Soliton solutions for two nonlinear partial differential equations using a Darboux transformation of the Lax pairs

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Two Darboux transformations of the (1+1)-dimensional Wu-Zhang (WZ) equation and the two-component Camassa-Holm (2CH) system with the reciprocal transformation are obtained. One-loop and two-loop soliton solutions and multisoliton(like) solutions of the 2CH system are obtained by using the Darboux transformations and selecting different seed solutions of the corresponding equations. The bidirectional soliton solutions of the (1+1)-dimensional WZ equation are also obtained. The interactions of two-soliton head-on and overtaking collisions for the WZ equation and the evolution of the two-soliton(-like) solutions for the 2CH system are studied.

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I. INTRODUCTION

A large number of useful methods have been proposed to construct soliton solutions and locally coherent structure solutions for nonlinear partial differential equations. Some of the most important methods are the inverse scattering transformation (IST) [1], the bilinear form [2], symmetry reduction [3], the Darboux transformation (DT) [5], the Painlevé analysis method [4], the Bäcklund transformation [6], the separated variable method [7], etc. The DT is one of the most powerful methods for constructing multisoliton and localized coherent structure solutions of nonlinear integrable equations in both 1+1 and 2+1 dimensions, such as the Korteweg–de Vries (KdV) equation, the Kadomtsev-Petviashvili equation, the Davey-Stewartson equation, the Nizhnik-Novikov-Veselov equation, and other integrable systems [5,8–12]. These DTs are based on the existence of Lax pairs which encode the nonlinear equations under consideration in the form of their compatibility conditions. The DT is a purely algebraic method which can be iterated step by step to generate infinitely many solutions (nontrivial) with the simplest (trivial) seed solution of the Lax pair.

It is well known that the KdV equation has N-soliton solutions and can be used to model overtaking collisions of water waves. But it is not a good model to describe the reflection of water waves on a vertical wall. In Ref. [13], sets of three nonlinear long-wave equations (Wu-Zhang equations) of the Boussinesq class for modeling nonlinear and dispersive long gravity waves traveling in two horizontal directions on shallow water were derived. For example, in the (1+1)-dimensional case, the equations are

$$\zeta_t+[(1+\zeta)\eta]_x+\frac{1}{4}\eta_{xxx}=0,$$

$$\eta_t+\eta\eta_x+\zeta_x=0, \tag{1.1}$$

derived by the scaling transformations $\frac{\sqrt{3}}{2}x\to x$ and $\frac{\sqrt{3}}{2}t\to t$ of the coordinate variables (x,t), where η is the surface velocity of the water wave and ζ is the wave elevation. The system (1.1) has a Lax pair. In fact, we can have another form of the system (1.1):

$$v_t+\frac{3}{2}vv_x-u_x=0,$$

$$u_t+\frac{1}{2}vu_x+uv_x-\frac{1}{4}v_{xxx}=0, \tag{1.2}$$

whose Lax pair is

$$\phi_{xx}=(\lambda^2+\lambda v+u)\phi, \tag{1.3}$$

$$\phi_t=\frac{1}{4}v_x\phi+\left(\lambda-\frac{1}{2}v\right)\phi_x, \tag{1.4}$$

obtained by the transformation

$$\eta=v,\quad \frac{1}{4}\eta^2-\zeta-1=\frac{1}{4}v^2-\zeta-1=u. \tag{1.5}$$

Bidirectional soliton solutions, multisoliton solutions, and periodic wave solutions of the (1+1)-dimensional Wu-Zhang (WZ) system are indirectly constructed by the Painlevé analysis method and the DT, by transforming the WZ system (1.1) into the Broer-Kaup (BK) equations and the Ablowitz-Kaup-Newell-Segur (AKNS) system [14–16]. In this paper, we try to directly get bidirectional soliton solutions of the WZ system (1.2) by the DT.

Another interesting system is the following two-component generalization of the well-known two-component Camassa-Holm (2CH) system which was derived by Liu and Zhang [17]:

$$m_s+Um_y+2mU_y-\rho\rho_y=0,$$

$$\rho_s+(\rho U)_y=0 \tag{1.6}$$

obtained by the deformation of bi-Hamiltonian structures of hydrodynamic type, where $m=U-U_{yy}+\frac{1}{2}\kappa$. Under the constraint $\rho=0$, the system (1.6) is reduced to the celebrated Camassa-Holm equation [18]

$$U_s + \kappa U_y - U_{yys} + 3UU_y = 2U_y U_{yy}, \tag{1.7}$$

which describes the fluid velocity of a shallow water wave in the y direction. The Camassa-Holm equation (1.7) is completely integrable because it has most of the integrable features (such as the Lax pair, the bi-Hamiltonian structure, solvability by the inverse scattering approach, etc.) of a nonlinear partial different equation [18–20]. Although multikink and peak-on solutions for the system (1.6) have been derived by Chen [16], the bilinear form of (1.6) has been constructed by Falqui [22], and multisoliton solutions have been obtained by changing (1.6) into the first negative flow of the AKNS hierarchy [23], there has been scarcely any investigation of the 2CH system (1.6). We try to get multisoliton solutions of (1.6) by DTs from the following Lax pair of the 2CH system (1.6):

$$\psi_{yy} + \left(-\frac{1}{4} + m\lambda - \rho^2\lambda\right)\psi = 0,$$

$$\psi_s = -\left(\frac{1}{2\lambda} + U\right)\psi_y + \frac{U_y}{2}\psi. \tag{1.8}$$

In order to investigate the solutions and integrability of the 2CH system (1.6), it is important that the system be transformed to a new form by the reciprocal transformation

$$dx = \rho\, dy - \rho u\, ds, \quad dt = ds.$$

The new form of the system (1.6) is

$$w_x + v_t = 0, \tag{1.9}$$

$$2u_t + wv_x + 2vw_x = 0, \tag{1.10}$$

$$2wu_x + 4uw_x - w_{xxx} = 0, \tag{1.11}$$

and its Lax pair is correspondingly rewritten as

$$\phi_{xx} = (\lambda^2 + \lambda v + u)\phi, \tag{1.12}$$

$$\phi_t = \frac{w_x}{4\lambda}\phi - \frac{w}{2\lambda}\phi_x, \tag{1.13}$$

in which

$$w = \rho, \quad v = -\frac{m}{w^2}, \quad u = \frac{1}{4w^2} + \frac{w_{xx}}{2w} - \frac{w_x^2}{4w^2}. \tag{1.14}$$

It is obvious that (1.12) is completely the same as (1.3). This equation is referred to as a linear Schrödinger equation with an energy-dependent potential (i.e., the potential $\lambda v + u$ is dependent on the energy λ). When the potential functions u and v depend only on the variable x, this spectral problem has been investigated by the inverse scattering method [24] and the Bäcklund transformation [25]. More general energy-dependent Schrödinger operators were studied using Miura maps [26], and infinite families of completely integrable nonlinear Hamiltonian equations have been derived for Schrödinger spectral problems [27]. Recently, two types of DT have been obtained, and abundant solutions for the eigenvalue problem (1.3) [or (1.12)] using different potential functions u and v [28].

In the present paper, we give two types of Darboux transformation for the WZ equation (1.2) and the 2CH system (1.6) in Sec. II. In Sec. III, we find a single-loop solution, a two-loop solution, and multisoliton(like) solutions of the two-component Camassa-Holm system (1.6) by applying the DTs and investigate the properties of multisoliton propagation. In Sec. IV, we focus on investigating bidirectional two-soliton solutions and two-soliton head-on and overtaking collisions for the $(1+1)$-dimensional WZ equation using the Darboux transformations. Finally, we conclude the paper in Sec. V.

II. DARBOUX TRANSFORMATIONS

In order to study the Darboux transformations for the spectral problem (1.3) and (1.12), first we rewrite Eq. (1.3) [or (1.12)] in the following matrix form:

$$\Phi_x = M\Phi, \quad \Phi = (\phi_x, \phi)^T,$$

$$M = \begin{pmatrix} 0 & \lambda^2 + \lambda v + u \\ 1 & 0 \end{pmatrix}, \tag{2.1}$$

and consider a Darboux transformation

$$\bar{\Phi} = T\Phi, \tag{2.2}$$

in which T satisfies

$$T_x + TM = \bar{M}T. \tag{2.3}$$

The spectral problem is now written

$$\bar{\Phi}_x = \bar{M}\bar{\Phi}; \tag{2.4}$$

here $\bar{M}$ has the same form as M except that v and u are replaced by v_1 and u_1. It is clear that v_1,u_1 is a solution of (1.2) and (1.9)–(1.11). According to the form of M, we suppose that T has the following form:

$$T = \begin{pmatrix} \lambda a_1 + a_0 & \lambda^2 b_2 + \lambda b_1 + b_0 \\ c_0 & \lambda d_1 + d_0 \end{pmatrix}, \tag{2.5}$$

where a_0, a_1, b_0, b_1, b_2, c_0, d_0, and d_1 are all undetermined functions with respect to the variables x and t.

Inserting (2.5), M, and $\bar{M}$ into (2.3), we obtain

$$\begin{pmatrix} \lambda^2 b_2 + \lambda b_1 + b_0 - (\lambda^2 + \lambda v_1 + u_1)c_0 & (\lambda a_1 + a_0)(\lambda^2 + \lambda v + u) - (\lambda^2 + \lambda v_1 + u_1)(\lambda d_1 + d_0) \\ \lambda d_1 + d_0 - \lambda a_1 - a_0 & c_0(\lambda^2 + \lambda v + u) - \lambda^2 b_2 - \lambda b_1 - b_0 \end{pmatrix}$$
$$+ \begin{pmatrix} \lambda a_{1x} + a_{0x} & \lambda^2 b_{2x} + \lambda b_{1x} + b_{0x} \\ c_{0x} & \lambda d_{1x} + d_{0x} \end{pmatrix} = 0. \qquad (2.6)$$

Because λ is a parameter, the coefficients of λ^j ($j = 3,2,1,0$) in every matrix element are equal to zero. Omitting the calculation for obtaining the DTs, we give them directly.

Proposition 1. If (u,v) [or (u,v,w)] is a known solution of the system (1.2) [or the system (1.9)–(1.11)] and c_0 and d_0 have the forms

$$d_0 = -\left(\frac{h_x}{h} - \lambda_1\right)c_0, \quad c_0 = \frac{1}{\sqrt{2h_x/h - 2\lambda_1 - v}}, \qquad (2.7)$$

then there is a Darboux transformation

$$\bar{\phi} = c_0\phi_x + (d_0 - \lambda c_0)\phi, \qquad (2.8)$$

and $(\bar{u},\bar{v})$ [or $(\bar{u},\bar{v},\bar{w})$] is also a solution of the system (1.2) [or (1.9)–(1.11)],

$$\bar{u} = u + \frac{2d_{0x} + c_{0xx}}{c_0}, \qquad (2.9)$$

$$\bar{v} = v - \frac{2c_{0x}}{c_0}, \qquad (2.10)$$

$$\left(\bar{w} = w + \frac{2c_{0t}}{c_0}\right); \qquad (2.11)$$

here the relation for $\bar{w}$ is only for the system (1.9)–(1.11).

Proposition 2. If (u,v) [or (u,v,w)] is a known solution of the system (1.2) [or (1.9)–(1.11)], and c_0 and d_0 have the forms

$$d_0 = -\left(\frac{h_x}{h} + \lambda_1\right)c_0, \quad c_0 = \frac{1}{\sqrt{-2h_x/h - 2\lambda_1 - v}}, \qquad (2.12)$$

then there is a Darboux transformation

$$\bar{\phi} = c_0\phi_x + (d_0 + \lambda c_0)\phi, \qquad (2.13)$$

and $(\bar{u},\bar{v})$ [or $(\bar{u},\bar{v},\bar{w})$] is also a solution of the system (1.2) [or (1.9)–(1.11)],

$$\bar{u} = u + \frac{2d_{0x} + c_{0xx}}{c_0}, \qquad (2.14)$$

$$\bar{v} = v + \frac{2c_{0x}}{c_0}, \qquad (2.15)$$

$$\left(\bar{w} = w - \frac{2c_{0t}}{c_0}\right), \qquad (2.16)$$

where h in (2.7) and (2.12) satisfies

$$h_{xx} = (\lambda_1^2 + \lambda_1 v + u)h, \qquad (2.17)$$

and the Lax pair Eq. (1.4) for the WZ equation [or (1.13) for the 2CH system] with $\lambda = \lambda_1$. Finally, additional solutions of the $(1+1)$-dimensional equation (1.1) and the 2CH system (1.6) can be obtained via the DTs and the corresponding transformations (1.5) and (1.14).

III. MULTISOLITON SOLUTIONS OF THE 2CH SYSTEM

According to the Darboux theorem, we can obtain different types of soliton solutions for the 2CH system (1.9)–(1.11) by two types of Darboux transformations and taking different seed solutions. Then, the solutions of the 2CH equation (1.6) with respect to the variables (x,t) give

$$\rho(x,t) = w, U(x,t) = -vw^2 - w\partial_{xt}^2(\ln w) - \frac{\kappa}{2}, \qquad (3.1)$$

via the multisoliton solutions v and w of (1.9)–(1.11). In principle, the solutions ρ and U of the 2CH system (1.6) with respect to the variables (y,s) are obtained by the reciprocal transformation

$$y = \int \frac{dx}{\rho(x,t)} + \int U(x,t)dt, \quad s = t. \qquad (3.2)$$

A. The multisoliton solutions for the first DT

Selecting the seed solution of the system (1.9)–(1.11) in the general form v_0, w_0 ($w_0 \neq 0$), and $u_0 = 1/(4w_0^2)$, and inserting the seed solution into (2.17) and (1.12), we get

$$h = \cosh \xi_1, \quad \phi = \cosh \xi, \qquad (3.3)$$

and the one-soliton-like wave solutions for (1.9)–(1.11) are

$$v_1 = v_0 + \frac{-8k_1^2\lambda_1^2 \operatorname{sech}^2 \xi_1}{2\lambda_1 + v_0 - 4k_1\lambda_1 \tanh \xi_1},$$

$$w_1 = w_0 + \frac{2\lambda_1 + v_0 + 4k_1\lambda_1 \tanh \xi_1 - 4k_1^2\lambda_1 \operatorname{sech}^2 \xi_1}{2\lambda_1 + v_0 - 4k_1\lambda_1 \tanh \xi_1}, \tag{3.4}$$

where

$$\xi_1 = k_1(2\lambda_1 x - w_0 t), \quad k_1 = \frac{\sqrt{4w_0^2\lambda_1^2 + 4v_0 w_0^2\lambda_1 + 1}}{4w_0\lambda_1},$$

and the function u_1 can be obtained from

$$u_1 = \frac{1}{4w_1^2} + \frac{w_{1xx}}{2w_1} - \frac{w_{1x}^2}{4w_1^2}.$$

The single-soliton-like solution of the system (1.6) is obtained with respect to the variables (x,t) from (3.1) as

$$\rho_1(x,t) = w_1,$$

$$\begin{aligned} U_1(x,t) = \frac{w_0^2}{G}\{ & -v_0[256\lambda_1 k_1^4 + 96\lambda_1^2 k_1^2(v_0+2\lambda_1)^2 + (v_0+2\lambda_1)^2] + 16v_0\lambda_1 k_1(v_0+2\lambda_1)(16\lambda_1^2 k_1^2 + 4\lambda_1^2 + 4v_0\lambda_1 + v_0^2)\tanh \xi_1 \\ & + 4k_1^2\lambda_1[-384k_1^4(v_0+2\lambda_1) + 8k_1^2(16\lambda_1^3 + 52\lambda_1^2 v_0 + 12\lambda_1 v_0^2 - v_0^3) + (v_0+2\lambda_1)^2(4\lambda_1^2 + 32\lambda_1 v_0 + 3v_0^2)]\operatorname{sech}^2 \xi_1 \\ & + 16k_1^3\lambda_1^2[128k_1^4\lambda_1^3 + 8k_1^2\lambda_1(8\lambda_1^2 + 6v_0\lambda_1 + 3v_0^2) - (v_0+2\lambda_1)(12\lambda_1^2 + 40\lambda_1 v_0 + 9v_0^2)]\tanh \xi_1 \operatorname{sech}^2 \xi_1 \\ & + 64k_1^4\lambda^3[k_1^2(48\lambda_1^2 + 24v_0\lambda_1) - 12\lambda_1^2 - 28\lambda_1 v_0 - 9v_0^2]\operatorname{sech}^4 \xi_1 - 256\lambda_1^4 k_1^5(8\lambda_1 k_1^2 - 2\lambda_1 - 3v_0)\tanh \xi_1 \operatorname{sech}^2 \xi_1\} - \frac{\kappa}{2}. \end{aligned} \tag{3.5}$$

Here

$$G = (v_0 + 2\lambda_1 - 4k_1\lambda_1 \tanh \xi_1 - 4\lambda_1 k_1^2 \operatorname{sech}^2 \xi_1) \times (v_0 + 2\lambda_1 - 4k_1\lambda_1 \tanh \xi_1)^3.$$

The corresponding Darboux transformation ϕ_1 is

$$\phi_1 = \frac{w_0(\lambda_1 - \lambda - 2k_1\lambda_1 \tanh \xi_1)\cosh \xi + 2k_2\lambda \sinh \xi}{\sqrt{w_0(4k_1\lambda_1 \tanh \xi_1 - 2\lambda_1 - v_0)}}, \tag{3.6}$$

where v_0, w_0, and λ_1 are constants that satisfy $4w_0^2\lambda_1^2 + 4v_0 w_0^2\lambda_1 + 1 > 0$. Actually, due to the good properties of the Darboux transformation, the function ϕ_1 with $\lambda = \lambda_1$ in (3.6) is just the function h_1 that satisfies the following spectral equation:

$$h_{1xx} = (\lambda_2^2 + \lambda_2 v_1 + u_1)h_1. \tag{3.7}$$

By applying the DT $(n+1)$ times, we can obtain the multisolitonlike solution of the system (1.9)–(1.11) and the corresponding solution of the 2CH system (1.6),

$$v_{n+1} = v_n + \frac{-2h_{nxx}h_n + 2h_{nx}^2 + v_{nx}h_n^2}{h_n(-2h_{nx} + 2\lambda_{n+1}h_n + v_n h_n)}, \tag{3.8}$$

$$w_{n+1} = \frac{2w_n h_{nx}^2 - 2(w_{nx} + 2\lambda_{n+1}w_n)h_n h_{nx} + (w_{nxx} - 2w_n u_n + 2\lambda_{n+1}w_n + 2\lambda_{n+1}w_{nx})h_n}{\lambda_{n+1}[(2\lambda_{n+1} + v_n)h_n - 2h_{nx}]h_n}, \tag{3.9}$$

and

$$\rho_{n+1}(x,t) = w_{n+1},$$

$$\begin{aligned} U_{n+1}(x,t) = & \frac{2w_n C_{nx}C_{ntt} + 2(C_{nxt} + C_n w_{nx})C_{ntt}}{C_n(C_n w_n + 2C_{nt})} - \frac{2C_{nxtt}}{C_n} - w_{nxt} - \frac{\kappa}{2} + \frac{2(6v_n w_n - w_{nx})C_{nt}^2 - 2(w_{nt} + 6w_n^2)}{C_n(C_n w_n + 2C_{nt})} \\ & + \frac{4(C_{nt} + w_n C_n)C_{nt}C_{nxt} + 8v_n C_n C_{nt}^3}{C_n^2(C_n w_1 + 2C_{nt})} + \frac{2w_{nt}C_{nxt} - 2w_n^3 C_{2x} + 6v_n w_n C_{nt}}{C_n w_n + 2C_{nt}} + \frac{(w_{nx}w_{nt} + v_n w_n^3)C_n}{C_n w_n + 2C_{nt}}, \end{aligned} \tag{3.10}$$

where

$$C_n = \frac{1}{\sqrt{2h_{nx}/h_n - 2\lambda_{n+1} - v_n}}, \quad n = 1,2,3, \ldots ,$$

and h_n is just the function ϕ_n of (2.8) obtained by applying the DT n times. Nevertheless, the multisoliton solutions (3.8) and (3.10) are functions with respect to the new variables x and t only, and not the original variables y and s. In order to obtain the multisoliton solutions of the 2CH system (1.6), we have to get the relation of the variables (y,s) to the variables (x,t). It is difficult to get this relation directly from (3.2) because complicated functions of ρ and U for multisoliton solutions are included in (3.2). Fortunately, there is a simple way to obtain the relation of the variables (y,s) to the variables (x,t) from (1.9). We suppose that

$$w = H_t, \quad v = -H_x, \tag{3.11}$$

where H is a function with respect to the variables x and t. So for H,

$$H = \int w\, dt - \int v\, dx,$$

and it must satisfy

$$H_{tt} - H_t^2 H_{xxtt} + H_t H_{tt} H_{xxt} + H_t H_{xt} H_{xtt} + H_{xt}^2 H_{tt} + H_t^4 H_{xx} + 2H_x H_t^3 H_{xt} = 0. \tag{3.12}$$

It is easy to get the function H from (1.10), (1.11), and (3.11). For example, we know the functions w_1 and v_1,

$$w_1 = w = w_0 + 2(\ln C_1)_t, \quad v_1 = v = v_0 - 2(\ln C_1)_x,$$

and

$$H_1 = H = -v_0 x + w_0 t + 2 \ln C_1 \tag{3.13}$$

from (3.11) when the DT is applied once, and

$$y = H_1$$

which was proved in Ref. [21]. Similarly, we can obtain

$$y = H_n = H = -v_0 x + w_0 t + 2 \ln(C_1 C_2 \cdots C_n), \tag{3.14}$$

when the DT is applied n times.

In principle, we can obtain single- and multisolitonlike solutions $U(y,s)$ and $\rho(y,s)$ of the 2CH equation (1.6) from (3.8), (3.10), and (3.14). When we select $h=\cosh \xi_1$ and $\phi=\sinh \xi_2$ for the DT of ϕ in (2.17) and (2.8), we have

$$v_1 = v_0 - \frac{8k_1^2\lambda_1^2 \operatorname{sech}^2 \xi_1}{v_0 + 2\lambda_1 - 4k_1\lambda_1 \tanh \xi_1},$$

$$w_1 = w_0 - \frac{4k_1^2\lambda_1 w_0 \operatorname{sech}^2 \xi_1}{v_0 + 2\lambda_1 - 4k_1\lambda_1 \tanh \xi_1},$$

and

$$h_1 = \frac{2k_2\lambda_2 \cosh \xi_2 + (\lambda_1 - \lambda_2 - 2k_1\lambda_1 \tanh \xi_1)\sinh \xi_2}{\sqrt{4k_1\lambda_1 \tanh \xi_1 - v_0 - 2\lambda_1}}.$$

Taking $n=1$ in (3.6)–(3.8) and substituting the above functions into (3.8) and (3.12), we get the two-loop soliton solution of the function ρ_2 with respect to y and s. As we know, the loop soliton of the 2CH system is first found. In Fig. 1, we plot the single-soliton solution and a two-loop soliton solution of ρ and U of the 2CH system (1.6) with respect to the variables (y,s). There is a loop soliton solution of ρ_1 and

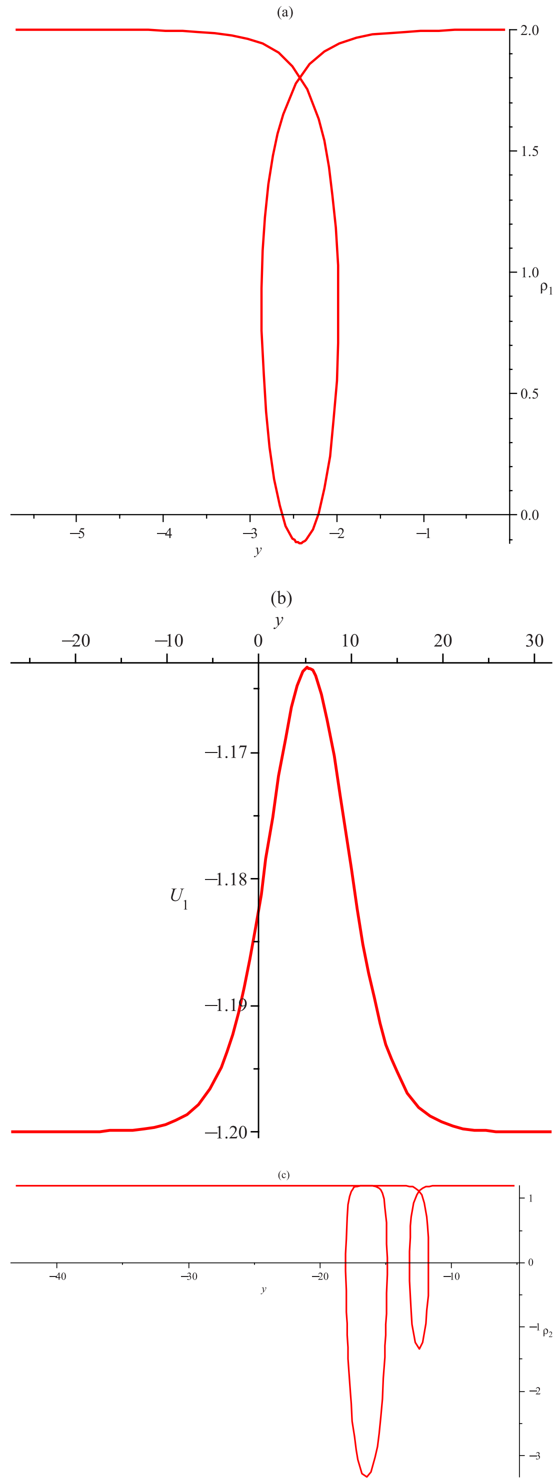

FIG. 1. (Color online) Single-soliton-like solution for ρ_1 and U_1 (3.5) with respect to y (3.14) and s when $v_0=-1.15$, $w_0=2$, $\lambda_1=-1$, $\kappa=0$, $t=-2$; and the two-loop soliton for ρ_2 (3.10) when $v_0=-1.2$, $w_0=1$, $\lambda_1=-1$, $\lambda_2=-2.0$, $t=-25$. (a) ρ_1, (b) U_1, and (c) ρ_2.

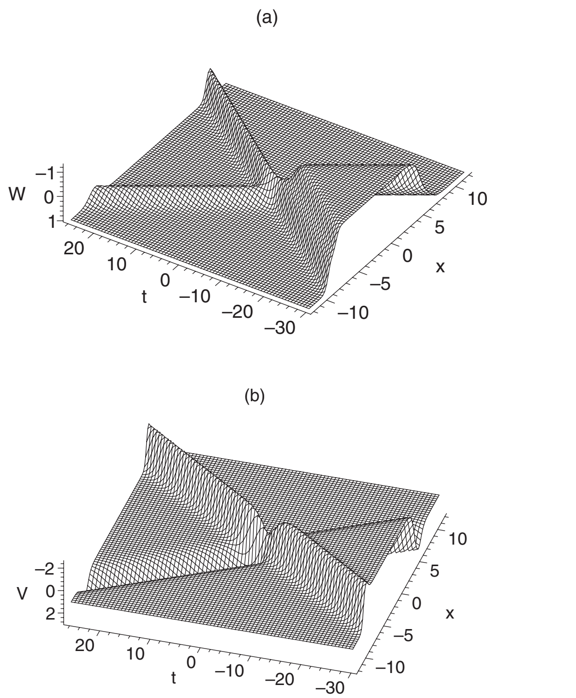

FIG. 2. Evolution of the two-soliton-like waves w_2 and v_2 in (3.8) and (3.9) with v_0=1.2, w_0=1, λ_1=1.5, λ_2=−1.8. (a) w_2 and (b) v_2.

a soliton of U in Figs. 1(a) and 1(b). Figure 1(c) shows the two-loop soliton solution of $\rho_2(y,s)$. In Fig. 2, the two-kink form evolves to the two-soliton form for the solution of w_2 while the two-soliton-like form evolves to the two-soliton form for the solution of v_2. The interaction of the two-kink (or two-soliton-like) solution is not an elastic collision. However, in Fig. 3, the interactions of the two-soliton solutions of w_2 and v_2 are elastic. The properties of the interaction for the multisoliton solutions are different for the different seed solutions v_0 and w_0.

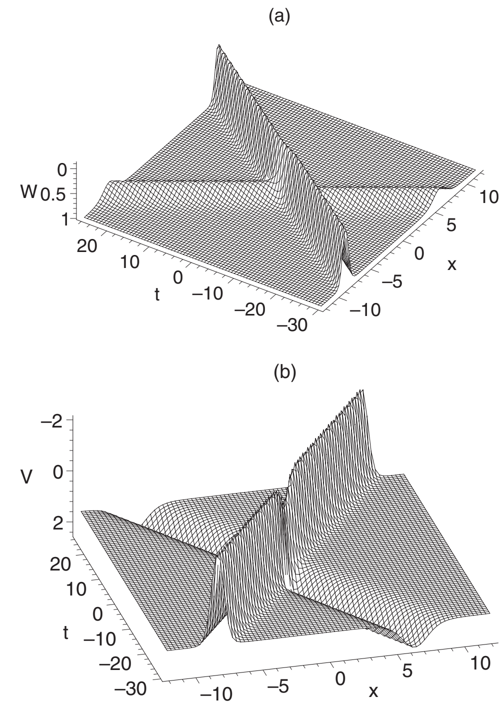

FIG. 3. Evolution of the two-soliton waves w_2 and v_2 in (3.8) and (3.9) with v_0=1.5, w_0=1, λ_1=1.6, λ_2=−1.8. (a) w_2 and (b) v_2.

B. The soliton solutions for the second Darboux transformation

As for the first Darboux transformation for the two-component CH equation (1.6), we can obtain multisoliton solutions of (1.6) by the second DT. Here, we only write the single-soliton solution of (1.6) with respect to the variables x and t:

$$\rho_1(x,t) = \frac{w_0(v_0 + 2\lambda_1 + 4k_1\lambda_1 \tanh \xi_1 - 4\lambda_1 k_1^2 \operatorname{sech}^2 \xi_1)}{v_0 + 2\lambda_1 + 4k_1\lambda_1 \tanh \xi_1}, \tag{3.15}$$

$$\begin{aligned} U_1(x,t) = {} & -\frac{\kappa}{2} + \frac{w_0^2}{G_1}\big(v_0[32k_1^2\lambda^2(8k_1^2\lambda_1^2 + 12\lambda_1^2 + 12v_0\lambda_1 + 3v_0^2) + (2\lambda_1 + v_0)^4] + 16v_0k_1\lambda_1(2\lambda_1 + v_0)[16k_1^2\lambda_1^2 + (2\lambda_1 + v_0)^2]\tanh \xi_1 \\ & + \{4\lambda_1k_1^2[384k_1^4\lambda_1^3(v_0 + 2\lambda_1) - 8\lambda_1(16\lambda_1^3 + 52v_0\lambda_1^2 - v_0^3 + 12v_0^2\lambda_1)k_1^2 - (4\lambda_1^2 + 32v_0\lambda_1 + 3v_0^2)(2\lambda_1 + v_0)^2] \\ & + 16k_1^3\lambda_1^2[128k_1^4\lambda_1^3 + 8\lambda_1(8\lambda_1^2 + 3v_0^2 + 6\lambda_1v_0)k_1^2 - (2\lambda_1 + v_0)(9v_0^2 + 40\lambda_1v_0 + 12\lambda_1^2)]\tanh \xi_1\}\operatorname{sech}^2 \xi_1 \\ & - \{64\lambda_1^3k_1^4[24\lambda_1k_1^2(2\lambda_1 + v_0) - 28\lambda_1v_0 - 9v_0^2 - 12\lambda_1^2] + 256\lambda_1^4k_1^5(8k_1^2\lambda_1 - 2\lambda_1 - 3v_0)\tanh \xi_1\}\operatorname{sech}^4 \xi_1\big). \end{aligned} \tag{3.16}$$

Here

$$G_1 = (v_0 + 2\lambda_1 + 4k_1\lambda_1 \tanh \xi_1)^3(4k_1^2\lambda_1 \operatorname{sech}^2 \xi_1 - 4k_1\lambda_1 \tanh \xi_1 - v_0 - 2\lambda_1).$$

IV. THE BIDIRECTIONAL SOLITONS OF THE (1+1)-DIMENSIONAL WZ EQUATION

We can get many multisoliton solutions for the (1+1)-dimensional WZ equation (1.2) by using the two DTs and selecting different seed solutions for (1.2). Here, we focus only on investigating the bidirectional soliton solutions of (1.2) because these solitons can be used to study water waves during coastal and harbor design. We can directly get bidirectional multisoliton solutions of (1.2) via the DT, whereas Zhang and Li [14] had to modify the WZ equation to AKNS to obtain bidirectional solitons. We discuss only the bidirectional solitons of (1.2) obtained by the first DT. Selecting the seed solution of (1.2) as $u=-1$, $v=0$, which is consistent with the condition for the propagation of water waves [14], and taking

$$h=\cosh \xi_1, \quad \phi=\cosh \xi,$$

$$\xi_1=k_1(x+\lambda_1 t), \quad k_1=\sqrt{\lambda_1^2-1}, \quad |\lambda_1|>1, \tag{4.1}$$

(ξ has the same form as ξ_1 except for replacing λ_1 with λ), the functions c_0 and d_0 can be obtained from (2.7). So the one-soliton solution of the system (1.2) is given by (2.9) and (2.10):

$$v_1=\frac{k_1^2 \operatorname{sech}^2 \xi_1}{k_1 \tanh \xi_1-\lambda_1},$$

$$u_1=\frac{4\lambda_1(4\lambda_1^2-3)-4k_1(4\lambda_1^2-1)\tanh \xi_1+8\lambda_1 k_1^2(\lambda_1^2-2)\operatorname{sech}^2 \xi_1-7\lambda_1 k_1^4 \operatorname{sech}^2 \xi_1}{4(k_1 \tanh \xi_1-\lambda_1)^3}-\frac{[4(2\lambda_1^2-1)-3k_1^2 \operatorname{sech}^2 \xi_1]\tanh \xi_1 \operatorname{sech}^2 \xi_1}{4(k_1 \tanh \xi_1-\lambda_1)^3}. \tag{4.2}$$

Finally, the single-soliton solutions of the $(1+1)$-dimensional WZ equation (1.1) have

$$\eta_1=v_1,$$

$$\zeta_1=\frac{2k_1^2\lambda_1(1-2\lambda_1^2+2k_1\lambda_1 \tanh \xi_1)\operatorname{sech}^2 \xi_1+k_1^4(3\lambda_1-k_1 \tanh \xi_1)\operatorname{sech}^4 \xi_1}{(k_1 \tanh \xi_1-\lambda_1)^3}. \tag{4.3}$$

When the DT is applied twice, the two-soliton solutions of the WZ system (1.1) are obtained:

$$\eta_2=\frac{2h_{1x}(-h_{1x}+v_1 h_1)+(v_{1x}-2\lambda_2^2+v_1^2-2u_1)h_1^2}{h_1(-2h_{1x}+2\lambda_2 h_1+v_1 h_1)}, \tag{4.4}$$

$$\begin{aligned}\zeta_2=&-\frac{h_1^2 v_{1x}^2+2h_1(2\lambda_2+v_1)(2\lambda_2^2+2\lambda_2 v_1+v_1^2-2u_1-4)h_{1x}}{2(-2h_{1x}+2\lambda_2 h_1+v_1 h_1)^2}+\frac{h_1[(\lambda_2^2+6\lambda_2 v_1+v_1^2+4u_1)h_1-4h_{1x}(\lambda_2+v_1)]v_{1x}}{2(-2h_{1x}+2\lambda_2 h_1+v_1 h_1)^2}\\&+\frac{h_1(v_{1xx}-2u_{1x})}{2(-2h_{1x}+2\lambda_2 h_1+v_1 h_1)}-\frac{2h_{1x}^2[h_{1x}^2-(2\lambda_2+v_1)h_1 h_{1x}+2h_1^2]}{h_1^2(-2h_{1x}+2\lambda_2 h_1+v_1 h_1)^2}\\&+\frac{[(4\lambda_2+v_1)^2-4]v_1^2+16\lambda_2 v_1(\lambda_2^2-u_1-1)-8u_1(2\lambda_2^2+u_1)+8\lambda_2(\lambda_2^2-2)}{4(-2h_{1x}+2\lambda_2 h_1+v_1 h_1)^2}.\end{aligned} \tag{4.5}$$

In order to illustrate that the solutions η_2 and ζ_2 are two-soliton solutions of the WZ equation (1.1), we show the evolution of the two-soliton solution ζ_2 with $\lambda_1=-1.5$, $\lambda_2=1.25$ in Fig. 4. This interaction of two solitons is a head-on collision.

As we know, the WZ equation (1.1) allows bidirectional water wave interaction. It is important to obtain the two-soliton solution that can express an overtaking collision. We do not obtain the two-soliton overtaking solution from (4.4) and (4.5) by selecting $\lambda_{1,2}>1$ (or $\lambda_{1,2}<-1$). Therefore, we need to select another solution of h in (2.17) in order to get the two-soliton overtaking solution of the WZ equation (1.1). We take h and ϕ in (2.17) and (1.3) as follows:

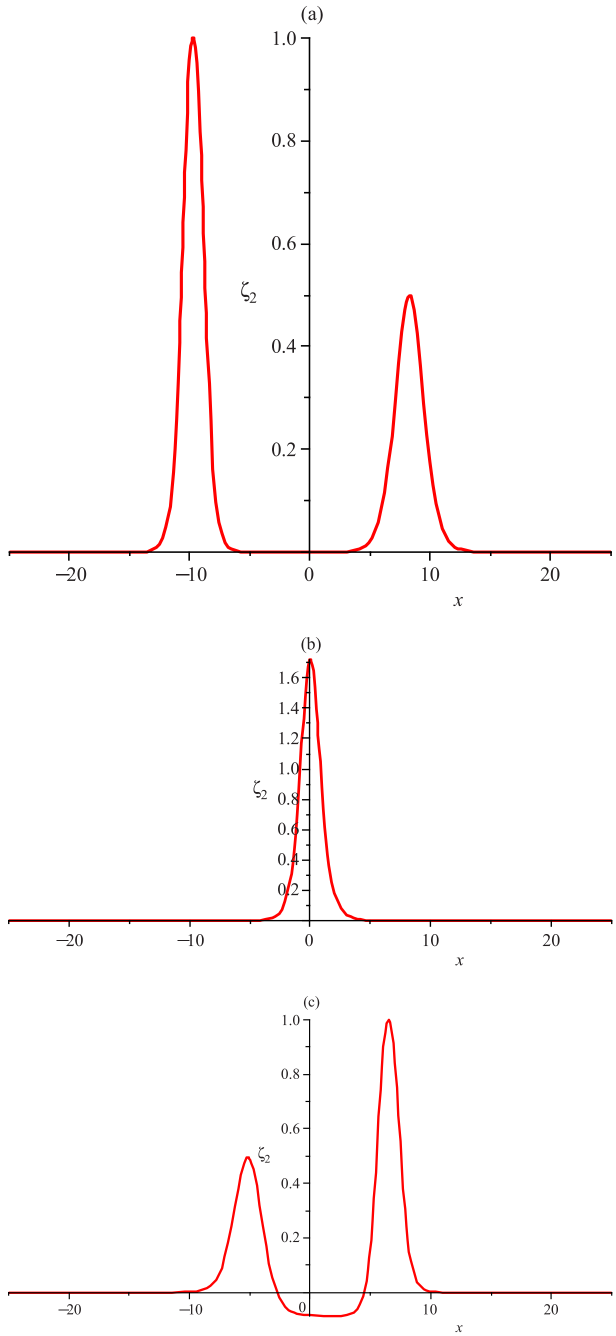

FIG. 4. (Color online) Evolution of the two-soliton head-on collision for ζ_2 (4.5) with $\lambda_1=-1.5$ and $\lambda_2=1.25$: (a) $t=-6$, (b) $t=0.5$, and (c) $t=6$.

$$h=\frac{11}{2}\cosh\,\xi_1-2\,\sinh\,\xi_1,\quad \phi=-\cosh\,\xi+3\,\sinh\,\xi, \tag{4.6}$$

and obtain the single-soliton solution of the system (1.2) as

$$v_1=\frac{105k_1^2\,\mathrm{sech}^2\,\xi_1}{(11-4\,\tanh\,\xi_1)[-4k_1-11\lambda_1+(11k_1+4\lambda_1)\tanh\,\xi_1]}, \tag{4.7}$$

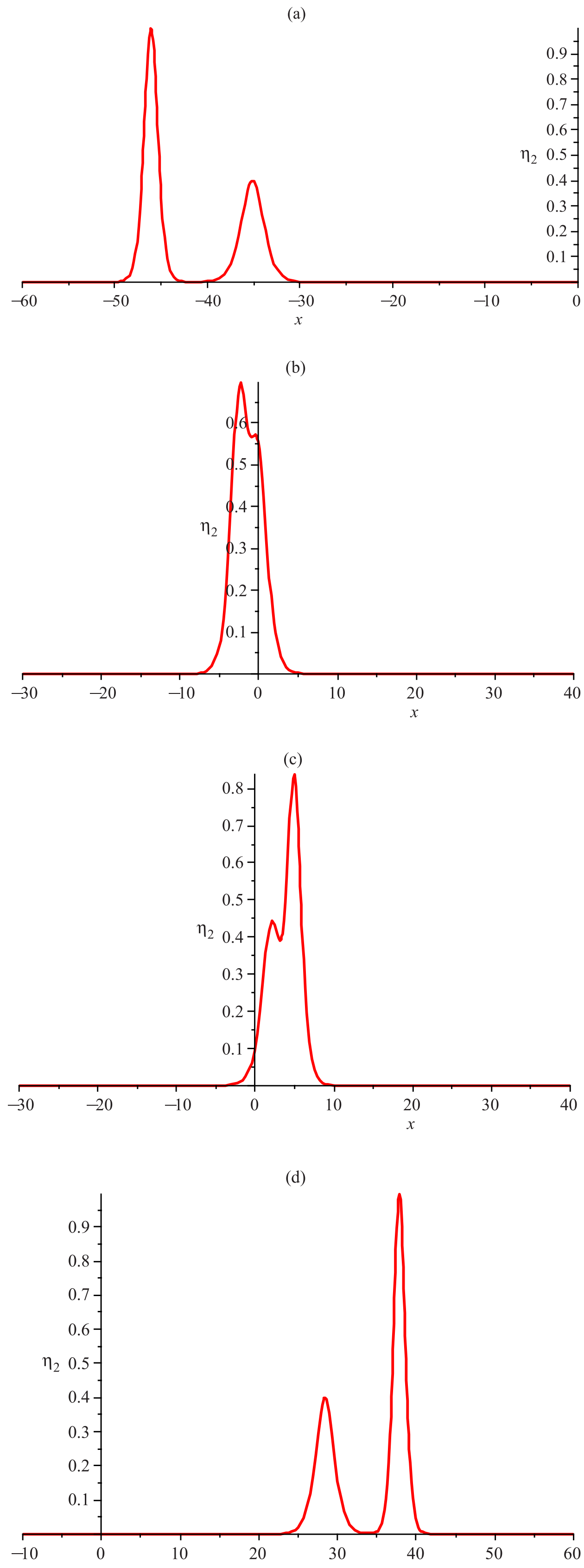

FIG. 5. (Color online) Evolution of the two-soliton overtaking collision for η_2 (4.4) with (4.7)–(4.9) and $\lambda_1=-1.2$ and $\lambda_2=-1.5$: (a) $t=-30$, (b) $t=-0.75$, (c) $t=3$, and (d) $t=25$.

$$u_1 = \frac{\chi_0 + \chi_1 \tanh \xi_1 + \chi_2 \operatorname{sech}^2 \xi_1 + \chi_3 \tanh \xi_1 \operatorname{sech}^2 \xi_1 + \chi_4 \operatorname{sech}^4 \xi_1}{(11 - 4 \tanh \xi_1)[4k_1 + 11\lambda_1 - (11k_1 + 4\lambda_1)\tanh \xi_1]}, \tag{4.8}$$

where $\chi_0 = 48\,224k_1\lambda_1 + 53\,026\lambda_1^2 - 26\,513$, $\chi_1 = 24\,112 - 48\,224\lambda_1^2 - 53\,026\lambda_1 k_1$, $\chi_2 = 88\lambda_1 k_1(105\lambda_1^2 - 548) + 26\,513 + 14\,385\lambda_1^4 - 53\,026\lambda_1^2$, $\chi_3 = (26\,513 - 14\,385\lambda_1^2)\lambda_1 k_1 - 9240\lambda_1^4 + 24\,112\lambda_1^2 - 12\,056$, $\chi_4 = -44(105\lambda_1^2 - 137)\lambda_1 k_1 - (39\,795\lambda_1^4 - 40\,819\lambda_1^2 - 40\,819)/4$, and

$$h_1 = -\frac{\{4k_1 - 22k_2 + 11(\lambda_1 - \lambda_2) + [8k_2 - 11k_1 + 4(\lambda_1 + \lambda_2)]\tanh \xi_1\}\cosh \xi_2}{\sqrt{[-8k_1 - 22\lambda_1 - (8\lambda_1 + 22k_1)\tanh \xi_1](11 - 4\tanh \xi_1)}} - \frac{\{11k_2 - 8k_1 - 22(\lambda_1 - \lambda_2) + [8k_2 - 11k_1 + 4(\lambda_1 + \lambda_2)]\tanh \xi_1\}\sinh \xi_2}{\sqrt{[-8k_1 - 22\lambda_1 - (8\lambda_1 + 22k_1)\tanh \xi_1](11 - 4\tanh \xi_1)}}. \tag{4.9}$$

Finally, we obtain the two-soliton overtaking solution of the WZ equation (1.1) by substituting (4.7)–(4.9) into (4.4) and (4.5). In Fig. 5, the evolution of the two-soliton overtaking collision is shown for the function η_2. Actually, the $2N$-soliton solution of the WZ system (1.1) is given when the DT is applied $2N$ times.

V. CONCLUSION

We investigated the Darboux transformations of the (1+1)-dimensional WZ equation and the 2CH system and obtained two types of DT for (1.1) and (1.6). For studying the DT of the two-component CH equation, we transformed (1.6) into (1.9)–(1.11) [whose Lax pair is almost the same as that of (1.2)] by the reciprocal transformation and found that (1.6) has the same DT as (1.1). We obtained a single-loop solution, a two-loop solution, and multisoliton (or multisolitonlike) solutions of the 2CH system with the two types of DT. We also discussed the evolutions of the two-soliton (or two-soliton-like) solutions of w_2 and v_2 with different seed solutions w_0 and v_0, and the evolutions of the soliton solutions. When the seed solution is $v_0 = 1.2$, $w_0 = 1.0$ and the wave numbers are $\lambda_1 = 1.5$, $\lambda_2 = -1.8$, the interaction of two solitons for v_2 and w_2 is inelastic because the soliton changes shape, while the interaction of two solitons is elastic when the seed solution is $v_0 = 1.5$, $w_0 = 1.0$ and the wave numbers are $\lambda_1 = 1.6$, $\lambda_2 = -1.8$. For the (1+1)-dimensional WZ equation, we directly obtained the bidirectional soliton solution of (1.2) by the DT. By selecting the seed solution $u = -1$, $v = 0$ which has the physical meaning of the (1+1)-dimensional WZ equation, we derived the multisoliton solutions of (1.1) by the first DT. We also obtained the two-soliton head-on and overtaking collisions of the WZ equation by taking different solutions h of the Lax pair Eq. (2.17) with different seed solutions. These two-soliton collisions can be used to illustrate the bidirectional propagation of water waves in shallow water.

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