Riemannian geometric approach to chaos in SU(2) Yang-Mills theory

Tetsuji Kawabe* and Shin'ichiro Koyanagi
Physics Department, Department of Acoustic Design, Kyushu University, Shiobaru, Fukuoka 815-8540, Japan

(Received 10 October 2007; published 26 March 2008)

Based on the Riemannian geometric approach to Hamiltonian systems with many degrees of freedom, we study a chaotic nature of the SU(2) Yang-Mills field. Particularly, we study the Lyapunov exponent of the Wu-Yang magnetic-monopole solution of the SU(2) Yang-Mills field equation by use of an analytic formula which is determined by the average Ricci curvature and its fluctuation on the Riemannian manifold. It is shown that the system is chaotic from the positive values of the Lyapunov exponent. Furthermore, we find that the energy dependence of Lyapunov exponents exhibits a crossover phenomenon. By using the linear stability analysis, we point out that this crossover is related to the instability of the monopole solution.

DOI: 10.1103/PhysRevE.77.036222 PACS number(s): 05.45.−a, 11.15.Tk, 02.40.Ky

I. INTRODUCTION

Recently there has been much attention to the investigation about Hamiltonian chaos from the Riemannian geometric approach (RGA) [1]. This approach allows us to relate chaotic dynamics to curvature properties of manifolds by regarding the trajectory of a dynamical system as a geodesic on the Riemannian manifold endowed with a suitable metric. The formalism based on the RGA has been applied to the two-dimensional Hamiltonian systems such as the Hénon-Heiles system [2], the homogeneous Yang-Mills-Higgs system [3], the Abelian-Higgs system [4], and other systems [5]. These systems have been also studied by the numerical approach using the standard techniques [6] of testing for chaotic motions. Henceforth we refer to this approach as the conventional approach. From the comparison between the results by the conventional approach and the RGA, it is confirmed that two different approaches yield consistent results on chaos in the dynamical systems with two degrees of freedom, $N=2$.

For the Hamiltonian systems with large N degrees of freedom, this RGA gives an analytic formula of the Lyapunov exponent (LE) in terms of the average quantities associated with curvature properties. Since the LE is a reliable indicator of chaos for dynamical systems, many efforts have been done to calculate the LE numerically in the conventional approach. The numerical estimation of the LE, however, has some inevitable problems on the CPU time and the accuracy of the numerical integration. Even for the estimation of one value of the LE at a given energy, it takes a lot of CPU time to obtain its asymptotic value through the long-term evolution of the systems with large N degrees of freedom. When we need to study the energy dependence of the LE, we have to consume tremendous CPU time in the conventional approach. Thus the analytic formula of the LE by the RGA is a very attractive tool that conveys more information on the chaos that goes beyond the conventional approach.

Indeed, this analytic formula has been applied to the systems with large N degrees of freedom such as the Fermi-Pasta-Ulam β (FPU-β) model [7,8], the lattice ϕ^4 classical model [8,9], and other models [10,11], and it is found that the behavior of LE exhibits a crossover as a function of the energy density $\varepsilon=E/N$. It has been pointed out that the crossover implies the existence of weak and strong chaos corresponding to the qualitatively different regimes in the dynamics. The transition between weak and strong chaos is a very important phenomenon to measure the degree of chaoticity and its relevance is also discussed in connection with the phase transition [9,11].

The purpose of this paper is to study the chaos of the SU(2) Yang-Mills (YM) gauge field depending on space and time by applying the analytic formula of the LE [1]. Mainly we focus on the Wu-Yang magnetic-monopole solution of the SU(2) YM classical field equation, and we study the behavior of its LE. We show that the dependence of the LE on the energy density ε has crossover as in the case of the FPU-β model and the lattice ϕ^4 classical model. This YM system has been studied for a long time in the hope that it may provide new knowledge on the vacuum structure of gauge field theories. Based on the various numerical simulations in the conventional approach, it has been shown that the YM system is intrinsically chaotic [12,13]. However, the above-mentioned crossover was not revealed because the simulation was not done for a wide range of energies enough to detect such a phenomenon. The chaotic properties of gauge fields also have been investigated by the method of the lattice gauge fields, and the role of chaotic gauge fields has been discussed, e.g., for the confinement of quarks and the phenomena in high energy physics [14,15]. Thus this RGA will bring a new feature of chaos of the SU(2) gauge field theories which might be difficult to find by the conventional approach, and which might connect to the findings in the lattice gauge fields.

This paper is organized as follows. In Sec. II, we give a brief review about the RGA for the Hamiltonian chaos and the analytic formula of the LE. In Sec. III, we present the Wu-Yang magnetic-monopole solution of the SU(2) YM classical field equation, and derive the quantities on the curvature of the Riemannian manifold necessary for estimating the LE. In Sec. IV, we show the numerical results for the LE of the YM system. Our conclusions are summarized in Sec. V.

*kawabe@design.kyushu-u.ac.jp

II. RIEMANNIAN GEOMETRIC DESCRIPTION OF INDICATOR FOR HAMILTONIAN CHAOS

In this section, we will briefly describe the Riemannian geometric description of a chaos indicator for a Hamiltonian with many degrees of freedom by closely following the works of Casetti, Pettini, and Cohen [1]. The basic idea of the RGA starts with the identity between the trajectories of a dynamical system and the geodesics in its configuration space on a suitable Riemannian manifold. The instability of the trajectories corresponds to the instability of the geodesics. This geodesic instability can be measured by time evolution of the geodesic variation field called the Jacobi field $\boldsymbol{J}$, which is defined by the spread between nearby geodesics. The evolution of the Jacobi field can be expressed by the Jacobi equation as follows [1]:

$$\frac{D^2\boldsymbol{J}}{Ds^2}+R(\boldsymbol{J},\boldsymbol{v})\boldsymbol{v}=0, \tag{1}$$

where s is the proper time, D/Ds is the covariant derivative along the geodesic, $R(\cdot,\cdot)$ is the Riemann curvature tensor, and $\boldsymbol{v}=d\boldsymbol{q}/ds$ is the velocity of the geodesic. The Jacobi equation (1) comes from the second-order variation of the geodesic equation with respect to a perturbed geodesic, and then relates the instability of the Jacobi field $\boldsymbol{J}$ to the curvature properties of the underlying manifold. Since the instability can be estimated by the norm $\|\boldsymbol{J}\|$ of the Jacobi field, it is effective to consider the norm equation by multiplying Eq. (1) by $\boldsymbol{J}$ as follows:

$$\left\langle \frac{D^2\boldsymbol{J}}{Ds^2},\boldsymbol{J}\right\rangle+\langle R(\boldsymbol{J},\boldsymbol{v})\boldsymbol{v},\boldsymbol{J}\rangle=0, \tag{2}$$

where $\langle X,Y\rangle\equiv g_{ij}X^iY^j$ stands for a scalar product. For the norm $\Psi\equiv\|\boldsymbol{J}\|=\sqrt{\langle \boldsymbol{J},\boldsymbol{J}\rangle}$ the norm Eq. (2) can be written as follows:

$$\frac{d^2\Psi}{ds^2}+k(s)\Psi=0, \tag{3}$$

where $k(s)=k(\boldsymbol{J},\boldsymbol{v})$ is the sectional curvature of the geodesic plane spanned by the directions $\boldsymbol{J}$ and $\boldsymbol{v}$. This expression of Eq. (3) is independent on a particular choice of the metric. Specializing to the Eisenhart metric for the arc length s, i.e., $ds^2=dt^2$, the norm Eq. (3) is rewritten in terms of the physical time t as follows:

$$\frac{d^2\Psi}{dt^2}+k(t)\Psi=0. \tag{4}$$

In the RGA, the norm Eq. (4) is assumed to be a stochastic oscillator equation with frequency $\sqrt{k(t)}$. Then the squared frequency $k(t)$ can be modeled by a Gaussian and δ-correlated stochastic process as follows [7]:

$$k(t)=k_0+\sqrt{\sigma_k^2}\,\eta(t), \tag{5}$$

where $\eta(t)$ is the fluctuation by the random Gaussian process, and the mean k_0 is given by the average of the Ricci curvature per degree of freedom, $k_R=K_R/N$, along a geodesic and the variance σ_k is by the rms fluctuation of k_R. The Ricci curvature K_R is defined as $K_R=R_{ij}v^iv^j$, where R_{ij} is the Ricci tensor and $v^i=dq^i/ds$ is velocity. Under the Eisenhart metric, the nonvanishing component of the Ricci tensor R_{ij} is R_{00} alone. Since the relation $q^0=t=s$ holds in this metric, the Ricci curvature reduces to $K_R=R_{00}v^0v^0=R_{00}$ because of $v^0=1$. The component R_{00} is determined by the potential function V of the Hamiltonian system as $R_{00}=\Delta V$, where Δ is the Laplacian operator. Thus the Ricci curvature k_R is given by

$$k_R(t)=\frac{K_R}{N}=\frac{\Delta V}{N}. \tag{6}$$

From Eq. (6) we can calculate the mean k_0 in Eq. (5) by the time average of k_R as

$$k_0\equiv\langle k_R(t)\rangle=\frac{1}{T}\int_t^{t+T}k_R(t')dt', \tag{7}$$

and the variance σ_k in Eq. (5) by the rms fluctuation of k_R as

$$\sqrt{\sigma_k^2}=\sqrt{\langle[k_R(t)-k_0]^2\rangle}. \tag{8}$$

In order to determine whether or not the YM system is chaotic, we evaluate the maximal LE λ. This exponent λ is defined as

$$\lambda=\lim_{t\to\infty}\frac{1}{t}\ln\frac{d(t)}{d(0)}, \tag{9}$$

where $d(t)=\sqrt{\Psi^2(t)+\dot{\Psi}^2(t)}$ is the distance between two neighboring Jacobi fields. Following the solving method for the stochastic oscillator Eq. (4), the LE λ can be given by [1,7]

$$\lambda(k_0,\sigma_k)=\frac{1}{2}\left(\Lambda-\frac{4k_0}{3\Lambda}\right), \tag{10}$$

where

$$\Lambda=\left(\sigma_k^2\tau+\sqrt{\left(\frac{4k_0}{3}\right)^3+\sigma_k^4\tau^2}\right)^{1/3}, \tag{11}$$

$$\tau=\frac{\pi\sqrt{k_0}}{2\sqrt{k_0(k_0+\sigma_k)}+\pi\sigma_k}.$$

Therefore in the RGA we can estimate the LE λ through the analytic formula (10) which is determined only by such geometric quantities as k_0 and σ_k.

III. APPLICATION TO WU-YANG MAGNETIC-MONOPOLE SOLUTION OF SU(2) YM FIELD EQUATION

We consider the SU(2) YM field equation as

$$\partial_\mu F^a_{\mu\nu}+\varepsilon^{abc}A^b_\mu F^c_{\mu\nu}=0, \tag{12}$$

where $F^a_{\mu\nu}$ is the field tensor as

$$F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu-\varepsilon^{abc}A^b_\mu A^c_\nu, \tag{13}$$

and the upper indices of the gauge field A^a_μ refer to the SU(2) group $(a=1,2,3)$, and the lower ones refer to the Lorentz group $(\mu=0,1,2,3)$. Under the time-dependent, spherically symmetric ansatz [16]

$$A_i^a(\boldsymbol{x},t) = -\varepsilon_{aij} x_j \frac{\phi(r,t)+1}{r^2}, \tag{14}$$

and the gauge condition $A_0^a=0$, the YM Eq. (12) reduces to a classical field equation in $1+1$ dimensions as follows:

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial r^2} = \frac{1}{r^2}(1-\phi^2)\phi, \tag{15}$$

where $r=|\boldsymbol{x}|$. The Hamiltonian of the system is given by

$$H = 4\pi \int_0^\infty dr \left[\frac{1}{2}\left(\frac{\partial \phi}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial \phi}{\partial r}\right)^2 + \frac{1}{4r^2}(1-\phi^2)^2 \right]. \tag{16}$$

It is noticed that this system (16) is equivalent to the nonlinear string under the Wu-Yang potential $U(\phi)=(1-\phi^2)^2/(4r^2)$.

The discretized version in space of the Hamiltonian (16) is given by

$$H = 4\pi h \left[\sum_{i=1}^{N} \frac{1}{2}\left(\frac{\partial \phi_i}{\partial t}\right)^2 + V(\phi_i) \right], \tag{17}$$

$$V(\phi_i) = \sum_{i=1}^{N+1} \frac{1}{2h^2}(\phi_i - \phi_{i-1})^2 + \sum_{i=1}^{N} \frac{1}{4(ih)^2}(1-\phi_i^2)^2, \tag{18}$$

where the coordinate r is subdivided into a small interval of length h as $r_i=ih$, with $i=1,2,\ldots,N$, and the function ϕ_i stands for $\phi(r_i,t)$. This Hamiltonian (17) describes a set of N nonlinear coupled oscillators under the potential V which consists of the harmonic potential and the Wu-Yang one. By taking the Laplacian of the potential V of Eq. (18) as $\Delta V = \partial^2 V/\partial \phi_i^2$, the Ricci curvature k_R of Eq. (6) can be calculated as follows:

$$\tilde{k}_R(t) \equiv h^2 k_R(t) = 2 + \frac{1}{N}\sum_{i=1}^{N} \frac{1}{i^2}(3\phi_i^2 - 1). \tag{19}$$

Here we define the rescaled Ricci curvature $\tilde{k}_R$ free of h. Similarly we define the rescaled variance $\tilde{\sigma}_k = h^2\sigma_k$ and the rescaled Lyapunov exponent $\tilde{\lambda}=h\lambda$.

It is well known that the YM equation (15) has five static solutions $\phi_0(r)$ [17]. Among them we focus on the Wu-Yang magnetic-monopole solution $\phi_{WY}(r)=0$ and study its chaotic property. In order to calculate the Ricci curvature $\tilde{k}_R(t)$ for the Wu-Yang monopole solution $\phi_{WY}(i)$ under perturbation, we approximate the fields $\phi(i,t)$ as a sum of $\phi_{WY}(i)$ and the perturbation around it as follows:

$$\phi(i,t) = \phi_{WY}(i) + \sqrt{\frac{2}{N}} \sum_{j=1}^{N-1} \psi(j,t) \sin\left(\frac{\pi i j}{N}\right). \tag{20}$$

The perturbation term of Eq. (20) is given by a combination of various harmonic modes by N coupled nonlinear oscillators. Following Ref. [13], the initial perturbation is given by exciting the $j=(N/2)$th mode alone in Eq. (20), i.e., $\psi(j,0)=0$ except for $j=N/2$, so that the strength of the initial perturbation is determined by the amplitude $A \equiv \psi(N/2,0)$. We

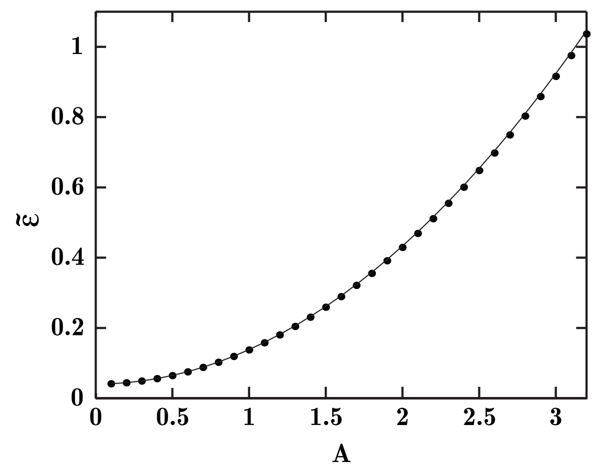

FIG. 1. Plot of the energy density $\tilde{\varepsilon}=\tilde{E}/N$ vs the amplitude A^2 of the perturbation. The solid line is depicted by $\tilde{\varepsilon}=4\pi(A^2+\pi^2/24)/N$ for $N=128$.

adopt the boundary and initial conditions as follows:

$$\phi(0,t) = \phi(N,t) = 0, \quad \dot{\phi}(i,0) = 0. \tag{21}$$

For the total energy E of the YM system, we define the rescaled energy $\tilde{E}=hE$ because we can rewrite Eqs. (17) and (18) as follows:

$$E = \frac{4\pi}{h}\left[\frac{1}{2}\sum_{i=1}^{N} \left(\frac{\partial \phi_i}{\partial \tau}\right)^2 + \frac{1}{2}\sum_{i=1}^{N+1} (\phi_i - \phi_{i-1})^2 + \frac{1}{4}\sum_{i=1}^{N} \frac{1}{i^2}(1-\phi_i^2)^2 \right], \tag{22}$$

where $\tau = t/h$. Under the perturbation of Eq. (20) with the initial conditions of Eq. (21), the initial energy $\tilde{E}_0$ can be expressed in terms of the amplitude A as follows:

$$\tilde{E}_0 = 4\pi\left\{ A^2 + \frac{1}{4}\left[\frac{1}{4}\sum_{k=1}^{N/2} \frac{1}{k^2} + \left(1 - \frac{2}{N}A^2\right) \sum_{k=1}^{2N/2} \frac{1}{(2k-1)^2} \right] \right\}, \tag{23}$$

where the first term A^2 in parentheses comes from the harmonic potential and the remainder comes from the Wu-Yang potential.

IV. NUMERICAL RESULTS

In order to study the behavior of the LE λ as a function of energy density $\tilde{\varepsilon}=\tilde{E}/N$, we take the amplitude A of the initial perturbation in the range [0.1, 3.2] with $\Delta A=0.1$. Figure 1 shows the relation between $\tilde{\varepsilon}=\tilde{E}_0/N$ and A for the case of $N=128$. For the case of $N \gg 1$, the dependence of $\tilde{\varepsilon}$ on A roughly obeys the parabolic curve $\tilde{\varepsilon}(A)=4\pi(A^2+\pi^2/24)/N$ as plotted with the solid line in Fig. 1, which can be derived from Eq. (23) for $N\to\infty$ because of $\sum_{k=1}^{\infty} 1/k^2 = \pi^2/6$ and $\sum_{k=1}^{\infty} 1/(2k-1)^2 = \pi^2/8$.

Figure 2 shows the result of the LE $\tilde{\lambda}=h\lambda$ of Eq. (10) vs $\tilde{\varepsilon}=\tilde{E}/N$ for the case of $N=128$. It is clear that this YM sys-

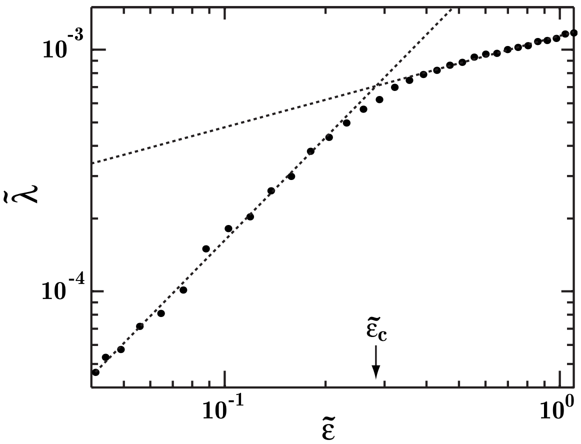

FIG. 2. Lyapunov exponent $\tilde{\lambda}$ of Eq. (10) vs the energy density $\tilde{\varepsilon}=\tilde{E}/N$ for $N=128$. The crossover occurs at $\tilde{\varepsilon}_c=0.283$.

tem is chaotic because of $\tilde{\lambda}>0$. We can see that $\tilde{\lambda}$ exhibits a crossover at the critical value of the threshold, $\tilde{\varepsilon}_c\approx 0.283$, whose value is determined by the crossing of the two asymptotes. The dependence of $\tilde{\lambda}$ on $\tilde{\varepsilon}$ is characterized by the power laws, i.e., $\tilde{\lambda}(\tilde{\varepsilon})\propto\tilde{\varepsilon}^{d_1}$ for $\tilde{\varepsilon}<\tilde{\varepsilon}_c$ and $\tilde{\lambda}(\tilde{\varepsilon})\propto\tilde{\varepsilon}^{d_2}$ for $\tilde{\varepsilon}>\tilde{\varepsilon}_c$, where $d_1\approx 1.41$ and $d_2\approx 0.378$. Since the analytic formula of the LE of Eq. (10) depends on the geometric quantities such as $\tilde{k}_0$ and $\tilde{\sigma}_k^2$, let us show the dependence of these quantities on $\tilde{\varepsilon}$ hereinafter.

Figure 3 shows the result of the mean of the Ricci curvature $\tilde{k}_0=h^2k_0$ of Eq. (7) vs the energy density $\tilde{\varepsilon}$ for the case of $N=128$, where we take $T=5\times 10^4$. The mean $\tilde{k}_0$ has a dip structure, i.e., it decreases with $\tilde{\varepsilon}$ and reaches its minimum $\tilde{k}_0^*\approx 1.999$ and then begins to increase. The value of $\tilde{k}_0^*$ seems to be reasonable from the lower limits determined by Eqs. (7) and (19) as

$$\tilde{k}_0 = 2 + \frac{1}{NT}\int_t^{t+T}\sum_{i=1}^{N}\frac{3\phi_i^2}{i^2}dt - \frac{1}{N}\left(\sum_{i=1}^{\infty}\frac{1}{i^2} - \sum_{i=N+1}^{\infty}\frac{1}{i^2}\right)$$

$$\geq 2 - \frac{1}{N}\sum_{i=1}^{\infty}\frac{1}{i^2} = 2 - \frac{\pi^2}{6N} \approx 1.987. \qquad (24)$$

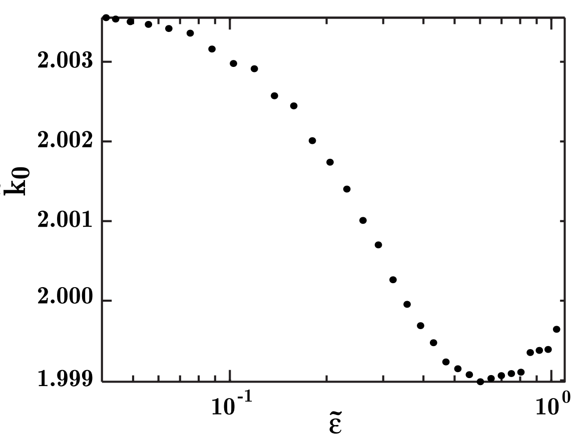

FIG. 3. Mean of the Ricci curvature $\tilde{k}_0$ of Eq. (7) vs the energy density $\tilde{\varepsilon}=\tilde{E}/N$ for $N=128$ and $T=5\times 10^4$.

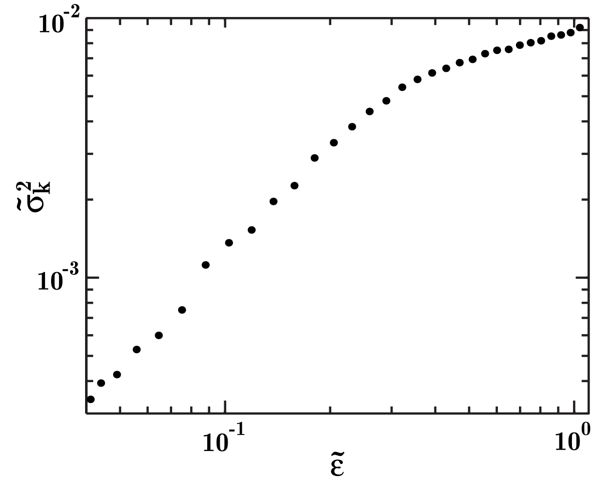

FIG. 4. Fluctuation $\tilde{\sigma}_k^2$ of Eq. (8) vs the energy density $\tilde{\varepsilon}=\tilde{E}/N$ for $N=128$.

Figure 4 shows the result of the fluctuation $\tilde{\sigma}_k^2=h^4\sigma_k^2$ of Eq. (8) vs $\tilde{\varepsilon}$ for the case of $N=128$. At first glance, we notice that the behavior of $\tilde{\sigma}_k^2$ is almost the same as one of the LE $\tilde{\lambda}$ in Fig. 2. This similarity between $\tilde{\sigma}_k^2$ and $\tilde{\lambda}$, however, is not surprising as long as the relation $\tilde{\sigma}_k\ll\tilde{k}_0$ holds. From the comparison between $\tilde{\sigma}_k^2$ of Fig. 4 and $\tilde{k}_0$ of Fig. 3, we can confirm that such a relation holds. In the limit $\tilde{\sigma}_k\ll\tilde{k}_0$ we can derive from Eq. (10) as [1]

$$\tilde{\lambda}\propto\tilde{\sigma}_k^2. \qquad (25)$$

Thus we understand that both behaviors of $\tilde{\lambda}$ and $\tilde{\sigma}_k^2$ become almost the same as shown in Figs. 2 and 4.

For the numerical integration for the equations of motion obtained from the Hamiltonian (17) and (18), we used two different methods to check the reliability of our results. One is a fourth-order Runge-Kutta (RK) method with a time step Δt of 1.25×10^{-3} and set $h=0.1$. The accuracy of the integration has been kept within 10^{-5} after 4×10^7 iterations, i.e., $T=5\times 10^4$. The other is a second-order symplectic method [18], whose accuracy of the integration is known to be better than the standard RK method. By using this symplectic method with the same values of Δt and h, we have executed a long-term evolution of the system within an accuracy of 10^{-6}, and obtained qualitatively the same results as those by the RK method. For this reason, we showed only the results obtained by the RK method. As for the lattice size N, we chose several values such as $N=32,64,128,256$ and $N=512$. As the qualitative results remain almost the same, we presented our results only for $N=128$. Therefore we think that the long-time properties obtained here will be characteristic of the real dynamics of the SU(2) YM field equation.

V. CONCLUDING REMARKS

We have studied the chaotic properties of the Wu-Yang magnetic-monopole solution of the SU(2) YM field equation from the RGA. From the positive values of the LE of Fig. 2, we see that this YM system is chaotic. This result is consistent with the result obtained by the conventional approach [13], where the YM system is shown to be intrinsically chaotic by numerical studies of the induction phenomenon and the behavior of the LE. However, the analytic formula (10) of the LE supplies more information that the crossover phenomenon exists in the behavior of the LE, which has not been revealed in Ref. [13] because the numerical analyses were done only up to the amplitude $A=1.0$ of the perturbation, i.e., $\tilde{\varepsilon}=0.138$, which was not energy enough to detect that the crossover phenomenon occurred at $\tilde{\varepsilon}_c$. Since the study of chaos in the gauge fields aims at clarifying the vacuum structure of gauge fields and the color confinement [14,19], this crossover connected with the phase transition might be an important phenomenon shedding light on this issue. Especially, it will be necessary to clarify the possible relation between the scaling of the LE with $\tilde{\varepsilon}$ and the similar behavior observed in the lattice gauge systems [20].

Let us comment on our results briefly. First, we would like to comment on a possible relation between the crossover structure of the LE and the instability of the Wu-Yang monopole solution. It will be reasonable to expect that the crossover structure of $\tilde{\sigma}_k^2$ comes from the dip of $\tilde{k}_0$ as shown in Fig. 3 because $\tilde{\sigma}_k^2$ is completely determined by $\tilde{k}_R(t)$ of Eq. (19). This dip stems from the second term of Eq. (19), which becomes negative in the region where $\phi_i^2<1/3$. We can show that this region is the same as one where the Wu-Yang monopole solution becomes unstable. For this purpose, let us approximate the solution $\phi(r,t)$ as a sum of the static solution $\phi_0(r)$ and the perturbation $\delta\phi(r,t)$ around it. By substituting $\phi(r,t)=\phi_0(r)+\delta\phi(r,t)$ to the YM equation (15), we obtain the following equation in the linear approximation:

$$\frac{\partial^2\delta\phi}{\partial t^2}-\frac{\partial^2\delta\phi}{\partial r^2}=\frac{1}{r^2}(1-3\phi_0^2)\delta\phi. \tag{26}$$

By assuming $\delta\phi(r,t)\propto\exp(ikr)\exp(i\omega t)$, we obtain the following relation from Eq. (26):

$$\omega^2=W(r)+k^2,\quad W(r)=\frac{1}{r^2}[3\phi_0^2(r)-1]. \tag{27}$$

For the Wu-Yang monopole solution $\phi_0=\phi_{WY}=0$, the potential becomes $W(r)=-1/r^2<0$ so that $\omega^2<0$ for small k, i.e., the perturbation grows exponentially with time as $\delta\phi(r,t)\propto\exp(|\omega|t)$. This triggers the time instability for the Wu-Yang monopole solution. Following Ref. [12], this instability occurs in the region where $W(r)<0$, i.e., $\phi_0^2(r)<1/3$, which is the same region as one for the dip mentioned above. Therefore, owing to Eq. (25), we could expect that the crossover of the LE correlates closely with this instability.

Second, we would like to point out the possible relation among the crossover, the curvatures, and chaos. In the RGA, the fluctuation of the curvature has been discussed in connection with the strength of chaos. In order to see this point, we plot the ratio of the normalized mean-square fluctuation (the variance) and the Ricci curvature, $\tilde{\sigma}_k/\tilde{k}_0$, vs $\tilde{\varepsilon}$ in Fig. 5. Since the mean $\tilde{k}_0$ is almost constant, $\tilde{k}_0\approx 2.00$, the behavior of this ratio is approximately $\tilde{\lambda}^{1/2}$ because of Eq. (25). Thus

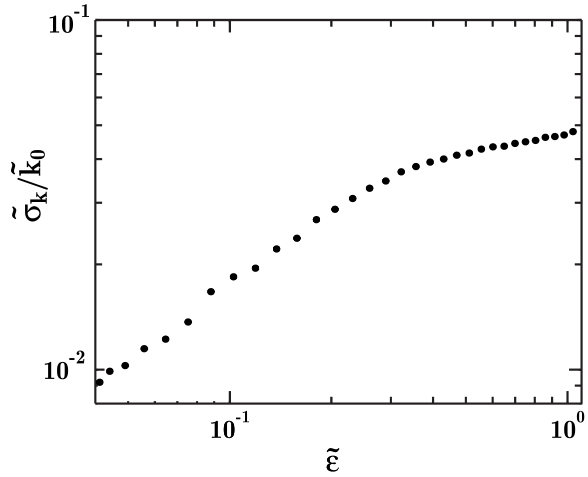

FIG. 5. Ratio $\tilde{\sigma}_k/\tilde{k}_0$ vs the energy density $\tilde{\varepsilon}$ for $N=128$.

we see from Fig. 5 that there are two characteristic regimes on the curvature fluctuation and chaos, i.e., the weakly chaotic regime corresponds to the small curvature fluctuation regime ($\tilde{\varepsilon}\ll\tilde{\varepsilon}_c$) while the strongly chaotic regime to the large fluctuation regime ($\tilde{\varepsilon}_c\ll\tilde{\varepsilon}$). This feature is qualitatively the same as one observed by other models such as the FPU-β model [7,8], the lattice ϕ^4 classical model [8,9], and the rare gas crystal model [10]. Therefore a crossover of $\tilde{\lambda}$ at $\tilde{\varepsilon}_c$ is a consequence of the behavior of $\tilde{k}_0$ and $\tilde{\sigma}_k$, and so the degree of chaoticity is controlled by the ratio $\tilde{\sigma}_k/\tilde{k}_0$.

Third, we comment on the origin of chaos in the YM system. In the RGA, this YM system is expressed by the norm Eq. (4) in the form of the Hill equation. Since the squared frequency $k(t)$ of Eq. (5) is positive definite due to Eq. (19), the oscillatory behavior of $k(t)$ might induce the parametric instability for the solution Ψ. When this instability can be activated, the norm Ψ is exponentially growing on the average. Thus the onset of chaos is due to the parametric instability of the system, which is the same as in the case of the FPU-β model [7,8] and the lattice ϕ^4 classical model [8,9].

Finally, we would like to discuss why the chaotic property of the YM system can be well described by the norm Eq. (3) describing the evolution of the Jacobi field. As explained thoroughly in Ref. [1], this equation is derived from the Jacobi equation (1) under several hypotheses and assumptions. Among them, the main one is the hypothesis that the manifold is quasi-isotropic. This hypothesis qualitatively implies that the manifold can be locally regarded as a deformed constant-curvature manifold. From this quasi-isotropy hypothesis, the full Riemann curvature tensor R in Eq. (1) can be replaced by an effective sectional curvature $k(s)$, i.e., $R_{ijkl}\approx k(s)(g_{ik}g_{jl}-g_{il}g_{jk})$ and the Ricci tensor R_{ij} can be written in the form as $R_{ij}\approx k(s)g_{ij}$. In other words, the Riemann curvature tensor and the Ricci tensor in the high-dimensional case are assumed to retain the same functional forms as in the case of the isotropic manifold with constant curvature K, i.e., $R_{ijkl}=K(g_{ik}g_{jl}-g_{il}g_{jk})$ and $R_{ij}=Kg_{ij}$. It is expected that this assumption holds when the fluctuations of the curvature are finite and small. Since our result in Fig. 4 shows that the fluctuations are rather small compared with the average of the Ricci curvature k_0, it seems to be consistent with this assumption. The other requisite hypothesis in the RGA is that the evolution of a generic geodesic is chaotic in the case of Hamiltonian systems with large N degrees of freedom. Owing to this hypothesis, the effective sectional curvature $k(s)$ in Eq. (3) can be modeled by a stochastic process so that $k(s)$ becomes free of the dynamics, i.e., of the evolution of the geodesic. Since the YM system is intrinsically chaotic as shown in Fig. 2, this system sufficiently satisfies the hypothesis on chaotic geodesics. From the fact that the present model satisfies the major hypotheses and assumptions in the RGA, we think that an important information on chaos in the gauge field can be obtained by analytically computing the largest LE as a function of ε. Here, we would like to emphasize that the LE of Fig. 2 is a theoretical and approximate estimate obtained from the analytic solution of the ordinary differential equation such as Eq. (4), and is not a true numerical estimate of the LE directly obtained from the numerical methods [6]. As long as the systems under study satisfy the hypotheses of the RGA, the analytic formula (10) of the LE will be effective and useful for the Hamiltonian systems with large N degrees of freedom.

The present work based on the RGA reveals the crossover structure of the LE in the Wu-Yang magnetic-monopole solution of the SU(2) YM field equation. The chaotic properties of the magnetic monopoles have been also studied in the lattice gauge systems, e.g., the LE structure of monopoles in the confinement phase and the Coulomb one [21], and the possible relation between the LE and the monopole density [22]. It will be important to address how the crossover of the LE relates to these findings in the lattice gauge systems. For the 't Hooft–Polyakov magnetic-monopole solution of the SU(2) Yang-Mills-Higgs field theory [23] and the vortex solution of the Abelian gauge-Higgs theories [24], which are more realistic than the pure YM field theories, it has been shown from the conventional approach that the order-to-chaos transition exists in the 't Hooft–Polyakov monopole solution [25,26], in the sphaleron solution [27], and in the vortex solution [28]. Since it is interesting to clarify the possible relation among the order-to-chaos transition, the crossover phenomenon, and the phase transition, it will be a very important issue to apply this RGA to the SU(2) Yang-Mills-Higgs field theories and the Abelian gauge-Higgs field theories.

ACKNOWLEDGMENTS

One of the authors (T.K.) would like to thank Tohru Hada, Hazime Mori, and the late Hirokazu Fujisaka for many helpful discussions.

[1] L. Casetti and M. Pettini, Phys. Rev. E **48**, 4320 (1993); L. Casetti, R. Livi, and M. Pettini, Phys. Rev. Lett. **74**, 375 (1995); L. Casetti, M. Pettini, and E. G. D. Cohen, Phys. Rep. **337**, 237 (2000).
[2] M. Cerruti-Sola and M. Pettini, Phys. Rev. E **53**, 179 (1996).
[3] K. Ramasubramanian and M. S. Sriram, Phys. Rev. E **64**, 046207 (2001).
[4] T. Kawabe, Phys. Rev. E **67**, 016201 (2003).
[5] T. Kawabe, Phys. Rev. E **71**, 017201 (2005).
[6] G. Benettin, L. Galgani, and J. M. Strelcyn, Phys. Rev. A **14**, 2338 (1976); G. Benettin, L. Galgani, A. Giorgilli, and J. M. Strelcyn, Meccanica **15**, 9 (1980); A. Wolf, J. B. Swift, H. L. Swinney, and J. A. Vastano, Physica D **16**, 285 (1985).
[7] L. Casetti, C. Clementi, and M. Pettini, Phys. Rev. E **54**, 5969 (1996).
[8] M. Pettini, Phys. Rev. E **47**, 828 (1993).
[9] L. Caiani, L. Casetti, C. Clementi, G. Pettini, M. Pettini, and R. Gatto, Phys. Rev. E **57**, 3886 (1998).
[10] L. Casetti and A. Macchi, Phys. Rev. E **55**, 2539 (1997).
[11] L. Caiani, L. Casetti, C. Clementi, and M. Pettini, Phys. Rev. Lett. **79**, 4361 (1997); L. Casetti, R. Gatto, and M. Pettini, J. Phys. A **32**, 3055 (1999).
[12] S. G. Matinyan, E. B. Prokhorenko, and G. K. Savvidy, Pis'ma Zh. Eksp. Teor. Fiz. **44**, 109 (1986) [JETP Lett. **44**, 138 (1986)]; S. G. Matinyan, E. B. Prokhorenko, and G. K. Savvidy, Nucl. Phys. B **298**, 414 (1988).
[13] T. Kawabe and S. Ohta, Phys. Rev. D **41**, 1983 (1990).
[14] B. Müller and A. Trayanov, Phys. Rev. Lett. **68**, 3387 (1992); T. S. Biró, S. G. Matinyan, and B. Müller, *Chaos and Gauge Field Theory* (World Scientific, Singapore, 1994); V. I. Kuvshinov and A. V. Kuzmin, Phys. Part. Nucl. **36**, 100 (2005).
[15] T. S. Biró, M. Feurstein, and H. Markum, Heavy Ion Phys. **7**, 235 (1998).
[16] E. Witten, Phys. Rev. Lett. **38**, 121 (1977).
[17] T. T. Wu and C. N. Yang, *Properties of Matter Under Unusual Conditions*, edited by H. Mark and S. Fernbach (Wiley-Interscience, New York, 1969), p. 349.
[18] H. Yoshida, Celestial Mech. **56**, 27 (1993).
[19] H. B. Nielsen and P. Olesen, Nucl. Phys. B **61**, 45 (1973).
[20] H. B. Nielsen, H. H. Rugh, and S. E. Rugh *ICHEP 96, Proceedings of the 28th International Conference, Warsaw, Poland*, edited by Z. Ajduk and A. Wroblewski (World Scientific, Singapore, 1997); e-print arXiv:hep-th/9611128; B. Müller, e-print arXiv:chao-dyn/9607001 (unpublished); U. W. Heinz, C. R. Hu, S. Leupold, S. G. Matinyan, and B. Müller, Phys. Rev. D **55**, 2464 (1997); J. Bolte, B. Müller, and A. Schafer, *ibid.* **61**, 054506 (2000); Á. Fülöp and T. S. Biró, Phys. Rev. C **64**, 064902 (2001).
[21] T. S. Biró, H. Markum, and R. Pullirsch, *Conference on Color Confinement and Hadrons in Quantum Chromodynamics*, edited by H. Suganuma *et al.* (RIKEN, Wako, Japan, 2003), p.439; e-print arXiv:hep-lat/0402013.
[22] H. Markum, R. Pullirsch, and W. Sakular, *Hep-Mad '01, Proceedings of the First Madagascar International Conference on High-Energy Physics*, edited by S. Narison (World Scientific, Singapore, 2002), p. 270; e-print arXiv:hep-lat/0201001.
[23] G. 't Hooft, Nucl. Phys. B **79**, 276 (1974); A. M. Polyakov, Zh. Eksp. Teor. Fiz. Pis'ma Red. **20**, 430 (1974) [JETP Lett. **20**, 194 (1974)].
[24] H. B. Nielsen and P. Olesen, Nucl. Phys. B **160**, 380 (1979).

[25] S. G. Matinyan, E. B. Prokhorenko, and G. K. Savvidy, Yad. Fiz. **50**, 284 (1989); Sov. J. Nucl. Phys. **50**, 178 (1989); M. P. Joy and M. Sabir, J. Phys. A **25**, 3721 (1992); C. N. Kumar and A. Khare, *ibid.* **22**, L849 (1989); B. A. Bambah, S. Lakshmibala, C. Mukku, and M. S. Sriram, Phys. Rev. D **47**, 4677 (1993); J. Segar and M. S. Sriram, *ibid.* **53**, 3976 (1996); D. Ebert, V. Ch. Zhukovsky, and M. V. Rogal, *ibid.* **65**, 065017 (2002).

[26] T. Kawabe and S. Ohta, Phys. Rev. D **44**, 1274 (1991); T. Kawabe, Phys. Lett. B **274**, 399 (1992); T. Kawabe and S. Ohta, *ibid.* **334**, 127 (1994).

[27] T. Kawabe, J. Phys. A **26**, L1131 (1993).

[28] T. Kawabe, Phys. Lett. B **343**, 254 (1995); T. Kawabe and S. Ohta, *ibid.* **392**, 433 (1997); S. Ohta and T. Kawabe, Phys. Rev. D **56**, 3660 (1997); T. Kawabe and S. Ohta, *ibid.* **61**, 105013 (2000).