Brownian motion of a classical harmonic oscillator in a magnetic field

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In this paper, the stochastic diffusion process of a charged classical harmonic oscillator in a constant magnetic field is exactly described through the analytical solution of the associated Langevin equation. Due to the presence of the magnetic field, stochastic diffusion takes place across and along the magnetic field. Along the magnetic field, the Brownian motion is exactly the same as that of the ordinary one-dimensional classical harmonic oscillator, which was very well described in Chandrasekhar's celebrated paper [Rev. Mod. Phys. **15**, 1 (1943)]. Across the magnetic field, the stochastic process takes place on a plane, perpendicular to the magnetic field. For internally Gaussian white noise, this planar-diffusion process is exactly described through the first two moments of the positions and velocities and their corresponding cross correlations. In the absence of the magnetic field, our analytical results are the same as those calculated by Chandrasekhar for the ordinary harmonic oscillator. The stochastic planar diffusion is also well characterized in the overdamped approximation, through the solutions of the Langevin equation.

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I. INTRODUCTION

In Chandrasekhar's celebrated paper [1], the study of Brownian motion of a free particle was very nicely described through a mathematical treatment relying upon both the Langevin and Fokker-Planck (FP) equations, and some general lines were suggested toward solving the same problem in the presence of an external field of force. The proposal has been extended to those situations for which the Brownian particle is considered to be electrically charged and also under the influence of an external magnetic field, in different situations [2–8]. In particular, in Ref. [8], the problem of stochastic diffusion of heavy ions in the presence of an electromagnetic field was solved through an alternative method of solution of the FP equation. This method was extended to solve the phase-space FP equation, describing the stochastic diffusion of heavy ions in the presence of an external electric field only [9]. Also in Chandrasekhar's paper, the solution to the problem of a one-dimensional classical harmonic oscillator describing a Brownian motion was given through the solutions of the Langevin and FP equations. After this work on the stochastic harmonic oscillator, many related topics have been reported in the literature; for example, we can mention that the study of a harmonic oscillator with random frequency is a subject that has been extensively investigated in different fields, including physics [10,11], biology [12], economics [13], and so on; the study of a harmonic oscillator with random damping [14] was also used for the problem of water waves influenced by a turbulent wind field [15]; the study of an underdamped harmonic oscillator for additive and multiplicative noise has been described in Ref. [16], showing the phenomenon of stochastic resonance [14,17,18]. Another branch that has been developed recently is that related to the so-called nonequilibrium work relations [19–23], also termed fluctuation theorems. Very recently, in Ref. [21], Jayannavar and Sahoo reported the analytical calculation of the distributions of work expended by a charged particle in the presence of a magnetic field in a two-dimensional harmonic well. This physical model is used to verify the Jarzynski equality (JE) [22], which relates nonequilibrium quantities with equilibrium free energies, in two different cases: in the first case (i), the center of the harmonic trap is dragged with a uniform velocity whereas in case (ii) it is subjected to an ac force. In both cases, the particle is moving in the presence of a time-dependent harmonic potential and the analytical results of [22] complement the Bohr–van Leeuwen theorem on the absence of diamagnetism in a classical system. An important point we would like to note here is that the JE is calculated through the explicit solution of the Langevin equation in the overdamped regime, not through the complete solution of this equation. This is so because in the large-friction limit or diffusive regime the fluctuations are dominant. However, as far as we know, the explicit and complete solution of the stochastic diffusion of a charged harmonic oscillator in the presence of an external constant magnetic field is a problem that has not been reported yet in the literature, and our purpose in this work is to give the complete solution of such a problem in a three-dimensional space-dependent harmonic potential well. Our proposal can be extended to a situation for which the charged particle is in a time-dependent harmonic potential well, or used for other possible applications on single nanosized systems in physical environments where fluctuations play a fundamental role [24–28]. As usual, in our work we assume that the magnetic field is allowed to point along the z axis of the Cartesian reference frame, and as a result two independent processes are taking place. One takes place along the z axis or along the magnetic field but is not affected by this field, and the particle is in a one-dimensional harmonic well. This case is exactly the same as that described by the ordinary one-dimensional harmonic oscillator describing a Brownian motion [1]. The other one takes place on the x-y plane across or

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perpendicular to the magnetic field and the particle is in a two-dimensional harmonic well. In this case, the Langevin equations associated with the x and y axes form a coupled pair of equations describing the planar-diffusion process, which is an interesting but not an easy problem to describe. To solve this planar harmonic oscillator's Brownian motion analytically, we use Landau's strategy [29], which relies upon a transformation, on the complex plane, of the Langevin equation. On the complex plane, the resulting complex Langevin equation strongly resembles that of the free harmonic oscillator and, therefore, its explicit solution is easily calculated using Chandrasekhar's ideas. Following the same strategy, we also study the overdamped limiting case corresponding to the large-friction limit. This case is easier than the preceding description, because the complex linear Langevin equation is not already of second order but linear and of first order. In fact, in the same large-friction limit, the same Landau strategy is used by Jayannavar and Sahoo [21] to solve the problem in a two-dimensional time-dependent harmonic well, and applied to calculate the JE. This work is organized as follows. In Sec. II, we establish the Langevin equation for the two independent process and solve only that associated with the planar-diffusion process in a two-dimensional harmonic potential well. In the absence of a magnetic field, our analytical results are compared with those calculated by Chandrasekhar. In Sec. III, we study the problem in the overdamped approximation and the conclusions are given in Sec. IV. Finally, we introduce three appendixes for the explicit calculations in our work.

II. THE LANGEVIN EQUATION OF A CLASSICAL HARMONIC OSCILLATOR IN A CONSTANT MAGNETIC FIELD

The Langevin equation of a charged particle embedded in a fluid describing a Brownian motion in a space-dependent potential $U(\mathbf{r})$ and in the presence of a constant magnetic field, can be written for the velocity vector $\mathbf{u}$ as

$$\dot{\mathbf{u}} = -\beta\mathbf{u} + \frac{q}{mc}\mathbf{u}\times\mathbf{B} - \frac{1}{m}\text{grad}\ U(\mathbf{r}) + \mathbf{A}(t), \tag{1}$$

where $\beta>0$ is the friction constant, q is the charge of the particle and m its mass, grad U is the gradient operator of potential U, and $\mathbf{A}(t)$ is the fluctuating force per unit mass, which satisfies the properties of Gaussian white noise with zero mean value $\langle A_i(t)\rangle=0$ and a correlation function given by

$$\langle A_i(t)A_j(t')\rangle = 2\lambda\,\delta_{ij}\delta(t-t'). \tag{2}$$

λ is a constant that measures the noise intensity and, according to the fluctuation-dissipation theorem, is related to the friction constant by $\lambda=\beta k_B T/m$ with k_B the Boltzmann constant and T the temperature of the surrounding medium. The overdot in Eq. (1) denotes the derivative with respect to time. Here we use $A_1\equiv A_x$, $A_2\equiv A_y$, and $A_3\equiv A_z$. If we assume a three-dimensional harmonic potential well $U(\mathbf{r})=k(x^2+y^2+z^2)/2$ with k a constant, and the magnetic field for simplicity pointing along the z axis of the Cartesian reference frame, that is, $\mathbf{B}=(0,0,B)$ with B a constant, then Eq. (1) can be described by means of two independent processes. One is described on the x-y plane perpendicular to the magnetic field and the charged particle is in a two-dimensional harmonic well $U(x,y)=k(x^2+y^2)/2$; the other is along the z axis or along the magnetic field in a one-dimensional harmonic well $U(z)=kz^2/2$. In these cases, Eq. (1) can be written in terms of its components as follows:

$$\ddot{x} + \beta\dot{x} + \omega^2 x - \Omega\dot{y} = A_x(t), \tag{3}$$

$$\ddot{y} + \beta\dot{y} + \omega^2 y + \Omega\dot{x} = A_y(t), \tag{4}$$

$$\ddot{z} + \beta\dot{z} + \omega^2 z = A_z(t), \tag{5}$$

where $\Omega=qB/mc$ is the Larmor frequency and $\omega^2=k/m$ is the characteristic frequency of the oscillator. As we can see, Eq. (5) is exactly the same as that of the ordinary classical harmonic oscillator describing a Brownian motion, which has already been solved by Chandrasekhar [1]. Thus we will focus on the two stochastic differential equations [Eqs. (3) and (4)], which describe the diffusion process on the x-y plane perpendicular to the magnetic field in a two-dimensional harmonic well. Due to the Gaussian characteristics of the processes here considered, there exist two ways of describing this planar-diffusion process explicitly. One is through the calculation of the first two moments of the variables x, y, u_x, and u_y, which can be achieved through the explicit solution of the Langevin equations (3) and (4). The other one is through the explicit solution of the Fokker-Planck equation associated with those Langevin equations. Here we follow the proposal of solving those Langevin equations and as a consequence the calculation of the moments of the relevant variables. Once those moments are calculated, all the transition probability densities (TPDs) $P(x,t|x_0)$,

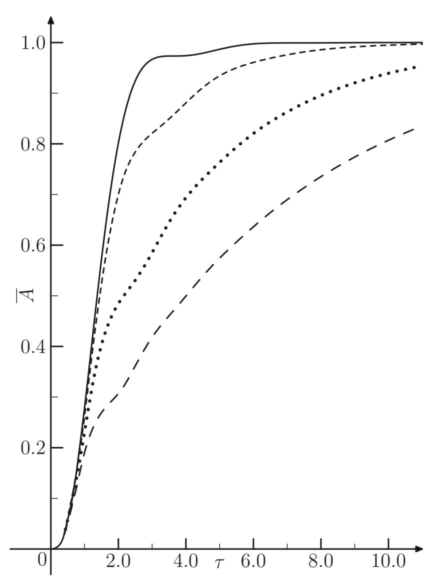

FIG. 1. Reduced correlation $\bar{A}\equiv\sigma^{\star}_{xx}\equiv\sigma_{xx}/(kT/m\omega^2)$ vs the reduced time $\tau\equiv\beta t$ for $\omega/\beta=1$ and different values of Ω/β. Solid line, $\Omega=0$; dashed line, $\Omega/\beta=1$; dotted line, $\Omega/\beta=2$; long-dashed line, $\Omega/\beta=3$.

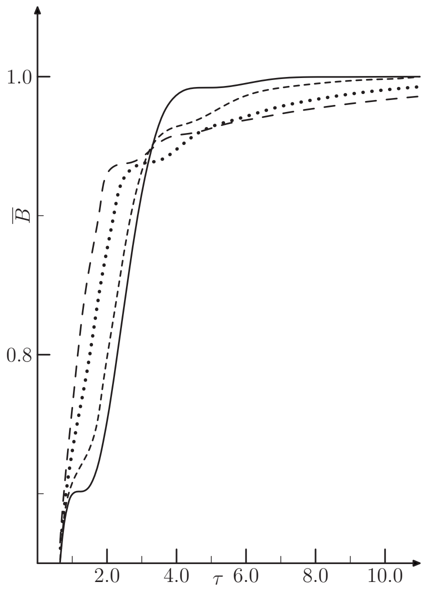

FIG. 2. Reduced correlation $\bar{B} \equiv \sigma^{\star}_{u_x u_x} \equiv \sigma_{u_x u_x}/(kT/m)$ vs the reduced time $\tau \equiv \beta t$ for $\omega/\beta = 1$ and different values of Ω/β. Solid line, $\Omega = 0$; dashed line, $\Omega/\beta = 1$; dotted line, $\Omega/\beta = 2$; long-dashed line, $\Omega/\beta = 3$.

$P(y,t|y_0)$, $P(u_x,t|x_0,u_{x_0})$, $P(u_y,t|y_0,u_{y_0})$, $P(x,u_x,t|x_0,u_{x_0})$, $P(y,u_y,t|y_0,u_{y_0})$, $P(x,u_y,t|x_0,u_{y_0})$, $P(y,u_x,t|y_0,u_{x_0})$, and $P(x,y,u_x,u_y,t|x_0,y_0,u_{x_0},u_{y_0})$ can readily be calculated from the general Gaussian distribution [30,31]

$$P(\{\xi\},t|\{\xi\}_0) = \frac{1}{(2\pi)^{n/2}(\text{Det } \sigma_{ij})^{1/2}} \times \exp\left(-\frac{1}{2}\sum_{i,j}(\sigma^{-1})_{ij}(\xi_i - \langle \xi_i \rangle)(\xi_j - \langle \xi_j \rangle)\right), \tag{6}$$

where $P(\{\xi\},0|\{\xi\}_0) = \delta(\{\xi\} - \{\xi\}_0)$ represents the initial condition, $\{\xi\} = (\xi_1, \ldots, \xi_n)$, and the variance $\sigma_{ij} = (\xi_i - \langle \xi_i \rangle)(\xi_j - \langle \xi_j \rangle)$. In our case $\xi_i = x, y, u_x, u_y$. The two differential equations (3) and (4) are clearly coupled and seem to be somewhat complicated to solve analytically. To avoid this mathematical difficulty, we use an alternative method of solution to the one proposed initially by Landau and Lifshitz [29], by solving the simple classical harmonic oscillator in the presence of a constant magnetic field. The proposal consists in mapping Eqs. (3) and (4) on the complex plane, by defining the complex function

$$\bar{\rho} = x + iy. \tag{7}$$

In this case, Eqs. (3) and (4) can be written as

$$\ddot{\bar{\rho}} + \bar{\beta}\dot{\bar{\rho}} + \omega^2 \bar{\rho} = \bar{\mathcal{A}}(t), \tag{8}$$

where $\bar{\beta} = \beta + i\Omega$, $\bar{\mathcal{A}}(t) = A_x(t) + iA_y(t)$. From now on, we will write any complex number and any complex function with an overbar, except for the correlations which appear in Figs. 1–6. As we can see, on the complex plane, Eq. (8) has a structure very similar to that of the ordinary one-dimensional harmonic oscillator describing a Brownian motion and,

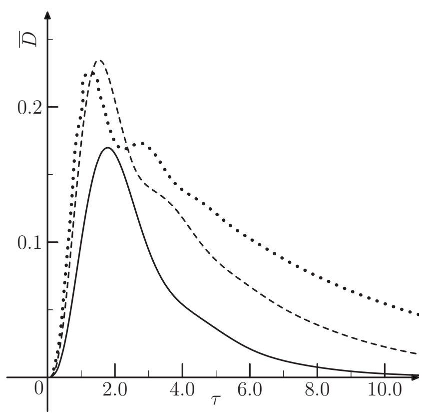

FIG. 4. Reduced correlation $\bar{D} \equiv \sigma^{\star}_{yu_x} \equiv \sigma_{yu_x}/(kT/m\omega)$ vs the reduced time $\tau \equiv \beta t$ for $\omega/\beta = 1$ and different values of Ω/β. Solid line, $\Omega = 1$; dashed line, $\Omega/\beta = 2$; dotted line, $\Omega/\beta = 3$.

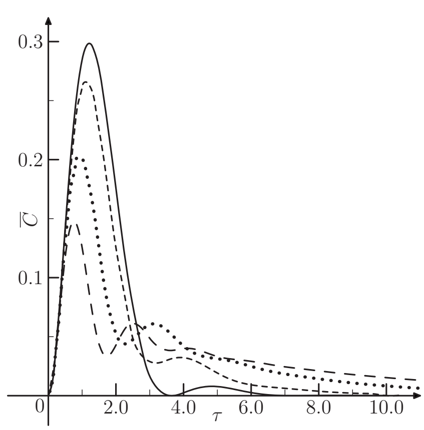

FIG. 3. Reduced correlation $\bar{C} \equiv \sigma^{\star}_{xu_x} \equiv \sigma_{xu_x}/(kT/m\omega)$ vs the reduced time $\tau \equiv \beta t$ for $\omega/\beta = 1$ and different values of Ω/β. Solid line, $\Omega = 0$; dashed line, $\Omega/\beta = 1$; dotted line, $\Omega/\beta = 2$; long-dashed line, $\Omega/\beta = 3$.

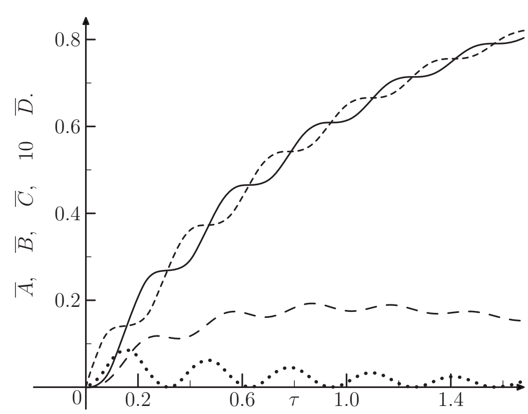

FIG. 5. Different reduced correlations (see Figs. 1–4 for their definitions) vs the reduced time $\tau \equiv \beta t$ for $\Omega/\beta = 1$ and $\omega/\beta = 10$. Solid line, $\bar{A} \equiv \sigma^{\star}_{xx}$; dashed line, $\bar{B} \equiv \sigma^{\star}_{u_x u_x}$; dotted line, $\bar{C} \equiv \sigma^{\star}_{xu_x}$; long-dashed line, $10\bar{D} \equiv 10 \times \sigma^{\star}_{yu_x}$.

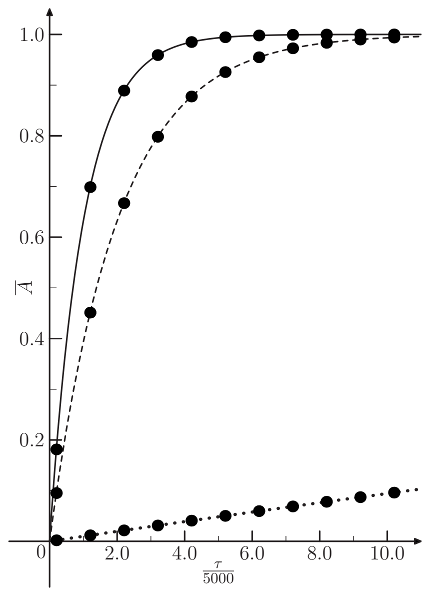

FIG. 6. Reduced correlation $\overline{A} \equiv \sigma_{xx}^{\star} \equiv \sigma_{xx}/(kT/m\omega^2)$ vs the reduced time $\tau \equiv \beta t$ divided by 5000 for $\omega/\beta = 0.01$ and different values of Ω/β. The solid line corresponds to the exact value given by Eq. (26) and $\Omega=0$; the dashed line is the exact result for $\Omega=1$; the dots the exact result for $\Omega=10$. The circles correspond to the overdamped approximation given by Eq. (50).

therefore, its solution is easy to calculate. In fact the solution of Eq. (8) is calculated in Appendix A, yielding the following result:

$$\bar{\rho}(t) = \frac{1}{\bar{\mu}_1 - \bar{\mu}_2}\left(e^{\bar{\mu}_1 t}\int_0^t e^{-\bar{\mu}_1 t'}\overline{\mathcal{A}}(t')dt' - e^{\bar{\mu}_2 t}\int_0^t e^{-\bar{\mu}_2 t'}\overline{\mathcal{A}}(t')dt'\right) + \bar{a}_{10}e^{\bar{\mu}_1 t} + \bar{a}_{20}e^{\bar{\mu}_2 t}, \tag{9}$$

where

$$\bar{\mu}_1 = -\frac{\bar{\beta}}{2} + \frac{\bar{\beta}_1}{2}, \quad \bar{\mu}_2 = -\frac{\bar{\beta}}{2} - \frac{\bar{\beta}_1}{2}, \tag{10}$$

and

$$\bar{\beta}_1 \equiv \sqrt{\bar{\beta}^2 - 4\omega^2} = \sqrt{\beta^2 - \Omega^2 - 4\omega^2 + i2\beta\Omega}. \tag{11}$$

On the other hand, the complex velocity is defined as $\bar{u}_\rho \equiv \dot{\bar{\rho}} = u_x + iu_y$. Therefore

$$\bar{u}_\rho(t) = \frac{1}{\bar{\mu}_1 - \bar{\mu}_2}\left(\bar{\mu}_1 e^{\bar{\mu}_1 t}\int_0^t e^{-\bar{\mu}_1 t'}\overline{\mathcal{A}}(t')dt' - \bar{\mu}_2 e^{\bar{\mu}_2 t}\int_0^t e^{-\bar{\mu}_2 t'}\overline{\mathcal{A}}(t')dt'\right) + \bar{\mu}_1\bar{a}_{10}e^{\bar{\mu}_1 t} + \bar{\mu}_2\bar{a}_{20}e^{\bar{\mu}_2 t}. \tag{12}$$

The constants $\bar{a}_{10}$ and $\bar{a}_{20}$ can be calculated in terms of the initial conditions $\bar{\rho}(0) \equiv \bar{\rho}_0 = x_0 + iy_0$, $\bar{u}_\rho(0) \equiv \bar{u}_{\rho_0} = u_{x_0} + iu_{y_0}$ and are given by

$$\bar{a}_{10} = -\frac{(\bar{\rho}_0\bar{\mu}_2 - \bar{u}_{\rho_0})}{\bar{\mu}_1 - \bar{\mu}_2}, \quad \bar{a}_{20} = \frac{\bar{\rho}_0\bar{\mu}_1 - \bar{u}_{\rho_0}}{\bar{\mu}_1 - \bar{\mu}_2}, \tag{13}$$

and therefore the solution of Eq. (8) and its corresponding velocity can be written, respectively, as

$$\bar{\rho}(t) + \frac{1}{\bar{\mu}_1 - \bar{\mu}_2}[(\bar{\rho}_0\bar{\mu}_2 - \bar{u}_{\rho_0})e^{\bar{\mu}_1 t} - (\bar{\rho}_0\bar{\mu}_1 - \bar{u}_{\rho_0})e^{\bar{\mu}_2 t}] = \int_0^t \overline{\mathcal{A}}(t')\bar{\psi}(t')dt', \tag{14}$$

$$\bar{u}_\rho(t) + \frac{1}{\bar{\mu}_1 - \bar{\mu}_2}[\bar{\mu}_1(\bar{\rho}_0\bar{\mu}_2 - \bar{u}_{\rho_0})e^{\bar{\mu}_1 t} - \bar{\mu}_2(\bar{\rho}_0\bar{\mu}_1 - \bar{u}_{\rho_0})e^{\bar{\mu}_2 t}] = \int_0^t \overline{\mathcal{A}}(t')\bar{\phi}(t')dt', \tag{15}$$

where the complex functions $\bar{\psi}(t')$ and $\bar{\phi}(t')$ read as follows:

$$\bar{\psi}(t') = \frac{1}{\bar{\mu}_1 - \bar{\mu}_2}(e^{\bar{\mu}_1(t-t')} - e^{\bar{\mu}_2(t-t')}), \tag{16}$$

$$\bar{\phi}(t') = \frac{1}{\bar{\mu}_1 - \bar{\mu}_2}(\bar{\mu}_1 e^{\bar{\mu}_1(t-t')} - \bar{\mu}_2 e^{\bar{\mu}_2(t-t')}). \tag{17}$$

The solution for the complex conjugate $\bar{\rho}^*(t)$ and its corresponding velocity $\bar{u}_\rho^*$ are also calculated in a completely similar way in Appendix A. In order to analyze the meaning of the solutions given in Eqs. (14) and (15) in terms of the original coordinates x, y, u_x, and u_y, we must separate the real and imaginary parts of Eqs. (14) and (15), as well as the real and imaginary parts of the roots $\bar{\mu}_1$ and $\bar{\mu}_2$, in such a way that the complex number $\bar{\beta}_1$ can be written as

$$\bar{\beta}_1 = \pm(\alpha + i\delta), \tag{18}$$

where

$$\alpha = \frac{1}{\sqrt{2}}\sqrt{\sqrt{a^2+b^2}+a}, \quad \delta = \frac{1}{\sqrt{2}}\sqrt{\sqrt{a^2+b^2}-a}, \tag{19}$$

and the parameters $a = \beta^2 - \Omega^2 - 4\omega^2$ and $b = 2\beta\Omega$. We show in Appendix B the solution and the complete expressions for the averages x, y, u_x, and u_y for arbitrary initial conditions, and only as an example we will write here the results for the particular case when $x_0 = y_0 = u_{y_0} = 0$ and $u_{x_0} \neq 0$. In this case,

$$\langle x\rangle = e^{-\beta t/2}\{B_0[\sinh(\alpha t/2)\cos(\delta t/2)\cos(\Omega t/2) + \cosh(\alpha t/2)\sin(\delta t/2)\sin(\Omega t/2)] + D_0[\cosh(\alpha t/2)\sin(\delta t/2)\cos(\Omega t/2) - \sinh(\alpha t/2)\cos(\delta t/2)\sin(\Omega t/2)]\}, \tag{20}$$

$$\langle y\rangle = e^{-\beta t/2}\{B_0[\cosh(\alpha t/2)\sin(\delta t/2)\cos(\Omega t/2) - \sinh(\alpha t/2)\cos(\delta t/2)\sin(\Omega t/2)] - D_0[\sinh(\alpha t/2)\cos(\delta t/2)\cos(\Omega t/2) + \cosh(\alpha t/2)\sin(\delta t/2)\sin(\Omega t/2)]\}, \tag{21}$$

$$\langle u_x \rangle = e^{-\beta t/2} u_{x_0}[\cosh(\alpha t/2)\cos(\delta t/2)\cos(\Omega t/2) + \sinh(\alpha t/2)\sin(\delta t/2)\sin(\Omega t/2)] - e^{-\beta t/2}\{F_0[\sinh(\alpha t/2)\cos(\delta t/2)\cos(\Omega t/2) + \cosh(\alpha t/2)\sin(\delta t/2)\sin(\Omega t/2)] + H_0[\cosh(\alpha t/2)\sin(\delta t/2)\cos(\Omega t/2) - \sinh(\alpha t/2)\cos(\delta t/2)\sin(\Omega t/2)]\}, \quad (22)$$

$$\langle u_y \rangle = - e^{-\beta t/2} u_{x_0}[\cosh(\alpha t/2)\cos(\delta t/2)\sin(\Omega t/2) - \sinh(\alpha t/2)\sin(\delta t/2)\cos(\Omega t/2)] + e^{-\beta t/2}\{H_0[\sinh(\alpha t/2)\cos(\delta t/2)\cos(\Omega t/2) + \cosh(\alpha t/2)\sin(\delta t/2)\sin(\Omega t/2)] - F_0[\cosh(\alpha t/2)\sin(\delta t/2)\cos(\Omega t/2) - \sinh(\alpha t/2)\cos(\delta t/2)\sin(\Omega t/2)]\}, \quad (23)$$

where now

$$B_0 \equiv \frac{2u_{x_0}\alpha}{\alpha^2+\delta^2}, \quad D_0 \equiv \frac{2u_{x_0}\delta}{\alpha^2+\delta^2}, \quad (24)$$

$$F_0 \equiv \frac{u_{x_0}\alpha(\beta^2+\delta^2)}{\beta(\alpha^2+\delta^2)}, \quad H_0 \equiv \frac{u_{x_0}\delta(\beta^2-\alpha^2)}{\beta(\alpha^2+\delta^2)}. \quad (25)$$

Equations (20)–(23) represent the deterministic solution of the problem since the average of the noise vanishes. The quantity α is such that $0<\alpha<\beta$ and the leading terms in Eqs. (20)–(23) when $t\to\infty$ decrease exponentially with a relaxation time $2/(\beta-\alpha)$, in such a way that the presence of the magnetic field makes the relaxation slower than in the case where $B=0$. As a second step in the analysis of this problem we calculate the two-time auto- and cross correlation of the relevant variables. In Appendix B, we provide the explicit expressions for all the correlations at different times t_1 and t_2. For equal times $t_1=t_2=t$, Eqs. (B47)–(B52) can be straightforwardly reduced to

$$\sigma_{xx} = \frac{k_BT}{m\omega^2}\left\{1 - e^{-\beta t}\left[1 + \frac{\beta^2+\delta^2}{\alpha^2+\delta^2}\times\left(2\sinh^2(\alpha t/2) + \frac{\alpha}{\beta}\sinh(\alpha t)\right) + \frac{\beta^2-\alpha^2}{\alpha^2+\delta^2}\left(2\sin^2(\delta t/2) + \frac{\delta}{\beta}\sin(\delta t)\right)\right]\right\}, \quad (26)$$

$$\sigma_{u_xu_x} = \frac{k_BT}{m}\left\{1 - e^{-\beta t}\left[1 + \frac{\beta^2+\delta^2}{\alpha^2+\delta^2}\times\left(2\sinh^2(\alpha t/2) - \frac{\alpha}{\beta}\sinh(\alpha t)\right) + \frac{\beta^2-\alpha^2}{\alpha^2+\delta^2}\left(2\sin^2(\delta t/2) - \frac{\delta}{\beta}\sin(\delta t)\right)\right]\right\}, \quad (27)$$

$$\sigma_{xu_x} = \frac{4\beta k_BT}{m(\alpha^2+\delta^2)}e^{-\beta t}[\sinh^2(\alpha t/2) + \sin^2(\delta t/2)], \quad (28)$$

$$\sigma_{yu_x} = \frac{2k_BT}{m(\alpha^2+\delta^2)}e^{-\beta t}[\delta\sinh(\alpha t) - \alpha\sin(\delta t)], \quad (29)$$

and $\sigma_{xy}=0$, $\sigma_{u_xu_y}=0$.

Now, let us define the reduced correlations as the corresponding dimensionless quantities, that is, $\sigma^\star_{xx}=\sigma_{xx}/(k_BT/m\omega^2)$, $\sigma^\star_{u_xu_x}=\sigma_{u_xu_x}/(k_BT/m)$, $\sigma^\star_{xu_x}=\sigma_{xu_x}/(k_BT/m\omega)$, and $\sigma^\star_{yu_x}=\sigma_{yu_x}/(k_BT/m\omega)$. Also the reduced time is $\tau=\beta t$ and in these new variables the behavior of correlations is shown in Figs. 1–4 for certain values of the magnetic field represented by the variable Ω/β and of ω/β. Several comments can be made about the behavior shown in Figs. 1–4. First we recall that the value $\Omega=0$ corresponds to Chandrasekhar's solution. In this case, in Fig. 4 the correlation $\sigma^\star_{yu_x}(\Omega=0)=0$, and it is not shown. Second, the relaxation when $t\to\infty$ is slower when $\Omega\neq 0$ than in the case without the magnetic field. This effect is enhanced as the normalized field Ω/β grows. In Figs. 1–3 the small-time behavior is essentially the same no matter the field value. Lastly, in Fig. 5 we can observe the behavior of the reduced correlations for $\Omega/\beta=1$ and $\omega/\beta=10$. In this case, due to the magnitude of the ω/β parameter, oscillations can be seen.

All our analytical results for the averages $\langle x\rangle$, $\langle y\rangle$, $\langle u_x\rangle$, and $\langle u_y\rangle$, and the correlation functions given in Eqs. (26)–(29), can be compared in the absence of magnetic field ($\Omega=0$) with those obtained by Chandrasekhar for the ordinary one-dimensional harmonic oscillator. In this case, the parameter $b=0$ and thus $\delta=0$, $a=\beta^2-4\omega^2$, which leads to $\alpha=\sqrt{a}$. If we define $\beta_1\equiv\sqrt{a}$, then $\alpha=\beta_1$. Under these conditions, we can clearly verify that

$$\langle x\rangle = x_0 e^{-\beta t/2}\left(\cosh(\beta_1 t/2) + \frac{\beta}{\beta_1}\sinh(\beta_1 t/2)\right) + \frac{2u_{x_0}}{\beta_1}e^{-\beta t/2}\sinh(\beta_1 t/2), \quad (30)$$

$$\langle u_x\rangle = u_{x_0} e^{-\beta t/2}\left(\cosh(\beta_1 t/2) - \frac{\beta}{\beta_1}\sinh(\beta_1 t/2)\right) - \frac{2x_0\omega^2}{\beta_1}e^{-\beta t/2}\sinh(\beta_1 t/2). \quad (31)$$

For the variances, we have

$$\sigma_{xx} = \frac{k_BT}{m\omega^2}\left[1 - e^{-\beta t}\left(2\frac{\beta^2}{\beta_1^2}\sinh^2(\beta_1 t/2) + \frac{\beta}{\beta_1}\sinh(\beta_1 t) + 1\right)\right], \quad (32)$$

$$\sigma_{u_xu_x} = \frac{k_BT}{m}\left[1 - e^{-\beta t}\left(2\frac{\beta^2}{\beta_1^2}\sinh^2(\beta_1 t/2) - \frac{\beta}{\beta_1}\sinh(\beta_1 t) + 1\right)\right], \quad (33)$$

$$\sigma_{xu_x} = \frac{4\beta k_B T}{m\beta_1^2} e^{-\beta t} \sinh^2(\alpha t/2). \tag{34}$$

The averages $\langle y\rangle$ and $\langle u_y\rangle$ have the same expressions as Eqs. (30) and (31), respectively, just changing the initial conditions $x_0 \to y_0$ and $u_{x_0} \to u_{y_0}$; both correspond exactly to Chandrasekhar's results. The auto- and cross correlations also reduce immediately to the corresponding expressions, when we recall that in this case $\sigma_{xu_y} = \sigma_{yu_x} = 0$, which means that in the absence of the magnetic field the processes (x, u_y) and (y, u_x) become statistically independent as expected.

III. THE OVERDAMPED LIMITING CASE

In the overdamped approximation, or in the limit of large friction, where $|\overline{\beta}|$ is large, we can neglect the inertial term in Eqs. (3) and (4) and therefore the resulting equation on the complex plane can be written as

$$\dot{\overline{\rho}} + \overline{\gamma}\overline{\rho} = \frac{\overline{\mathcal{A}}(t)}{\overline{\beta}}, \tag{35}$$

where

$$\overline{\gamma} \equiv \frac{\omega^2}{\overline{\beta}} = \gamma_R - i\gamma_I, \quad \overline{\gamma}^* \equiv \frac{\omega^2}{\overline{\beta}^*} = \gamma_R + i\gamma_I, \tag{36}$$

and

$$\gamma_R \equiv \frac{\beta\omega^2}{\beta^2 + \Omega^2}, \quad \gamma_I \equiv \frac{\Omega\omega^2}{\beta^2 + \Omega^2}, \tag{37}$$

which represent the real and imaginary parts of $\overline{\gamma}$. The solution of Eq. (35) and its corresponding velocity are immediately calculated, yielding the following results:

$$\overline{\rho} = \langle\overline{\rho}\rangle + \int_0^t \overline{\mathcal{A}}(t')\overline{\chi}(t')dt', \tag{38}$$

$$\overline{u}_\rho = \langle\overline{u}_\rho\rangle + \int_0^t \overline{\mathcal{A}}(t')\overline{\zeta}(t')dt' + \frac{\overline{\mathcal{A}}(t)}{\overline{\beta}}, \tag{39}$$

where the deterministic solutions $\langle\overline{\rho}\rangle$ and $\langle\overline{u}_\rho\rangle$ are given by

$$\langle\overline{\rho}\rangle = e^{-\overline{\gamma}t}\overline{\rho}_0, \quad \langle\overline{u}_\rho\rangle = -\overline{\gamma}e^{-\overline{\gamma}t}\overline{\rho}_0. \tag{40}$$

The complex functions $\overline{\chi}(t')$ and $\overline{\zeta}(t')$ are defined as

$$\overline{\chi}(t') = \frac{1}{\overline{\beta}} e^{-\overline{\gamma}(t-t')}, \quad \overline{\zeta}(t') = -\frac{\overline{\gamma}}{\overline{\beta}} e^{-\overline{\gamma}(t-t')}. \tag{41}$$

In a similar way, the solutions for the complex conjugates of $\overline{\rho}(t)$ and $\overline{u}_\rho(t)$, as well as for $x(t)$, $y(t)$, $u_x(t)$, and $u_y(t)$ are given in Eqs. (C24)–(C27) of Appendix C. So the statistical properties of the variables of the system will be calculated in a similar way as before. The x, y, u_x, and u_y averages are shown to be

$$\langle x\rangle = e^{-\gamma_R t}[x_0 \cos(\gamma_I t) - y_0 \sin(\gamma_I t)], \tag{42}$$

$$\langle y\rangle = e^{-\gamma_R t}[y_0 \cos(\gamma_I t) + x_0 \sin(\gamma_I t)], \tag{43}$$

$$\langle u_x\rangle = \frac{e^{-\gamma_R t}}{\beta^2 + \Omega^2}\{\omega^2[y_0\beta - x_0\Omega]\sin(\gamma_I t) - \omega^2[x_0\beta + y_0\Omega]\cos(\gamma_I t)\}, \tag{44}$$

$$\langle u_y\rangle = -\frac{e^{-\gamma_R t}}{\beta^2 + \Omega^2}\{\omega^2[x_0\beta + y_0\Omega]\sin(\gamma_I t) + \omega^2[y_0\beta - x_0\Omega]\cos(\gamma_I t)\}. \tag{45}$$

Similarly as in the preceding section, we now evaluate the integrals given in Eqs. (C30)–(C34) to obtain the following expressions for the variances at two times, that is,

$$\sigma_{xx} = \frac{k_B T}{m\omega^2}(e^{\gamma_R|t_1-t_2|} - e^{-\gamma_R(t_1+t_2)})\cos\gamma_I(t_1-t_2), \tag{46}$$

$$\sigma_{xu_x} = \frac{k_B T}{m\beta_e}(e^{\gamma_R|t_1-t_2|} - e^{-\gamma_R(t_1+t_2)}) \times\left(\cos\gamma_I(t_1-t_2) - \frac{\Omega}{\beta}\sin\gamma_I(t_1-t_2)\right), \tag{47}$$

$$\sigma_{yu_x} = \frac{k_B T}{m\beta_e}(e^{\gamma_R|t_1-t_2|} - e^{-\gamma_R(t_1+t_2)}) \times\left(\frac{\Omega}{\beta}\cos\gamma_I(t_1-t_2) + \sin\gamma_I(t_1-t_2)\right), \tag{48}$$

$$\sigma_{xy} = \frac{k_B T}{m\omega^2}(e^{\gamma_R|t_1-t_2|} - e^{-\gamma_R(t_1+t_2)})\sin\gamma_I(t_1-t_2), \tag{49}$$

where $\beta_e \equiv \beta(1 + \frac{\Omega^2}{\beta^2})$ and, according to Eq. (37), $\gamma_R = \omega^2/\beta_e^2$ and $\gamma_I = \omega^2\Omega/\beta\beta_e$. Again, if $t_1 = t_2 = t$ then the variances reduce to

$$\sigma_{xx} = \frac{k_B T}{m\omega^2}(1 - e^{-2\omega^2 t/\beta_e}), \tag{50}$$

$$\sigma_{xu_x} = \frac{k_B T}{m\beta_e}(1 - e^{-2\omega^2 t/\beta_e}), \tag{51}$$

$$\sigma_{yu_x} = \frac{k_B T}{m}\frac{\Omega}{\beta\beta_e}(1 - e^{-2\omega^2 t/\beta_e}). \tag{52}$$

Here β_e accounts for a redefinition of the friction coefficient β when the magnetic field is present. This situation can be understood in the following way. In the absence of the magnetic field, it is evident that $\beta_e = \beta$ and therefore expressions (50)–(52) reduce, respectively, to

$$\sigma_{xx} = \frac{k_B T}{m\omega^2}(1 - e^{-2\omega^2 t/\beta}), \tag{53}$$

$$\sigma_{xu_x} = \frac{k_B T}{m\beta}(1 - e^{-2\omega^2 t/\beta}), \tag{54}$$

and $\sigma_{xu_y} = \sigma_{yu_x} = 0$. Thus, the description of the Brownian motion through Eqs. (50)–(52) suggest a redefinition of the friction coefficient by the value β_e.

The reader may notice that we did not give the expressions for the correlation of the velocity; actually is easy to see that this correlation must have a term proportional to Dirac's δ. This follows from Eq. (39) since the velocity is proportional to a combination of the Gaussian white noise, which is δ correlated. A Dirac δ in the correlations for the velocity is a direct result of the approximation done, and does not correspond to what we obtained without the approximation. It is to be expected that, although the correlations for x-u_x and y-u_x remain finite, the comparison with the exact results may not show agreement. This is actually the case if one considers the large-time behavior for the exact result and the overdamped approximation. In Fig. 6 a comparison between the exact and the overdamped approximation results for $\sigma^{\star}_{xx}$ is shown for $\omega/\beta=0.01$ and different values of Ω/β. As is evident from the figure the agreement is rather good. Notice that for the free-field case $\sigma^{\star}_{xx}$ is given equally well by the exact result and the overdamped approximation; in contrast there is a noticeable difference in the values for $\sigma^{\star}_{xu_x}$ given by the exact result and the overdamped approximation.

IV. CONCLUDING REMARKS

In this work, we have been able to solve the problem of Brownian motion of a charged particle in the presence of a constant magnetic field in a time-independent harmonic well. The magnetic field is allowed to point along the z axis of a Cartesian reference frame and, as a consequence of this fact, two independent processes for the charged harmonic oscillator occur. One process takes place along the magnetic field and is described by the Langevin equation (5). This process was exactly solved by Chandrasekhar [1] in 1943. The other one takes place on the x-y plane perpendicular to the magnetic field and is described by two coupled Langevin equations. To solve these equations, we use an alternative method of solution relying upon a transformation of these two equations through the change of variables given in Eq. (7). This method allows us to establish Eq. (8), which is then easier to solve than Eqs. (3) and (4). We have established the general auto- and cross correlations at two different times, which are given in Eqs. (B47)–(B52). At equal times $t_1=t_2=t$ Eqs. (B47)–(B52) reduce, respectively, to Eqs. (26)–(29) whereas $\sigma_{xy}=0$ and $\sigma_{u_x u_y}=0$. Our analytical results have been compared with those calculated by Chandrasekhar, in the absence of the magnetic field. In this case, the planar-diffusion process, on the x-y plane, takes place in an independent way for each one of the processes, along the x and y axes, as expected. In the overdamped approximation we have calculated the same statistical properties, which is easier than in the full description. Our complete solution can be extended to that situation for which the particle is in a two-dimensional time-dependent harmonic well, similar to that studied in Ref. [21] in the overdamped approximation. Finally, in this large-friction limit the explicit solution of the Smoluchowski equation associated with the Jayannavar and Sahoo problem is in progress.

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APPENDIX A: SOLUTIONS FOR $\bar{\rho}(t)$ AND $\bar{\rho}^*(t)$ FUNCTIONS

To calculate the solution of Eq. (8), we follow the same method used by Chandrasekhar to solve the one-dimensional harmonic oscillator [1]; thus, we first calculate its homogeneous solution, which is given by

$$\bar{\rho}_h = \bar{a}_1 e^{\bar{\mu}_1 t} + \bar{a}_2 e^{\bar{\mu}_2 t}, \tag{A1}$$

where $\bar{a}_1$ and $\bar{a}_2$ are constant complex numbers, $\bar{\mu}_1$ and $\bar{\mu}_2$ are the roots of the homogeneous equation associated with Eq. (8), i.e.,

$$\bar{\mu}^2 + \bar{\beta}\bar{\mu} + \omega^2 = 0, \tag{A2}$$

and therefore the roots are given by

$$\bar{\mu}_1 = -\frac{\bar{\beta}}{2} + \frac{\bar{\beta}_1}{2}, \quad \bar{\mu}_2 = -\frac{\bar{\beta}}{2} - \frac{\bar{\beta}_1}{2}, \tag{A3}$$

and

$$\bar{\beta}_1 \equiv \sqrt{\bar{\beta}^2 - 4\omega^2} = \sqrt{\beta^2 - \Omega^2 - 4\omega^2 + i2\beta\Omega}. \tag{A4}$$

Now, we suppose that the solution of Eq. (8) is also of the form of Eq. (A1), but now assuming that $\bar{a}_1$ and $\bar{a}_2$ are time-dependent functions, that is,

$$\bar{\rho}(t) = \bar{a}_1(t) e^{\bar{\mu}_1 t} + \bar{a}_2(t) e^{\bar{\mu}_2 t}, \tag{A5}$$

and also satisfy the following condition:

$$\frac{d\bar{a}_1}{dt} e^{\bar{\mu}_1 t} + \frac{d\bar{a}_2}{dt} e^{\bar{\mu}_2 t} = 0. \tag{A6}$$

Substituting Eq. (A5) into Eq. (8), we have

$$\bar{\mu}_1 \frac{d\bar{a}_1}{dt} e^{\bar{\mu}_1 t} + \bar{\mu}_2 \frac{d\bar{a}_2}{dt} e^{\bar{\mu}_2 t} = \overline{\mathcal{A}}(t). \tag{A7}$$

The solution of the system of equations (A6) and (A7) leads to

$$\bar{a}_1(t) = \frac{1}{\bar{\mu}_1 - \bar{\mu}_2} \int_0^t e^{-\bar{\mu}_1 t'} \overline{\mathcal{A}}(t') + \bar{a}_{10}, \tag{A8}$$

$$\bar{a}_2(t) = -\frac{1}{\bar{\mu}_1 - \bar{\mu}_2} \int_0^t e^{-\bar{\mu}_2 t'} \overline{\mathcal{A}}(t') + \bar{a}_{20}, \tag{A9}$$

where $\bar{a}_{10}$ and $\bar{a}_{20}$ are constant. So the solution of Eq. (8) is then

$$\bar{\rho}(t) = \frac{1}{\bar{\mu}_1 - \bar{\mu}_2} \left(e^{\bar{\mu}_1 t} \int_0^t e^{-\bar{\mu}_1 t'} \overline{\mathcal{A}}(t') dt' - e^{\bar{\mu}_2 t} \int_0^t e^{-\bar{\mu}_2 t'} \overline{\mathcal{A}}(t') dt' \right) + \bar{a}_{10} e^{\bar{\mu}_1 t} + \bar{a}_{20} e^{\bar{\mu}_2 t}. \tag{A10}$$

In a completely similar way, we can calculate the solution of the Langevin equation for the conjugate variable $\bar{\rho}^*(t) \equiv x - iy$, which is given by

$$\ddot{\bar{\rho}}^* + \bar{\beta}^* \dot{\bar{\rho}}^* + \omega^2 \bar{\rho}^* = \bar{\mathcal{A}}^*(t), \tag{A11}$$

where the asterisk stands for the complex conjugate. In this case, the complex function $\bar{\rho}^*(t)$ and its corresponding velocity $\bar{u}_\rho^*(t) = u_x - iu_y$ are given by

$$\bar{\rho}^*(t) + \frac{1}{\bar{\mu}_1^* - \bar{\mu}_2^*}[(\bar{\rho}_0^* \bar{\mu}_2^* - \bar{u}_{\rho_0}^*)e^{\bar{\mu}_1^* t} - (\bar{\rho}_0^* \bar{\mu}_1^* - \bar{u}_{\rho_0}^*)e^{\bar{\mu}_2^* t}] = \int_0^t \bar{\mathcal{A}}^*(t')\bar{\psi}^*(t')dt' \tag{A12}$$

and

$$\bar{u}_\rho^*(t) + \frac{1}{\bar{\mu}_1^* - \bar{\mu}_2^*}[\bar{\mu}_1^*(\bar{\rho}_0^* \bar{\mu}_2^* - \bar{u}_{\rho_0}^*)e^{\bar{\mu}_1^* t} - \bar{\mu}_2^*(\bar{\rho}_0^* \bar{\mu}_1^* - \bar{u}_{\rho_0}^*)e^{\bar{\mu}_2^* t}] = \int_0^t \bar{\mathcal{A}}^*(t')\bar{\phi}^*(t')dt', \tag{A13}$$

where now the roots

$$\bar{\mu}_1^* = -\frac{\bar{\beta}^*}{2} + \frac{\bar{\beta}_1^*}{2}, \quad \bar{\mu}_2^* = -\frac{\bar{\beta}^*}{2} - \frac{\bar{\beta}_1^*}{2}, \tag{A14}$$

and

$$\bar{\beta}_1^* = \sqrt{\bar{\beta}^{*2} - 4\omega^2} = \sqrt{\beta^2 - \Omega^2 - 4\omega^2 - i2\beta\Omega}. \tag{A15}$$

Here the functions $\bar{\psi}^*(t')$ and $\bar{\phi}^*(t')$ are now

$$\bar{\psi}^*(t') = \frac{1}{\bar{\mu}_1^* - \bar{\mu}_2^*}(e^{\bar{\mu}_1^*(t-t')} - e^{\bar{\mu}_2^*(t-t')}), \tag{A16}$$

$$\bar{\phi}^*(t') = \frac{1}{\bar{\mu}_1^* - \bar{\mu}_2^*}(\bar{\mu}_1^* e^{\bar{\mu}_1^*(t-t')} - \bar{\mu}_2^* e^{\bar{\mu}_2^*(t-t')}). \tag{A17}$$

Under these conditions, the aforementioned solutions can be written in shorthand as

$$\bar{\rho} = \langle \bar{\rho} \rangle + \int_0^t \bar{\mathcal{A}}(t')\bar{\psi}(t')dt', \tag{A18}$$

$$\bar{\rho}^* = \langle \bar{\rho}^* \rangle + \int_0^t \bar{\mathcal{A}}^*(t')\bar{\psi}^*(t')dt', \tag{A19}$$

and

$$\bar{u}_\rho = \langle \bar{u}_\rho \rangle + \int_0^t \bar{\mathcal{A}}(t')\bar{\phi}(t')dt', \tag{A20}$$

$$\bar{u}_\rho^* = \langle \bar{u}_\rho^* \rangle + \int_0^t \bar{\mathcal{A}}^*(t')\bar{\phi}^*(t')dt', \tag{A21}$$

where

$$\langle \bar{\rho} \rangle = \bar{\rho}_0 e^{-\bar{\beta}t/2}\cosh(\bar{\beta}_1 t/2) + \frac{(\bar{\rho}_0\bar{\beta} + 2\bar{u}_{\rho_0})}{\bar{\beta}_1}e^{-\bar{\beta}t/2}\sinh(\bar{\beta}_1 t/2), \tag{A22}$$

$$\langle \bar{u}_\rho \rangle = \bar{u}_{\rho_0} e^{-\bar{\beta}t/2}\cosh(\bar{\beta}_1 t/2) - \frac{(\bar{u}_{\rho_0}\bar{\beta} + 2\bar{\rho}_0\omega^2)}{\bar{\beta}_1}e^{-\bar{\beta}t/2}\sinh(\bar{\beta}_1 t/2), \tag{A23}$$

$$\langle \bar{\rho}^* \rangle = \bar{\rho}_0^* e^{-\bar{\beta}^* t/2}\cosh(\bar{\beta}_1^* t/2) + \frac{(\bar{\rho}_0^*\bar{\beta}^* + 2\bar{u}_{\rho_0}^*)}{\bar{\beta}_1^*}e^{-\bar{\beta}^* t/2}\sinh(\bar{\beta}_1^* t/2), \tag{A24}$$

$$\langle \bar{u}_\rho^* \rangle = \bar{u}_{\rho_0}^* e^{-\bar{\beta}^* t/2}\cosh(\bar{\beta}_1^* t/2) - \frac{(\bar{u}_{\rho_0}^*\bar{\beta}^* + 2\bar{\rho}_0^*\omega^2)}{\bar{\beta}_1^*}e^{-\bar{\beta}^* t/2}\sinh(\bar{\beta}_1^* t/2). \tag{A25}$$

APPENDIX B: SOLUTIONS FOR $x(t)$ AND $y(t)$ FUNCTIONS

By defining the complex functions $\bar{\psi}(t) \equiv \psi_R(t) + i\psi_I(t)$, with $\psi_R(t)$ and $\psi_I(t)$ the real and imaginary parts of $\bar{\psi}(t)$, respectively, and similarly for $\bar{\phi}(t) \equiv \phi_R(t) + i\phi_I(t)$, and also

$$\bar{\Psi}(t') \equiv \bar{\mathcal{A}}(t')\bar{\psi}(t'), \quad \bar{\Psi}^*(t') \equiv \bar{\mathcal{A}}^*(t')\bar{\psi}^*(t'), \tag{B1}$$

$$\bar{\Phi}(t') \equiv \bar{\mathcal{A}}(t')\bar{\phi}(t'), \quad \bar{\Phi}^*(t') \equiv \bar{\mathcal{A}}^*(t')\bar{\phi}^*(t'), \tag{B2}$$

we can easily show that

$$\bar{\Psi}(t') = \mathrm{Re}\bar{\Psi}(t') + i\mathrm{Im}\bar{\Psi}(t'), \tag{B3}$$

$$\bar{\Psi}^*(t') = \mathrm{Re}\bar{\Psi}(t') - i\mathrm{Im}\bar{\Psi}(t'), \tag{B4}$$

$$\bar{\Phi}(t') = \mathrm{Re}\bar{\Phi}(t') + i\mathrm{Im}\bar{\Phi}(t'), \tag{B5}$$

$$\bar{\Phi}^*(t') = \mathrm{Re}\bar{\Phi}(t') - i\mathrm{Im}\bar{\Phi}(t'), \tag{B6}$$

with

$$\mathrm{Re}\bar{\Psi}(t') = A_x\psi_R - A_y\psi_I, \tag{B7}$$

$$\mathrm{Im}\bar{\Psi}(t') = A_x\psi_I + A_y\psi_R, \tag{B8}$$

$$\mathrm{Re}\bar{\Phi}(t') = A_x\phi_R - A_y\phi_I, \tag{B9}$$

$$\mathrm{Im}\bar{\Phi}(t') = A_x\phi_I + A_y\phi_R. \tag{B10}$$

However, from Eqs. (16) and (17), we can also show that

$$\psi_R(t') = \frac{1}{\alpha^2+\delta^2}[\alpha\eta_R(t,t') + \delta\eta_I(t,t')], \tag{B11}$$

$$\psi_I(t') = \frac{1}{\alpha^2+\delta^2}[\alpha\eta_I(t,t') - \delta\eta_R(t,t')], \tag{B12}$$

$$\phi_R(t') = \frac{1}{\alpha^2+\delta^2}[\alpha\varphi_R(t,t') + \delta\varphi_I(t,t')], \tag{B13}$$

$$\phi_I(t') = \frac{1}{\alpha^2+\delta^2}[\alpha\varphi_I(t,t') - \delta\varphi_R(t,t')], \tag{B14}$$

where

$$\eta_R(t,t') = \frac{1}{2}(e^{\bar{\mu}_1(t-t')} - e^{\bar{\mu}_2(t-t')}) + \frac{1}{2}(e^{\bar{\mu}_1^*(t-t')} - e^{\bar{\mu}_2^*(t-t')}), \tag{B15}$$

$$\eta_I(t,t') = -\frac{i}{2}(e^{\bar{\mu}_1(t-t')} - e^{\bar{\mu}_2(t-t')}) + \frac{i}{2}(e^{\bar{\mu}_1^*(t-t')} - e^{\bar{\mu}_2^*(t-t')}), \tag{B16}$$

and

$$\varphi_R(t,t') = \frac{1}{2}(\bar{\mu}_1 e^{\bar{\mu}_1(t-t')} - \bar{\mu}_2 e^{\bar{\mu}_2(t-t')}) + \frac{1}{2}(\bar{\mu}_1^* e^{\bar{\mu}_1^*(t-t')} - \bar{\mu}_2^* e^{\bar{\mu}_2^*(t-t')}), \tag{B17}$$

$$\varphi_I(t,t') = -\frac{i}{2}(\bar{\mu}_1 e^{\bar{\mu}_1(t-t')} - \bar{\mu}_2 e^{\bar{\mu}_2(t-t')}) + \frac{i}{2}(\bar{\mu}_1^* e^{\bar{\mu}_1^*(t-t')} - \bar{\mu}_2^* e^{\bar{\mu}_2^*(t-t')}). \tag{B18}$$

If we now take into account that $\langle\bar{\rho}\rangle=\langle x\rangle+i\langle y\rangle$, $\langle\bar{\rho}^*\rangle=\langle x\rangle-i\langle y\rangle$, $\langle\bar{u}_\rho\rangle=\langle u_x\rangle+i\langle u_y\rangle$, and $\langle\bar{u}_\rho^*\rangle=\langle u_x\rangle-i\langle u_y\rangle$, then, from Eqs. (A18)–(A21) and (B1)–(B6), we can conclude that the solutions of Eqs. (3) and (4) will be given, respectively, by

$$x(t) = \langle x\rangle + \int_0^t \mathrm{Re}\bar{\Psi}(t')dt', \tag{B19}$$

$$y(t) = \langle y\rangle + \int_0^t \mathrm{Im}\bar{\Psi}(t')dt', \tag{B20}$$

$$u_x(t) = \langle u_x\rangle + \int_0^t \mathrm{Re}\bar{\Phi}(t')dt', \tag{B21}$$

$$u_y(t) = \langle u_y\rangle + \int_0^t \mathrm{Im}\bar{\Phi}(t')dt', \tag{B22}$$

where the averages for x, y, u_x, and u_y will be

$$\begin{aligned}\langle x\rangle = {}& e^{-\beta t/2}x_0[\cosh(\alpha t/2)\cos(\delta t/2)\cos(\Omega t/2) \\ &+ \sinh(\alpha t/2)\sin(\delta t/2)\sin(\Omega t/2)] \\ &+ e^{-\beta t/2}y_0[\cosh(\alpha t/2)\cos(\delta t/2)\sin(\Omega t/2) \\ &- \sinh(\alpha t/2)\sin(\delta t/2)\cos(\Omega t/2)] + e^{-\beta t/2}(B_0+C_0) \\ &\times[\sinh(\alpha t/2)\cos(\delta t/2)\cos(\Omega t/2) \\ &+ \cosh(\alpha t/2)\sin(\delta t/2)\sin(\Omega t/2)] + e^{-\beta t/2}(D_0-E_0) \\ &\times[\cosh(\alpha t/2)\sin(\delta t/2)\cos(\Omega t/2) \\ &- \sinh(\alpha t/2)\cos(\delta t/2)\sin(\Omega t/2)],\end{aligned} \tag{B23}$$

$$\begin{aligned}\langle y\rangle = {}& e^{-\beta t/2}y_0[\cosh(\alpha t/2)\cos(\delta t/2)\cos(\Omega t/2) \\ &+ \sinh(\alpha t/2)\sin(\delta t/2)\sin(\Omega t/2)] \\ &- e^{-\beta t/2}x_0[\cosh(\alpha t/2)\cos(\delta t/2)\sin(\Omega t/2) \\ &- \sinh(\alpha t/2)\sin(\delta t/2)\cos(\Omega t/2)] - e^{-\beta t/2}(D_0-E_0) \\ &\times[\sinh(\alpha t/2)\cos(\delta t/2)\cos(\Omega t/2) \\ &+ \cosh(\alpha t/2)\sin(\delta t/2)\sin(\Omega t/2)] + e^{-\beta t/2}(B_0+C_0) \\ &\times[\cosh(\alpha t/2)\sin(\delta t/2)\cos(\Omega t/2) \\ &- \sinh(\alpha t/2)\cos(\delta t/2)\sin(\Omega t/2)],\end{aligned} \tag{B24}$$

$$\begin{aligned}\langle u_x\rangle = {}& e^{-\beta t/2}u_{x_0}[\cosh(\alpha t/2)\cos(\delta t/2)\cos(\Omega t/2) \\ &+ \sinh(\alpha t/2)\sin(\delta t/2)\sin(\Omega t/2)] \\ &+ e^{-\beta t/2}u_{y_0}[\cosh(\alpha t/2)\cos(\delta t/2)\sin(\Omega t/2) \\ &- \sinh(\alpha t/2)\sin(\delta t/2)\cos(\Omega t/2)] - e^{-\beta t/2}(F_0+G_0) \\ &\times[\sinh(\alpha t/2)\cos(\delta t/2)\cos(\Omega t/2) \\ &+ \cosh(\alpha t/2)\sin(\delta t/2)\sin(\Omega t/2)] - e^{-\beta t/2}(H_0-I_0) \\ &\times[\cosh(\alpha t/2)\sin(\delta t/2)\cos(\Omega t/2) \\ &- \sinh(\alpha t/2)\cos(\delta t/2)\sin(\Omega t/2)],\end{aligned} \tag{B25}$$

$$\begin{aligned}\langle u_y\rangle = {}& e^{-\beta t/2}u_{y_0}[\cosh(\alpha t/2)\cos(\delta t/2)\cos(\Omega t/2) \\ &+ \sinh(\alpha t/2)\sin(\delta t/2)\sin(\Omega t/2)] \\ &- e^{-\beta t/2}u_{x_0}[\cosh(\alpha t/2)\cos(\delta t/2)\sin(\Omega t/2) \\ &- \sinh(\alpha t/2)\sin(\delta t/2)\cos(\Omega t/2)] + e^{-\beta t/2}(H_0-I_0) \\ &\times[\sinh(\alpha t/2)\cos(\delta t/2)\cos(\Omega t/2) \\ &+ \cosh(\alpha t/2)\sin(\delta t/2)\sin(\Omega t/2)] - e^{-\beta t/2}(F_0+G_0) \\ &\times[\cosh(\alpha t/2)\sin(\delta t/2)\cos(\Omega t/2) \\ &- \sinh(\alpha t/2)\cos(\delta t/2)\sin(\Omega t/2)],\end{aligned} \tag{B26}$$

where

$$B_0 \equiv \frac{\alpha}{\alpha^2+\delta^2}\left(\frac{x_0}{\beta}(\beta^2+\delta^2) + 2u_{x_0}\right), \tag{B27}$$

$$C_0 \equiv \frac{\delta}{\alpha^2+\delta^2}\left(\frac{y_0}{\beta}(\beta^2-\alpha^2) + 2u_{y_0}\right), \tag{B28}$$

$$D_0 \equiv \frac{\delta}{\alpha^2+\delta^2}\left(\frac{x_0}{\beta}(\beta^2-\alpha^2)+2u_{x_0}\right), \tag{B29}$$

$$E_0 \equiv \frac{\alpha}{\alpha^2+\delta^2}\left(\frac{y_0}{\beta}(\beta^2+\delta^2)+2u_{y_0}\right), \tag{B30}$$

$$F_0 \equiv \frac{\alpha}{\alpha^2+\delta^2}\left(\frac{u_{x_0}}{\beta}(\beta^2+\delta^2)+2\omega^2 x_0\right), \tag{B31}$$

$$G_0 \equiv \frac{\delta}{\alpha^2+\delta^2}\left(\frac{u_{y_0}}{\beta}(\beta^2-\alpha^2)+2\omega^2 y_0\right), \tag{B32}$$

$$H_0 \equiv \frac{\delta}{\alpha^2+\delta^2}\left(\frac{u_{x_0}}{\beta}(\beta^2-\alpha^2)+2\omega^2 x_0\right), \tag{B33}$$

$$I_0 \equiv \frac{\alpha}{\alpha^2+\delta^2}\left(\frac{u_{y_0}}{\beta}(\beta^2+\delta^2)+2\omega^2 y_0\right). \tag{B34}$$

On the other hand, for the two-time auto- and cross correlations we use the definition established in the text. According to Eqs. (B7)–(B14) and (B19)–(B22), it is possible to show that the variances at two times are

$$\sigma_{xx} = \langle x(t_1)x(t_2)\rangle - \langle x(t_1)\rangle\langle x(t_2)\rangle = \sigma_{yy} = \langle y(t_1)y(t_2)\rangle - \langle y(t_1)\rangle\langle y(t_2)\rangle = \mathbb{I}(t_1,t_2), \tag{B35}$$

$$\sigma_{u_xu_x} = \langle u_x(t_1)u_x(t_2)\rangle - \langle u_x(t_1)\rangle\langle u_x(t_2)\rangle = \sigma_{u_yu_y} = \langle u_y(t_1)u_y(t_2)\rangle - \langle u_y(t_1)\rangle\langle u_y(t_2)\rangle = \mathbb{J}(t_1,t_2), \tag{B36}$$

$$\sigma_{xu_x} = \langle x(t_1)u_x(t_2)\rangle - \langle x(t_1)\rangle\langle u_x(t_2)\rangle = \sigma_{yu_y} = \langle y(t_1)u_y(t_2)\rangle - \langle y(t_1)\rangle\langle u_y(t_2)\rangle = \mathbb{K}(t_1,t_2), \tag{B37}$$

$$\sigma_{xu_y} = \langle x(t_1)u_y(t_2)\rangle - \langle x(t_1)\rangle\langle u_y(t_2)\rangle = -\sigma_{yu_x} = \langle y(t_1)u_x(t_2)\rangle - \langle y(t_1)\rangle\langle u_x(t_2)\rangle = \mathbb{L}(t_1,t_2), \tag{B38}$$

$$\sigma_{xy} = \langle x(t_1)y(t_2)\rangle - \langle x(t_1)\rangle\langle y(t_2)\rangle = -\sigma_{yx} = \langle y(t_1)x(t_2)\rangle - \langle y(t_1)\rangle\langle x(t_2)\rangle = \mathbb{M}(t_1,t_2), \tag{B39}$$

$$\sigma_{u_xu_y} = \langle u_x(t_1)u_y(t_2)\rangle - \langle u_x(t_1)\rangle\langle u_y(t_2)\rangle = -\sigma_{u_yu_x} = \langle u_y(t_1)u_x(t_2)\rangle - \langle u_y(t_1)\rangle\langle u_x(t_2)\rangle = \mathbb{N}(t_1,t_2), \tag{B40}$$

where

$$\mathbb{I}(t_1,t_2) = \frac{2\lambda}{\Lambda^2}\int_0^{t_1}\int_0^{t_2}[\eta_R(t_1,t_1')\eta_R(t_2,t_2') + \eta_I(t_1,t_1')\eta_I(t_2,t_2')]dt_1'dt_2', \tag{B41}$$

$$\mathbb{J}(t_1,t_2) = \frac{2\lambda}{\Lambda^2}\int_0^{t_1}\int_0^{t_2}[\varphi_R(t_1,t_1')\varphi_R(t_2,t_2') + \varphi_I(t_1,t_1')\varphi_I(t_2,t_2')]dt_1'dt_2', \tag{B42}$$

$$\mathbb{K}(t_1,t_2) = \frac{2\lambda}{\Lambda^2}\int_0^{t_1}\int_0^{t_2}[\eta_R(t_1,t_1')\varphi_R(t_2,t_2') + \eta_I(t_1,t_1')\varphi_I(t_2,t_2')]dt_1'dt_2', \tag{B43}$$

$$\mathbb{L}(t_1,t_2) = \frac{2\lambda}{\Lambda^2}\int_0^{t_1}\int_0^{t_2}[\eta_R(t_1,t_1')\varphi_I(t_2,t_2') - \eta_I(t_1,t_1')\varphi_R(t_2,t_2')]dt_1'dt_2', \tag{B44}$$

$$\mathbb{M}(t_1,t_2) = \frac{2\lambda}{\Lambda^2}\int_0^{t_1}\int_0^{t_2}[\eta_R(t_1,t_1')\eta_I(t_2,t_2') - \eta_I(t_1,t_1')\eta_R(t_2,t_2')]dt_1'dt_2', \tag{B45}$$

$$\mathbb{N}(t_1,t_2) = \frac{2\lambda}{\Lambda^2}\int_0^{t_1}\int_0^{t_2}[\varphi_R(t_1,t_1')\varphi_I(t_2,t_2') - \varphi_I(t_1,t_1')\varphi_R(t_2,t_2')]dt_1'dt_2', \tag{B46}$$

where $\Lambda^2 \equiv \alpha^2+\delta^2$. These integrals can be evaluated by using Eqs. (B15)–(B18); the result is the following:

$$\sigma_{xx} = \frac{2\lambda}{\Lambda^2}\left(\frac{E_1(t_1,t_2)}{\beta-\alpha}\cos[\Theta_1(t_1-t_2)] + \frac{E_2(t_1,t_2)}{\beta+\alpha}\times\cos[\Theta_2(t_1-t_2)] - \frac{2}{\beta^2+\delta^2}[\mathcal{F}_R\mathcal{B}_R - \mathcal{F}_I\mathcal{B}_I]\right), \tag{B47}$$

$$\sigma_{u_xu_x} = \frac{2\lambda}{\Lambda^2}\left(\bar{\mu}_1\bar{\mu}_1^*\frac{E_1(t_1,t_2)}{\beta-\alpha}\cos[\Theta_1(t_1-t_2)] + \bar{\mu}_2\bar{\mu}_2^*\frac{E_2(t_1,t_2)}{\beta+\alpha}\cos[\Theta_2(t_1-t_2)] - \frac{2}{\beta^2+\delta^2}[\mathrm{Re}(\bar{\mu}_1\bar{\mu}_2^*)\times(\mathcal{F}_R\mathcal{B}_R - \mathcal{F}_I\mathcal{B}_I) - \mathrm{Im}(\bar{\mu}_1\bar{\mu}_2^*)(\mathcal{F}_R\mathcal{B}_I + \mathcal{F}_I\mathcal{B}_I)]\right), \tag{B48}$$

$$\sigma_{xu_x} = \frac{2\lambda}{\Lambda^2}\left[-\frac{E_1(t_1,t_2)}{2}\left(\cos[\Theta_1(t_1-t_2)] + \frac{\delta-\Omega}{\beta-\alpha}\times\sin[\Theta_1(t_1-t_2)]\right) + \frac{E_2(t_1,t_2)}{2}\left(\frac{\delta+\Omega}{\beta+\alpha}\sin[\Theta_2(t_1-t_2)] - \cos[\Theta_2(t_1-t_2)]\right) - \frac{1}{\beta^2+\delta^2}(\mathcal{F}_R\mathcal{C}_R - \mathcal{F}_I\mathcal{C}_I)\right], \tag{B49}$$

$$\sigma_{yu_x} = \frac{2\lambda}{\Lambda^2}\Bigg[\frac{E_1(t_1,t_2)}{2}\left(\frac{\delta-\Omega}{\beta-\alpha}\cos[\Theta_1(t_1-t_2)] + \sin[\Theta_1(t_1-t_2)]\right) - \frac{E_2(t_1,t_2)}{2}\left(\frac{\delta+\Omega}{\beta+\alpha}\cos[\Theta_2(t_1-t_2)] + \sin[\Theta_2(t_1-t_2)]\right) + \frac{1}{\beta^2+\delta^2}[\mathcal{F}_R\mathcal{D}_I + \mathcal{F}_I\mathcal{D}_R]\Bigg], \quad \text{(B50)}$$

$$\sigma_{xy} = \frac{2\lambda}{\Lambda^2}\left(-\frac{E_1(t_1,t_2)}{\beta-\alpha}\sin[\Theta_1(t_1-t_2)] + \frac{E_2(t_1,t_2)}{\beta+\alpha}\times\sin[\Theta_2(t_1-t_2)] + \frac{2}{\beta^2+\delta^2}(\mathcal{F}_I\mathcal{E}_R - \mathcal{F}_R\mathcal{E}_I)\right), \quad \text{(B51)}$$

$$\sigma_{u_xu_y} = \frac{2\lambda}{\Lambda^2}\Bigg(-\bar{\mu}_1\bar{\mu}_1^*\frac{E_1(t_1,t_2)}{\beta-\alpha}\sin[\Theta_1(t_1-t_2)] + \bar{\mu}_2\bar{\mu}_2^*\frac{E_2(t_1,t_2)}{\beta+\alpha}\sin[\Theta_2(t_1-t_2)] + \frac{2}{\beta^2+\delta^2}[\mathrm{Re}(\bar{\mu}_1\bar{\mu}_2^*)\times(\mathcal{F}_R\mathcal{E}_I + \mathcal{F}_I\mathcal{E}_R) + \mathrm{Im}(\bar{\mu}_1\bar{\mu}_2^*)(\mathcal{F}_R\mathcal{E}_I - \mathcal{F}_I\mathcal{E}_R)]\Bigg), \quad \text{(B52)}$$

where $\Lambda^2 \equiv \alpha^2+\delta^2$, $\Theta_1 \equiv \frac{1}{2}(\delta-\Omega)$, $\Theta_2 \equiv \frac{1}{2}(\delta+\Omega)$, and

$$E_1(t_1,t_2) \equiv (e^{-(\beta-\alpha)|t_1-t_2|/2} - e^{-(\beta-\alpha)(t_1+t_2)/2}), \quad \text{(B53)}$$

$$E_2(t_1,t_2) \equiv (e^{-(\beta+\alpha)|t_1-t_2|/2} - e^{-(\beta+\alpha)(t_1+t_2)/2}), \quad \text{(B54)}$$

$$F_1(t_1,t_2) \equiv (e^{-(\beta-i\delta)|t_1-t_2|/2} - e^{-(\beta-i\delta)(t_1+t_2)/2}), \quad \text{(B55)}$$

$$F_2(t_1,t_2) \equiv (e^{-(\beta+i\delta)|t_1-t_2|/2} - e^{-(\beta+i\delta)(t_1+t_2)/2}), \quad \text{(B56)}$$

$$\mathcal{F}_R \equiv \beta\mathrm{Re}\bar{F}_1 - \delta\mathrm{Im}\bar{F}_1, \quad \mathcal{F}_I \equiv \delta\mathrm{Re}\bar{F}_1 + \beta\mathrm{Im}\bar{F}_1, \quad \text{(B57)}$$

$$\mathrm{Re}\bar{F}_1 = e^{-\beta|t_1-t_2|/2}\cos\delta|t_1-t_2|/2 - e^{-\beta(t_1+t_2)/2}\cos\delta(t_1+t_2)/2, \quad \text{(B58)}$$

$$\mathrm{Im}\bar{F}_1 = e^{-\beta|t_1-t_2|/2}\sin\delta|t_1-t_2|/2 - e^{-\beta(t_1+t_2)/2}\sin\delta(t_1+t_2)/2, \quad \text{(B59)}$$

$$\mathrm{Re}(\bar{\mu}_1\bar{\mu}_2^*) = \frac{1}{4}(\beta^2-\alpha^2) - \frac{1}{4}(\delta^2-\Omega^2), \quad \text{(B60)}$$

$$\mathrm{Im}(\bar{\mu}_1\bar{\mu}_2^*) = -\frac{1}{2}(\beta\delta-\alpha\Omega), \quad \text{(B61)}$$

$$\mathcal{B}_R \equiv \cos[\Omega(t_1-t_2)/2]\cosh[\alpha(t_1-t_2)/2], \quad \text{(B62)}$$

$$\mathcal{B}_I \equiv \sin[\Omega(t_1-t_2)/2]\sinh[\alpha(t_1-t_2)/2], \quad \text{(B63)}$$

$$\mathcal{C}_R \equiv \sin[\Omega(t_1-t_2)/2]\{\delta\sinh[\alpha(t_1-t_2)/2] + \Omega\cosh[\alpha\times(t_1-t_2)/2]\} - \cos[\Omega(t_1-t_2)/2]\{\beta\cosh[\alpha(t_1-t_2)/2] + \alpha\sinh[\alpha(t_1-t_2)/2]\}, \quad \text{(B64)}$$

$$\mathcal{C}_I \equiv \sin[\Omega(t_1-t_2)/2]\{\alpha\cosh[\alpha(t_1-t_2)/2] + \beta\sinh[\alpha\times(t_1-t_2)/2]\} + \cos[\Omega(t_1-t_2)/2]\{\delta\cosh[\alpha(t_1-t_2)/2] + \Omega\sinh[\alpha(t_1-t_2)/2]\}, \quad \text{(B65)}$$

$$\mathcal{D}_R \equiv \sin[\Omega(t_1-t_2)/2]\{\delta\cosh[\alpha(t_1-t_2)/2] + \Omega\sinh[\alpha\times(t_1-t_2)/2]\} - \cos[\Omega(t_1-t_2)/2]\{\alpha\cosh[\alpha(t_1-t_2)/2] + \beta\sinh[\alpha(t_1-t_2)/2]\}, \quad \text{(B66)}$$

$$\mathcal{D}_I \equiv \sin[\Omega(t_1-t_2)/2]\{\beta\cosh[\alpha(t_1-t_2)/2]\} + \alpha\sinh[\alpha(t_1-t_2)/2] + \cos[\Omega(t_1-t_2)/2]\{\delta\sinh[\delta(t_1-t_2)/2] + \Omega\cosh[\alpha(t_1-t_2)/2]\}, \quad \text{(B67)}$$

$$\mathcal{E}_R \equiv \cos[\Omega(t_1-t_2)/2]\sinh[\alpha(t_1-t_2)/2], \quad \text{(B68)}$$

$$\mathcal{E}_I \equiv \sin[\Omega(t_1-t_2)/2]\cosh[\alpha(t_1-t_2)/2]. \quad \text{(B69)}$$

APPENDIX C: SOLUTIONS FOR $x(t)$ AND $y(t)$ FUNCTIONS FOR LARGE FRICTION

The Langevin equation for the complex function $\bar{\rho}^*(t)$ reads

$$\dot{\bar{\rho}}^* + \bar{\gamma}^*\bar{\rho}^* = \frac{\bar{\mathcal{A}}^*(t)}{\bar{\beta}^*}. \quad \text{(C1)}$$

Thus its solution and its corresponding velocity $\bar{u}_\rho^*$ read

$$\bar{\rho}^* = \langle\bar{\rho}^*\rangle + \int_0^t \bar{\mathcal{A}}^*(t')\bar{\chi}^*(t')dt', \quad \text{(C2)}$$

$$\bar{u}_\rho^* = \langle\bar{u}_\rho^*\rangle + \int_0^t \bar{\mathcal{A}}^*(t')\bar{\zeta}^*(t')dt' + \frac{\bar{\mathcal{A}}^*(t)}{\bar{\beta}}, \quad \text{(C3)}$$

such that

$$\langle\bar{\rho}^*\rangle = e^{-\bar{\gamma}^*t}\bar{\rho}_0^*, \quad \langle\bar{u}_\rho^*\rangle = -\bar{\gamma}^*e^{-\bar{\gamma}^*t}\bar{\rho}_0^*, \quad \text{(C4)}$$

$$\bar{\chi}^*(t') = \frac{1}{\bar{\beta}^*}e^{-\bar{\gamma}^*(t-t')}, \quad \bar{\zeta}^*(t') = -\frac{\bar{\gamma}^*}{\bar{\beta}^*}e^{-\bar{\gamma}^*(t-t')}. \quad \text{(C5)}$$

Again, if we define the complex functions $\bar{\chi}(t') = \chi_R + i\chi_I$ and $\bar{\zeta}(t') = \zeta_R + i\zeta_I$, and also

$$\bar{\Xi}(t') \equiv \bar{\mathcal{A}}(t')\bar{\chi}(t'), \quad \bar{\Xi}^*(t') \equiv \bar{\mathcal{A}}^*(t')\bar{\chi}^*(t'), \quad \text{(C6)}$$

$$\bar{\Pi}(t') \equiv \bar{\mathcal{A}}(t')\bar{\zeta}(t'), \quad \bar{\Pi}^*(t') \equiv \bar{\mathcal{A}}^*(t')\bar{\zeta}^*(t'), \quad \text{(C7)}$$

we can show that

$$\bar{\Xi}(t') = \mathrm{Re}\bar{\Xi}(t') + i\mathrm{Im}\bar{\Xi}(t'), \tag{C8}$$

$$\bar{\Xi}^*(t') = \mathrm{Re}\bar{\Xi}(t') - i\mathrm{Im}\bar{\Psi}(t'), \tag{C9}$$

$$\bar{\Pi}(t') = \mathrm{Re}\bar{\Pi}(t') + i\mathrm{Im}\bar{\Pi}(t'), \tag{C10}$$

$$\bar{\Pi}^*(t') = \mathrm{Re}\bar{\Pi}(t') - i\mathrm{Im}\bar{\Pi}(t'), \tag{C11}$$

where

$$\mathrm{Re}\bar{\Xi}(t') = A_x\chi_R - A_y\chi_I, \tag{C12}$$

$$\mathrm{Im}\bar{\Xi}(t') = A_x\chi_I + A_y\chi_R, \tag{C13}$$

$$\mathrm{Re}\bar{\Pi}(t') = A_x\zeta_R - A_y\zeta_I, \tag{C14}$$

$$\mathrm{Im}\bar{\Pi}(t') = A_x\zeta_I + A_y\zeta_R. \tag{C15}$$

Here the real and imaginary parts of $\bar{\chi}$ and $\bar{\zeta}$ can be written as

$$\chi_R(t') = \frac{1}{\beta^2+\Omega^2}[\beta\kappa_R(t,t') + \Omega\kappa_I(t,t')], \tag{C16}$$

$$\chi_I(t') = \frac{1}{\beta^2+\Omega^2}[\beta\kappa_I(t,t') - \Omega\kappa_R(t,t')], \tag{C17}$$

$$\zeta_R(t,t') = -\frac{1}{\beta^2+\Omega^2}[\beta\tau_R(t,t') + \Omega\tau_I(t,t')], \tag{C18}$$

$$\zeta_I(t') = -\frac{1}{\beta^2+\Omega^2}[\beta\tau_I(t,t') - \Omega\tau_R(t,t')], \tag{C19}$$

where

$$\kappa_R(t,t') = \frac{1}{2}(e^{-\bar{\gamma}(t-t')} + e^{-\bar{\gamma}^*(t-t')}), \tag{C20}$$

$$\kappa_I(t,t') = -\frac{i}{2}(e^{-\bar{\gamma}(t-t')} - e^{-\bar{\gamma}^*(t-t')}), \tag{C21}$$

$$\tau_R(t,t') = \frac{1}{2}(\bar{\gamma}e^{-\bar{\gamma}(t-t')} + \bar{\gamma}^*e^{-\bar{\gamma}^*(t-t')}), \tag{C22}$$

$$\tau_I(t,t') = -\frac{i}{2}(\bar{\gamma}e^{-\bar{\gamma}(t-t')} - \bar{\gamma}^*e^{-\bar{\gamma}^*(t-t')}). \tag{C23}$$

Also, in a similar way as was done in Appendix B, the solutions for the real functions $x(t)$, $y(t)$, $u_x(t)$, and $u_y(t)$ are then

$$x(t) = \langle x\rangle + \int_0^t \mathrm{Re}\bar{\Xi}(t')dt', \tag{C24}$$

$$y(t) = \langle y\rangle + \int_0^t \mathrm{Im}\bar{\Xi}(t')dt', \tag{C25}$$

and

$$u_x(t) = \langle u_x\rangle + \int_0^t \mathrm{Re}\bar{\Pi}(t')dt' + \vartheta_x(t), \tag{C26}$$

$$u_y(t) = \langle u_y\rangle + \int_0^t \mathrm{Im}\bar{\Pi}(t')dt' + \vartheta_y(t), \tag{C27}$$

where

$$\vartheta_x(t) = \frac{\beta A_x(t) + \Omega A_y(t)}{\beta^2+\Omega^2}, \tag{C28}$$

$$\vartheta_y(t) = \frac{\beta A_y(t) - \Omega A_x(t)}{\beta^2+\Omega^2}. \tag{C29}$$

Also, from Eqs. (C12)–(C19) and (C24)–(C29), we can show analogously as in Appendix B that at two times $\sigma_{xx}=\sigma_{yy}=\mathbb{P}(t_1,t_2)$, $\sigma_{u_xu_x}=\sigma_{u_yu_y}=\mathbb{Q}(t_1,t_2)$, $\sigma_{xu_x}=\sigma_{yu_y}=\mathbb{R}(t_1,t_2)$, and $\sigma_{xu_y}=-\sigma_{yu_x}=\mathbb{S}(t_1,t_2)$, and $\sigma_{xy}=-\sigma_{yx}=\mathbb{T}$,where now

$$\mathbb{P}(t_1,t_2) = \frac{2\lambda}{\Gamma^2}\int_0^{t_1}\int_0^{t_2}[\kappa_R(t_1,t_1')\kappa_R(t_2,t_2') + \kappa_I(t_1,t_1')\kappa_I(t_2,t_2')]dt_1'dt_2', \tag{C30}$$

$$\mathbb{Q}(t_1,t_2) = \frac{2\lambda}{\Gamma^2}\Big(\delta(t_1-t_2) - [\tau_R(t_1,t_1') + \tau_R(t_2,t_1')] + \int_0^{t_1}\int_0^{t_2}[\tau_R(t_1,t_1')\tau_R(t_2,t_2') + \tau_I(t_1,t_1')\tau_I(t_2,t_2')]dt_1'dt_2'\Big), \tag{C31}$$

$$\mathbb{R}(t_1,t_2) = \frac{2\lambda}{\Gamma^2}\int_0^{t_1}\int_0^{t_2}[\kappa_R(t_1,t_1')\tau_R(t_2,t_2') + \kappa_I(t_1,t_1')\tau_I(t_2,t_2')]dt_1'dt_2', \tag{C32}$$

$$\mathbb{S}(t_1,t_2) = \frac{2\lambda}{\Gamma^2}\int_0^{t_1}\int_0^{t_2}[\kappa_R(t_1,t_1')\tau_I(t_2,t_2') - \kappa_I(t_1,t_1')\tau_R(t_2,t_2')]dt_1'dt_2', \tag{C33}$$

$$\mathbb{T}(t_1,t_2) = \frac{2\lambda}{\Gamma^2}\int_0^{t_1}\int_0^{t_2}[\kappa_R(t_1,t_1')\kappa_I(t_2,t_2') - \kappa_I(t_1,t_1')\kappa_R(t_2,t_2')]dt_1'dt_2', \tag{C34}$$

where $\Gamma^2 \equiv \beta^2+\Omega^2$. These integrals can be evaluated with the help of Eqs. (C20)–(C23).

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