees a sharp occupancy distribution in the minima of net potentials. In this situation the probability flux from the left (right) potential well to the right (left) one is given by $W_0/T_0(-W_1/T_1)$, where T_0 and T_1 are the mean first passage times from the left and right potential wells, respectively. In the stationary case, the total probability flux between the left and right potential wells must vanish, implying

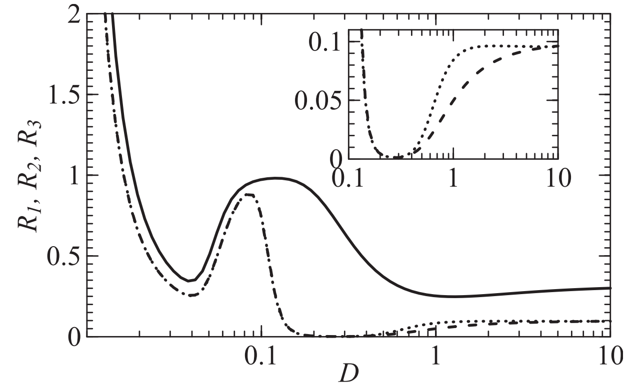

FIG. 4. Ratios $R_1 \equiv W_0/W_1$, $R_2 \equiv T_0/T_1$, $R_3 \equiv \tilde{T}_0/\tilde{T}_1$ versus the dimensionless temperature D in the case of dichotomous noise, $q = 1/2$. Solid line: the function $R_1(D)$ computed from Eq. (14). Dashed line: the function $R_2(D)$ computed from Eqs. (A2) and (A3). Dotted line: the function $R_3(D)$ computed by means of the kinetic approximation Eq. (18). Parameter values: $a=2$, $\varepsilon=1$, $d=0.24$, $\nu=10^{-3}$. The inset depicts the region of significant difference between R_2 and R_3.

$$\frac{W_0}{W_1} = \frac{T_0}{T_1}. \tag{16}$$

A comparison of the above results with calculations for the mean first passage time shows that the highly nonlinear behavior of W_0 and W_1 at low and moderate temperatures is related to resonant activation [6,26–29]. First, we consider the regime (i), $\nu > \max(K_n^{(i)})$, which corresponds to temperatures that are lower than the temperature D_m corresponding to the local maximum of W_0. The MFPT T_i depends on the initial occupancy probabilities $\rho_n^{(i)}$ of the net potential wells $V_n^{(i)}$. More precisely, $\rho_n^{(i)}$ denotes the conditional probability that, if the particle is in the potential well (i), then it is in the potential configuration V_n. In the case of $\nu > \max(K_n^{(i)})$, barrier fluctuations are much faster than barrier passage; there is, between two crossings over the barrier, enough time for intrawell relaxation of the particle probability distribution and the particle spends most of the time in the stationary probability distribution corresponding to the stationary fluctuations of the potential profile.

Considering that the stationary probabilities for the occurrence of the potential configurations V_1, V_2, and V_3 are q, $1-2q$, and q, respectively, we obtain

$$\rho_1^{(i)} = q \quad \rho_2^{(i)} = 1 - 2q, \quad \rho_3^{(i)} = q. \tag{17}$$

With the initial conditions (17), the exact formulas for T_0 and T_1, being complex and cumbersome, will be presented in the Appendix [Eqs. (A2) and (A3)].

In Fig. 4 the ratios W_0/W_1 and T_0/T_1 are compared for dichotomous noise, $q = \frac{1}{2}$. When comparing the whole curves of $T_0(D)/T_1(D)$ and $W_0(D)/W_1(D)$, one can distinguish two regions. For $D < D_m = 0.1$ the curve $T_0(D)/T_1(D)$ follows $W_0(D)/W_1(D)$ quite well [30]. Thus, in this region formula (16) with Eqs. (17) applies, and the occupation process of the potential wells can be characterized by the mean first passage times over the potential barrier. For higher values of D, however, a significant discrepancy occurs. This reflects the fact that for $D>D_m$ the time scale considered, $\nu>\max(K_n^{(i)})$, is no longer applicable [Eq. (17) is no longer valid]. Furthermore, the dotted line in Fig. 4 shows clearly that for $D<D_m$ the kinetic approximation is valid [27],

$$T_i \approx \tilde{T}_i = \frac{2\nu + K_1^{(i)} + K_3^{(i)}}{2K_1^{(i)}K_3^{(i)} + \nu(K_1^{(i)} + K_3^{(i)})}, \quad i=0,1, \tag{18}$$

and captures, with Eq. (15), the qualitative behavior of $W_0(D)/W_1(D)$ accurately. Moreover, for a fixed ν and sufficiently low temperatures it follows from Eq. (15) that the inequalities $\nu \gg K_3^{(0)} \gg K_2^{(0)} \gg K_1^{(0)}$ and $\nu \gg K_1^{(1)} \gg K_2^{(1)} \gg K_3^{(1)}$ are very strong, and from Eq. (18) we can deduce a simpler expression, $T_0 \approx 2/K_3^{(0)}$, $T_1 \approx 2/K_1^{(1)}$, i.e., the escape process is determined by passage over the lowest of the three potential barriers. It should be noted that in the case of trichotomous noise, $q<\frac{1}{2}$, an analogous expression is also valid:

$$T_0 \approx \frac{1}{qK_3^{(0)}}, \quad T_1 \approx \frac{1}{qK_1^{(1)}}. \tag{19}$$

With Eq. (19) we then obtain from Eq. (16) the small-D approximation

$$\frac{W_0}{W_1} \approx \left(\frac{d\Delta V_1^{(1)}}{(1-d)\Delta V_3^{(0)}}\right)^2 e^{(\Delta V_3^{(0)} - \Delta V_1^{(1)})/D}. \tag{20}$$

As $\Delta V_3^{(0)} > \Delta V_1^{(1)}$, this result implies $W_0 \gg W_1$ for sufficiently small temperatures [$D<D_{\min}$, where $D_{\min}$ corresponds to the first local minimum of $W_0(D)$]. For increasing values of D the occupancy probability of the left potential well W_0 decreases, attaining $W_0=W_1=\frac{1}{2}$ at $D=D_{\rm cr}$. It is remarkable that the approximate Eq. (20) is acceptable also for the estimation of the "critical" temperature $D_{\rm cr}$ within a very broad range of noise correlation times τ_c, i.e., in this range of τ_c the temperature $D_{\rm cr}$ depends only on the potential profile but not on the noise correlation time. Actually this interval of the switching rate values coincides with the condition for the regime (i):

$$\max(K_n^{(i)}) < \nu < \min\left(\frac{\Delta V_n^{(i)}}{L_i^2}\right).$$

A comparison of these inequalities with Eqs. (15) and (20) in the case of the potential parameters displayed in Fig. 3 shows that Eq. (20) is acceptable for the estimation of $D_{\rm cr}$, if $1>\nu>10^{-12}$. This result is in accordance with Fig. 3. For example, in Fig. 3 the actual value of $D_{\rm cr}$ is 0.018, exceeding only slightly the value $D_{\rm cr}\approx 0.016$ that can be found from Eq. (20). We emphasize that if the time scale considered, i.e., the regime (i), is not applicable at $D=D_{\rm cr}$, the dependence of $D_{\rm cr}$ on ν is remarkable (cf. Fig. 5). The fact that the small-D approximation (20) is applicable in a broad range of switching rate values demonstrates that the noise-induced stability in the low-D region is related to resonant activation [16,26,27]. In particular, a general feature of the resonant activation phenomenon for linear ramp, which is similar to our situation, is that with increasing barrier height (or decreasing temperature) a long flat region develops around the

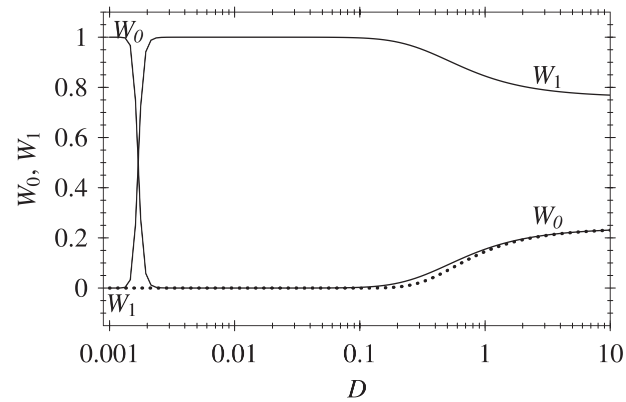

FIG. 5. Dependence of the probabilities W_0 and W_1 on the dimensionless temperature D [Eq. (14)] in the case of fast fluctuations, $\nu=40$. Parameter values: $a=2$, $\varepsilon=1$, $d=0.24$, $q=0.45$. The dotted line depicts the function $W_0(D)$ in the case of a nonfluctuating average potential $V_2(x)$.

resonant switching rate $\nu_{\rm res}$ ($\nu_{\rm res}$ corresponds to the minimum of the MFPT versus ν) [26,27].

Regime (ii): $K_1^{(0)} < K_2^{(0)} < \nu < K_3^{(0)}$, $K_3^{(1)} < K_2^{(1)} < \nu < K_1^{(1)}$. In this case, when the above inequalities are sufficiently strong and if, in addition, $\nu \ll \min(\Delta V_n^{(i)}/L_i^2)$, the following approximations are valid:

$$T_0 \approx T_1 \approx \frac{1}{q\nu} = \frac{1}{q}\tau_c. \tag{21}$$

A surprising observation about this result is that the mean first passage times T_0, T_1, and also the ratio $W_0/W_1 \approx T_0/T_1 \approx 1$, do not depend on the white noise intensity D, i.e., there exists a finite interval of values of temperature where both (left and right) potential wells are equally occupied [cf. Fig. 3(c)].

Let us now consider the derivation of an approximate Eq. (21) for the MFPT T_0 (the derivation of T_1 is analogous to that). For the regime (ii), the particle locked in the noise state $n=1$ at the right net potential minimum (see Fig. 1) will move, at the initial time $t=0$, to the left net potential minimum $V_1^{(0)}(0)$. As the inequalities presented above are sufficiently strong, the particle can escape over the potential barrier back to the right potential minimum only in the noise state $z_3=-a$. In this state the depth of the left net potential well is small, and the corresponding Kramers time is much shorter than the noise correlation time $\tau=1/\nu$. In the case of trichotomous fluctuations $Z(t)$, the probability $\bar{W}(t)$ that in a certain time interval $(0,\ t)$ transition to the noise state $z_3=-a$ does not occur is given by $\bar{W}(t)=\exp(-q\nu t)$ [5,24]. The probability that the transition to $z_3=-a$ occurs within the time interval $(t,t+dt)$ is $\nu q\, dt$. Consequently, the MFPT from the left potential well to the right one is approximately given by $T_0 \approx q\nu\int_0^\infty te^{-q\nu t}dt = 1/(q\nu)$. Thus, we have obtained an earlier result, namely, Eq. (21).

Regime (iii): $K_1^{(0)} < \nu < K_2^{(0)} < K_3^{(0)}$, $K_3^{(1)} < K_2^{(1)} < \nu < K_1^{(1)}$. Now, the MFPT for the right potential well is the same as in the case of the regime (ii), $T_1 \approx 1/(q\nu)$. The MFPT for the left potential well can be expressed as $T_0 \approx 1/[(1-q)\nu]$. Using Eq. (16) one obtains the occupancy probabilities

$$W_0 \approx q, \quad W_1 \approx 1-q. \tag{22}$$

Thus, in this regime the probabilities W_0 and W_1 do not depend on the temperature D either. The only distinction from the case of (ii) is the dependence on the noise kurtosis $\varphi=[1/(2q)-3]$, which reflects the three-level structure of the noise. A comparison with Fig. 3(c) shows that for the temperature interval considered our approximation (22) captures the exact results extremely well. The physical mechanism appropriate to generate the result $T_0 \approx 1/[(1-q)\nu]$ is analogous to those considered in the case of (ii). The key factor is that a particle in the left net potential well $V_1^{(0)}$ is not able, before switching to the state $n=2$ or 3, to pass the potential barrier by a thermally activated escape. Hence now the probability $\bar{W}(t)=\exp[-(1-q)\nu t]$ and the probability that the transition $z_1=a \to z_2=0$ or $z_1=a \to z_3=-a$ occurs within the time interval $(t,t+dt)$ is $(1-q)\nu dt$. So the leading contribution to the MFPT T_0 in this regime is $T_0 \approx (1-q)\nu\int_0^\infty t e^{-(1-q)\nu t}dt = 1/[(1-q)\nu]$.

Regime (iv): $\nu < K_1^{(0)}$. In this case, for increasing values of D, the occupancy probability W_0 first decreases to a minimum and next grows to the value d (see Fig. 3). Such a behavior of W_0 can be intuitively understood. The more we increase the temperature, the more jumping events over the barrier of the net potential V_1 from left to right can take place during the noise correlation time. As a result the MFPT for the left potential well must decrease. As long as the temperature is such that $K_2^{(1)} < \nu < K_1^{(0)}$, the MFPT for the right potential well is approximately constant: $T_1 \sim 1/(q\nu)$. Consequently, the occupation of the left potential well must decrease with the growth of D. As we enter the high-temperature area, $\nu < K_3^{(1)}$, the influence of temperature is twofold. First, during the noise correlation time the particles are able to pass all net potential barriers in both directions and, second, the distributions of particles within the potential wells tend to become more uniform, as the temperature grows. For very high temperatures, $D \gg \Delta V_3^{(1)}$, the Brownian particles "fail to see" the structure of the potential profile and move as in a simple rectangular potential well. Hence, the probability W_0 saturates at the value d as D increases. This result is due to the uniform distribution of Brownian particles as well as to the fact that the width of the left potential well is d.

Finally, we will briefly consider the behavior of the probability W_0 in the high-frequency regime, $\nu > \min(\Delta V_n^{(i)}/L_i^2)$. A general feature of our solution is that with an increasing switching rate ν the resonancelike phenomenon, i.e., the local maximum [see Figs. 3(a) and 3(b)], becomes less and less sharp and disappears at $\nu \sim \max(\Delta V_n^{(i)}/L_i^2)$. For large values of the switching rate ν, two characteristic regions can be discerned for the temperature D (see Fig. 5): first, the region of low intrawell diffusion levels $D\nu < \min[(\Delta V_n^{(i)}/L_i)^2]$, for which the characteristic distance of intrawell thermal diffusion $\sqrt{D\tau_c}$ is much smaller than the typical deterministic distances of the driven particles during the noise correlation time $\tau_c=1/\nu$, and, second, the regime $D\nu \gg \min[(\Delta V_n^{(i)}/L_i)^2]$, where thermal diffusion dominates. In the regime of low diffusion the behavior of W_0 is similar to that considered by Eq. (20), but the critical temperature D_{cr}, at which $W_0=W_1=\frac{1}{2}$, decreases as ν increases. Note that D_{cr} becomes zero at the limit $\nu \to \infty$. In this case, the response of W_0 to small variations of temperature is extremely pronounced, leading to an infinite derivative dW_0/dD at $D=0$. In the region of strong diffusion, the Brownian particle is subject to the average potential $V_2(x)$ in the case of fast fluctuations. Hence, in this regime the occupancy probability W_0 depends on temperature in the same way as in the case of the nonfluctuating potential $V_2(x)$, i.e., with increasing temperature W_0 increases monotonically up to the value d (see Fig. 5).

IV. CONCLUSIONS

In the present work, we analyze the behavior of one-dimensional overdamped Brownian motion in a sawtoothlike asymmetric bistable potential driven by a trichotomous noise and an additive thermal noise. Using the corresponding master equation, we obtained an exact expression for the occupancy probability of the metastable state, and demonstrated the phenomenon of noise-induced stability. Our major result is the effect of multiply enhanced stability of a metastable state versus temperature. Notably, enhancement of the stability also occurs at moderate temperatures, i.e., where the temperature D is such that the lowest barrier height of the system is just a few D, which is relevant for cell biology [31]. In the case of dichotomous noise, which is a special case of trichotomous noise, a qualitatively similar model has been studied in [16]. However, neither the phenomenon of multiply temperature-enhanced stability nor the existence of a corresponding resonancelike peak versus temperature at moderate values of D has been recognized or discussed before in the context of the stability analysis of such simple models with a double-well potential. Furthermore, for certain system parameters, the occupancy of the metastable potential well as a function of the temperature has two relatively large plateaus at moderate temperatures, i.e., in this system parameter region thermal noise is effectively suppressed. This noise suppression appears when the escape process strongly correlates with potential fluctuations [see Eq. (21)].

Comparison of the exact results with the approximations considered in Sec. III suggests that interesting phenomena, such as noise-induced stability, resonancelike enhancement of stability at moderate temperatures, and the existence of finite temperature regions (plateaus) where the occupancy of the metastable state is insensitive to temperature, are quite robust to variation of the kurtosis and the noise correlation time (over a broad range) as well as the potential profile. Note that, in a more general case, if the potential is smooth, in Eq. (15) for the Kramers rates $K_n^{(i)}$ only a modification of the prefactor is necessary, while the exponent, which mainly determines the dependence of $K_n^{(i)}$ on temperature, depends on the depth of the potential well. The key factors for the appearance of the above-mentioned effects are that most of the depths $\Delta V_n^{(i)}$ of the net potential wells are sufficiently distinct so as to lead to situations where the rates $K_n^{(i)}$ allow consideration of well-defined separations of time scales, and that the lowest barrier height for the metastable state (left potential well, see Fig. 1), $\Delta V_3^{(0)}$, slightly exceeds the lowest barrier height for the right potential well, $\Delta V_1^{(1)}$. The major advantage of the reported phenomena is that the control parameter is temperature, which can easily be varied in experiments.

Our exact analytical results concerning the enhancement of stability of a metastable state in a fluctuating bistable potential can be a good starting point for investigations of more realistic systems. Here, we briefly mention three possible directions. First, it would be interesting to investigate the behavior of W_0 on continuous transformation of the piecewise linear potential into a smooth one. Second, in the dynamics of soft-spin Ising magnets, the spin projections can be modeled as different states of trichotomous or dichotomous noise. We believe that the model discussed in this paper can be expanded, along the lines described in [21], to one that is suitable for studying metastability in a two-dimensional Ising model with dynamic impurities. Finally, our paper is restricted to the case of a well-defined potential flipping rate determined by the noise correlation time. However, in many physical systems, fluctuations have power-law correlations (a well-defined noise correlation time is absent). Thus, it is important to investigate, by numerical simulations, the occurrence of the resonant phenomena described in this paper with those strongly correlated fluctuations.

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APPENDIX: FORMULAS FOR THE MEAN FIRST PASSAGE TIME

Using the standard methods described in [32], from the backward equation for the master equation (5), the equation for the MFPT can be deduced:

$$(h_i - z_n)\frac{\partial}{\partial x}T_i(x,z_n) + D\frac{\partial^2}{\partial x^2}T_i(x,z_n) - \nu T_i(x,z_n) + \nu T_i(x) = -1, \tag{A1}$$

where $i=0,1$, $n=1,2,3$, and

$$T_i(x) = \sum_{n=1}^{3} \rho_n^{(i)} T_i(x,z_n)$$

with the initial probabilities $\rho_n^{(i)}$ given by Eq. (17). After solving Eq. (A1) with the boundary conditions

$$T_i(d,z_n) = 0, \quad \frac{d}{dx}T_i(x,z_n)|_{x=L_i} = 0,$$

where $L_0=0$ and $L_1=1$, a straightforward calculation gives for the mean first passage times $T_0 \equiv T_0(0)$ and $T_1 \equiv T_1(1)$

$$T_i = \gamma_i + \sum_{j=1}^{5} C_{ij}. \tag{A2}$$

Here the constants γ_i and C_{ij} are determined by a nonhomogeneous system of six linear algebraic equations:

$$\sum_{j=1}^{5} C_{ij}\lambda_{ij}A_{nij} = \frac{D}{h_i},$$

$$\gamma_i + \sum_{j=1}^{5} C_{ij}A_{nij}e^{\lambda_{ij}l_i/D} = \frac{l_i}{h_i} - \frac{z_n}{\nu h_i}, \tag{A3}$$

where $l_0=d$, $l_1=-(1-d)$, and the quantities A_{nij}, λ_{ij} are the same as in Eq. (7). Hence, the problem is solved and the evaluation of the MFPT can be handled by linear algebra.

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