Problem of transport in billiards with infinite horizon

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We consider particles transport in the Sinai billiard with infinite horizon. The simulation shows that the transport is superdiffusive in both continuous and discrete time. Also, it is shown that the moments do not converge to the Gaussian moments even in the logarithmically renormalized time scale, at least for a fairly long computational time. These results are discussed with respect to the existent rigorous theorems. Similar results are obtained for the stadium billiard.

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The considered model of scattering billiards was initiated in [1] to demonstrate an importance of mixing and dispersion of trajectories for foundation of statistical physics. Rigorous consideration of the problem, based on the dynamical chaos theory, was started from the paper [2] (see also [3]). The problem considered in [2] is known now as the Sinai billiard with infinite horizon (SBIH). After about 40 years statistical properties of this model and, particularly, the problem of particles transport, are still unclear being the subject of numerous publications. The Sinai billiard is a square billiard table with a circular scatterer in the center and absolutely elastic collisions of a point particle (ball) with the scatterer. Its periodic continuation in both x and y directions forms a square lattice and is known as the Lorentz gas (Fig. 1). There are three different approaches to the problem with well distinguished results: (a) The coordinates of the ball can be considered as functions of time $\mathbf{r}(t)=(x(t),y(t))$; (b) the same coordinates $\mathbf{r}(n)$ as a function of the number n of collisions with scatterers; (c) angle ψ_n of the velocity of a ball $\mathbf{v}_n$, ($|\mathbf{v}|=1$) at the collision point on the circle with the normal vector at this point, and the angle coordinate θ_n on the circle of the point of collision as functions of n (not considered here). The main difficulty in studying the SBIH is the existence of the corridors within which a ball can propagate infinite time without scattering. Such corridors make scars of zero measure on the phase plane (see, for example, in [4]) raising a question of how these scars influence the long term characteristics of dynamical and statistical processes of trajectories that do exhibit scatterings. Some important results on the studying of the problem are as follows. It was mentioned in [5,6] that the presence of corridors leads to an algebraic decay of the velocity correlation

$$\langle(\mathbf{v}(0),\mathbf{v}(t))\rangle \sim \text{const}/t \tag{1}$$

that gives for the second moment of the displacement

$$\langle|\mathbf{R}(t)-\mathbf{R}(0)|^2\rangle \sim \text{const}\times(t\ln t). \tag{2}$$

Similar estimates were confirmed in [7] although the simulation performed in [7] was not able to establish the presence of $\ln t$ in two. It was obtained in [8] under some assumptions that the limit distribution of the particles displacement on the plane is Gaussian, but the normalization factor is $(t\ln t)^{1/2}$ and not $t^{1/2}$ as in the usual Gaussian case. Simulations performed in [9] confirmed Eq. (1) for the SBIH and the exponential decay of the velocity correlation for the finite horizon when the corridors are absent. It is worthwhile to mention that while the map for the scattering angles (ψ_n,ϑ_n) possesses fast decay of correlations for the finite horizon when the number of collisions $n\to\infty$ (stretched exponential decay was obtained in [10] and then improved to the exponential decay in [11]), the case of the infinite horizon creates severe difficulties for theoretical analysis, probably because of nonexponential decay of distribution of Poincaré recurrences [9,12]. Numerical simulations in [12] show that

$$\langle|\mathbf{R}(t)|^{2m}\rangle \sim \text{const}\times(t\ln t)^{\mu(m)}, \quad m\in\mathbf{N} \tag{3}$$

and that $\mu(1)=1$, while for $m>1$ it does not follow the Gaussian law, but corresponds to a strong superdiffusion

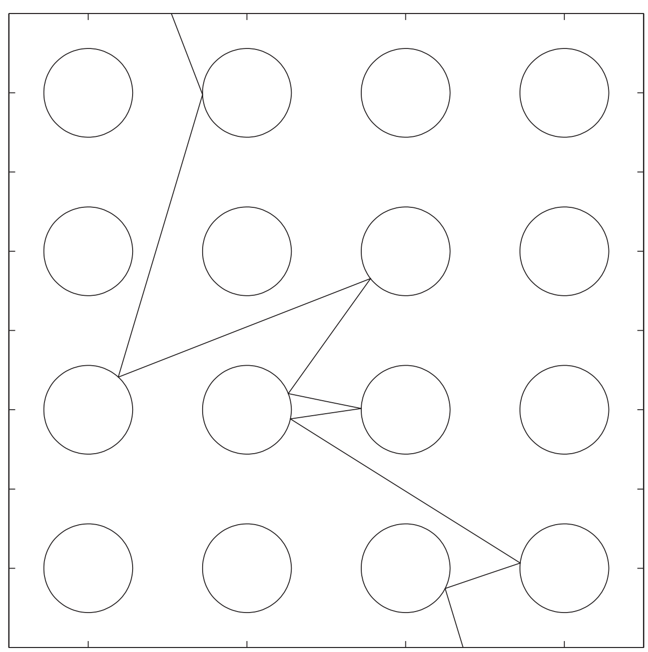

FIG. 1. Sample of a trajectory in the periodic Lorentz gas.

with $\mu(m) > m$ $(m > 1)$. In [13] the results on superdiffusion for the moments with $m > 1$ and normal diffusion for $m \leqslant 1$ were confirmed and a possible origin of the discrepancy between numerical results and proposition in [8] was discussed in details. The main point, as it was suggested in [13], is that convergence for distribution function of particle displacements to the Gaussian distribution does not imply similar convergence for the moments. It was also estimated in [13] that $\mu(m) = 2m-1$ for $m \geqslant 1$ with fairly good agreement with simulation and up to the rescaling factor $\ln t$. Rigorous results were obtained in [14,15]. Particularly, in the recent publication [15] a rigorous proof is given for the Bleher's conjecture on the convergence to the Gaussian distribution of the limit distribution function of the displacements in SBIH. This remarkable result makes the problem of the moments' behavior with time even sharper since physical observables typically are moments' dynamics, and their simulations does not show the Gaussianity.

The goal of this paper is to provide renewal results of massive computations and to show that some well defined transport properties of particles do not follow the Gaussian law, exhibit superdiffusion, and, for the time of observation, do not display a "normal" approach to the limit distribution. We consider two models: SBIH (periodic Lorentz gas) and stadium (Bunimovich) billiard [16]. We confirm these observations as a result of particles' long "flights" in the corridors and, as a result of the flights, persistent fluctuations [17] that do not have a finite time of relaxation as it exists for the Gaussian fluctuations. Comparing to [17] the computations here use larger time and number of trajectories. We also consider dependence of the moments on the number of collisions n and compare this dependence with the moment dependence on time. It will be confirmed that in all considered cases the moments with $m > 1$ do not converge to the Gaussian law and some important details of the moment's dynamics will be presented and compared to [17,12].

First, we describe the data related to the moments (3). We use two presentations:

$$\langle |\mathbf{R}(t)|^{2m} \rangle \equiv \frac{1}{N} \sum_{k=1}^{N} |\mathbf{R}_k(t)|^{2m} \sim \text{const} \times (t \ln t)^{\mu(m)}, \quad (4)$$

$$\langle |\mathbf{R}(n)|^{2m} \rangle \equiv \frac{1}{N} \sum_{k=1}^{N} |\mathbf{R}_k(n)|^{2m} \sim \text{const} \times (n \ln n)^{\bar{\mu}(m)}. \quad (5)$$

In all computations we consider $m = 1,2,3,4$ or $m = 0.25, 0.5, 1, 2, 3, 4$; N is a number of trajectories that are formed by an ensemble of N initial conditions over which the data are averaged. A large number of trajectories $\{\mathbf{R}(t, \mathbf{R}_k(t=0)), k=1,\ldots,N\}$ or $\{\mathbf{R}(n, \mathbf{R}_k(n=0))\}$ generates histograms $\mathcal{P}(t, \mathbf{R}(t))$ or $\mathcal{P}(n, \mathbf{R}(n))$. The ensemble moments (4) and (5) are just the mean values obtained after averaging over these histograms. Since the computational time is finite, the values of $\mathcal{P}(t, \mathbf{R}(t))$, $\mathcal{P}(n, \mathbf{R}(n))$ make sense within the intervals

$$0 < t < t_{\max}, \quad 0 < n < n_{\max}. \quad (6)$$

The meaning of condition (6) is to cut the largest flights for which statistics always will be insufficient and fluctuations could be enormously large compared to the small values of $\mathcal{P}(t, \mathbf{R}(t))$ and $\mathcal{P}(n, \mathbf{R}(n))$. The corresponding simulation data are presented in Fig. 2. The moments' $\langle |\mathbf{R}(t)|^{2m} \rangle$ and $\langle |\mathbf{R}(n)|^{2m} \rangle$ dependence correspondingly on $(t \ln t)$ and $(n \ln n)$ for the SBIH (periodic Lorentz gas) are given for two different sets of conditions. The upper row is for scatterers with radius 3.86 and period 13 and the data are averaged over 250 000 trajectories. The lower row is for scatterers with radius 0.56 and period 2 and the data are averaged over 1 344 000 trajectories. The results for both sets are close to each other and the results for the dependence on $(t \ln t)$ are consistent with the previous results in [12] obtained with less trajectories. For $m \leqslant 2$ the exponents of the time dependence are "normal" and correspond to the assumption of Gaussianity considered in [8]. For $m > 2$ there is clear superdiffusion with a strong deviation for the values μm as they are supposed to be for the Gaussian law. The plots display "jumps" due to the particle flights along the corridors. The larger the moment power m, the larger are the jumps. The presence of jumps implies the absence of limits $t \to \infty$ or $n \to \infty$ for the moments, at least for the calculation time 10^7 which is fairly big. These results are consistent with the previous calculations in [17,12]. The time dependence of the moments follows the approximate law

$$\mu(m) = 1 + a(m-1), \quad m > 1, \quad a \approx 1.8. \quad (7)$$

This law is close to the estimates in [13] where $a = 2$. The difference could be explained by a nonuniform distribution

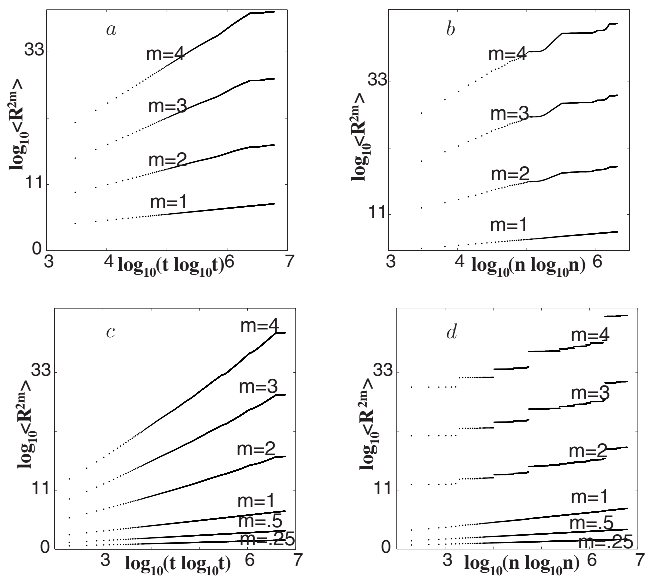

FIG. 2. Moments' dependence on $(t \ln t)$ and $(n \ln n)$ for the SBIH (periodic Lorentz gas) for two different sets of conditions. The upper row is for scatterers with radius 3.86 and period 13. The data are averaged over 250 000 trajectories. The lower row is for scatterers with radius 0.56 and period 2. The data are averaged over 1 344 000 trajectories. The slopes for $m = 0.25$ is 0.24 and for $m = 0.5$ is 0.48. For $m = 1,2,3,4$ the slopes are correspondingly 1, 2.8, 4.7, 6.4 for the dependence on $t \ln t$, and 1, 2.5, 3.9, 5.2 for the dependence on $n \ln n$ (lower case).

TABLE I. Values of the moments of x at $n_{\max}=1\,000\,000$.

D/a	M_1	M_2	M_3	M_4
2/0.405	3.9×10^{-6}	1.00	1.7	4.17
1/0.405	2.5×10^{-5}	1.00	−0.15	3.67
1/0.473	5.1×10^{-5}	1.00	0.0013	3.37

TABLE II. Values of the moments of y at $n_{\max}=1\,000\,000$.

D/a	M_1	M_2	M_3	M_4
2/0.405	-1.03×10^{-6}	1.00	0.122	5.33
1/0.405	2.4×10^{-5}	0.9999	0.006	2.86
1/0.473	7.01×10^{-5}	1.001	−0.13	3.73

of particle flights within the corridors with respect to the angles.

This issue was studied separately in order to understand a convergence to the Gaussian distribution of the moments of particle displacements does or does not exist. For this goal the central moments were studied in independent calculations and by a different code with averaging over $N=500$ trajectories indexed by k below. Let us define the central moments as

$$M_m(n) = \frac{1}{N}\sum_{k=1}^{N}\left[\frac{1}{\sigma}\left(\frac{x_k(n)-x_k(0)}{(n\ln n)^{1/2}}-\bar{\mu}\right)\right]^m, \qquad (8)$$

where $x_k(n)$ is displacement along x of the kth trajectory after n collisions, with $n\leqslant n_{\max}$, and σ, μ are defined as follows:

$$\bar{\mu} = M_1' = \frac{1}{N}\sum_{k=1}^{N}\frac{x_k(n_{\max})-x_k(0)}{\sqrt{n_{\max}\ln n_{\max}}} \qquad (9)$$

and

$$\sigma^2 = M_2'(n_{\max}) - [M_1'(n_{\max})]^2 = M_2'(n_{\max}) - \bar{\mu}^2. \qquad (10)$$

Here $M_2'(n)$ and $[M_1'(n)]$ are the raw moments; the mth raw moment is defined as

$$M_m'(n) = \frac{1}{N}\sum_{k=1}^{N}\left[\frac{x_k(n)-x_k(0)}{(n\ln n)^{1/2}}\right]^m. \qquad (11)$$

Similar expressions can be introduced for coordinate y if we replace $x_k(n)$ by $y_k(n)$.

The results of the calculations are presented in Table I and Table II for three different cases of the period D and radius of the scatterer a. While the values of M_1 are close to zero as they should be for the central Gaussian moment, the values for M_3 are too far from zero. Moreover, the fourth central Gaussian moments should take the values $M_4=3$ while the values in the tables deviate, sometimes significantly, from that. More comments could be done from the plots in Fig. 3 and Fig. 4. They display jumps and they do not display a good convergence for some cases as is seen from Fig. 4 for M_2. Although we use the notions of "good" or "bad" convergence in Fig. 4 not in a rigorously defined way, it should be an important intrinsic feature of the SBIH that leads to large jumps of, for example, M_2, after time as long as 10^6 collisions. These simulations give a preliminary idea of what happens. Later they will be followed by more exhaustive statistical estimation of the asymptotic behavior of the process.

The stadium billiard [16] is different from the SBIH model of no-local-dispersion billiard that nevertheless possesses similar ergodic features with a zero measure phase space domain of bouncing trajectories. The anomalous properties of escape time from the open windows of the billiard were studied in many details in [18] demonstrating the power law distribution of the escape time. Anomalous properties of the recurrences and transport were considered in [17]. A y-periodic extension of the billiard could be introduced to show the existence of an infinite horizon for trajectories in the corridor (Fig. 5). It was numerically demonstrated in [17] that the presence of the flights imposes superdiffusive transport of particles in the y direction. More precisely, it was shown that

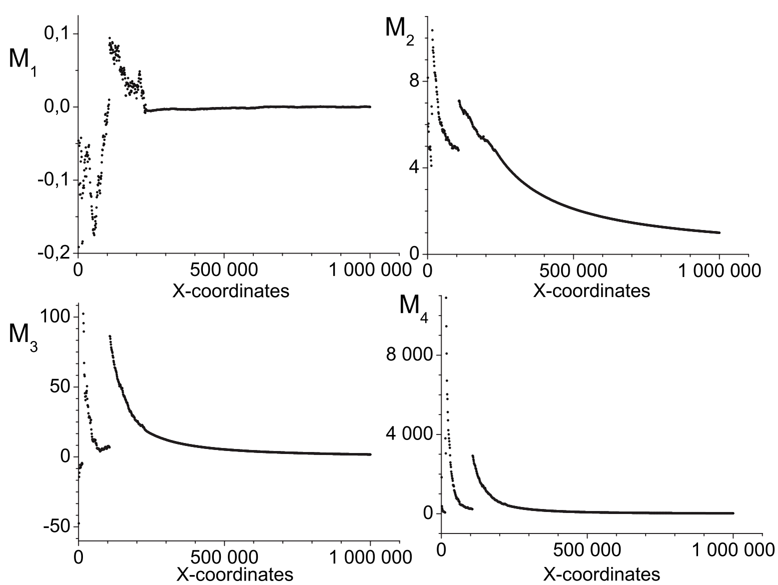

FIG. 3. Dependence of the central moments on $(n\ln n)$ for $D=2$, $a=0.405$ after 10^6 collisions.

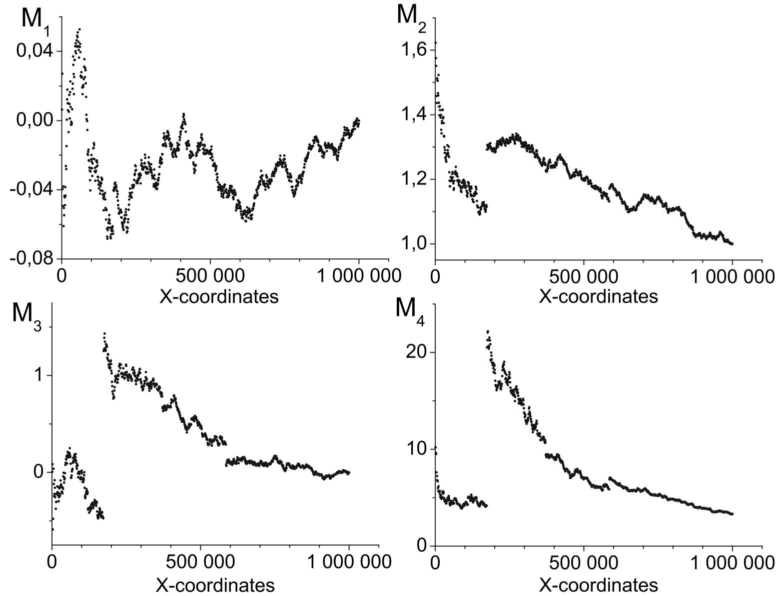

FIG. 4. Dependence of the central moments on $(n \ln n)$ for $D=1$, $a=0.473$ after 10^6 collisions.

$$\langle |y(t)|^{2m} \rangle \sim \text{const} \times (t \ln t)^{\mu(m)} \tag{12}$$

with $\mu(m) > m$, i.e., with superdiffusive transport. Here we have performed more extensive computations by studying both dependences: on $(t \ln t)$ and $(n \ln n)$. The corresponding results are presented in Fig. 6. The results display the anomalous (superdiffusive) transport for the time dependence of moments with $m > 2$ and fairly strong deviation from the Gaussian law $\mu(m) = m$, while, for a similar dependence on n,

$$\langle |y(n)|^{2m} \rangle \sim \text{const} \times (n \ln n)^{\mu(m)}, \tag{13}$$

the diffusion is slightly subdiffusive. The jumps are stronger than in the case of SBIH and the flights are longer. Despite the extremely long simulation (time $\sim 10^9$) there was no indication of a good convergence to the limit values, at least for $m > 2$. The obtained results on the existence of the anomalous transport for the stadium billiard are similar to Eqs. (4) and (5) for the Lorentz gas.

In conclusion, described results can be summarized as follows.

(a) Trajectories in the coordinate space $\mathbf{R} = (x, y)$ perform random walk in time t or number of collisions n, which does not show time (or n) dependence typical for the Gaussian processes even after rescaling $t \to t \ln t$ $(n \to n \ln n)$.

(b) This could be a result of either the chaotic dynamics of the flow being not Gaussian (after rescaling) or the real limit having not been achieved during the computational time. The probability of the latter is very small since the computational

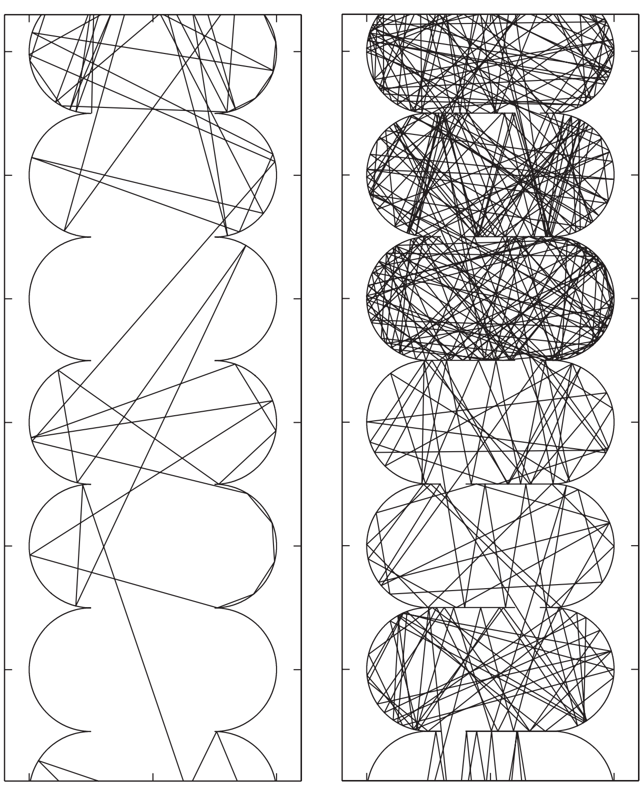

FIG. 5. Sample of a trajectory in the stadium billiard with infinite horizon.

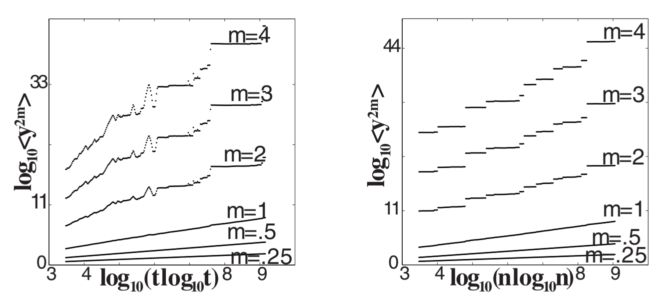

FIG. 6. Dependence of the y moments for the stadium billiard obtained after averaging over 28 864 trajectories (left) and 131 072 trajectories (right). The slopes (μ) are 0.25, 0.5, 1 for $m = 0.25, 0.5, 1$ in both cases and 2.1, 3.2, 4.3 (left) and 1.8, 2.7, 3.6 (right) for correspondingly $m = 2, 3, 4$.

time is fairly large and the results do not show a tendency to achieve the Gaussian limit.

(c) Most probably, the limit may exist in a weak sense for the moments' dependence on n as one can see for the SBIH and stadium billiards (Figs. 2 and 6), $m>1$.

(d) The dynamics is ergodic in full phase space excluding zero measure domains of the bouncing trajectories. Just these domains are responsible for the anomalous properties of the transport in the SBIH (or the Lorentz gas) or in the stadium billiard.

(e) More specific discussion concerns the use of simulation to test convergence of moments of $R(t)$ and $R(n)$ since these moments are more descriptive characteristics of the transport. It is obvious that the moments of $R(n)/\sqrt{n \ln n}$ do not converge to the moments of the Gaussian distribution. Although this may seem to contradict the results of Szasz and Varju who obtained a central limit theorem for the SBIH, we have to take into account that in this theorem the convergence is in distribution. In general, the convergence in distribution of a sequence of random variables does not imply the convergence of the moments. An example of a sequence of random variables X_n that converges in distribution to the Dirac distribution while the moments of all orders diverge is given in [19]. This occurs when the probability of large values decreases slowly as n goes to infinity. For the central limit theorem with independent random variables, the convergence of the moments of the normalized sum to the Gaussian moments holds only under the additional Lindberg conditions [20] which guarantees that the variance of one step is small in comparison with the total variance. In the SBIH case, the sequence of free flights between collisions is not independent. The convergence of moments in the central limit theorem for a more general stochastic process has also been studied for martingales [21] or under strong mixing conditions [22] using some Lindberg-like conditions. As far as we know, no such results have been established for the SBIH and less as to convergence of moments to the Gaussian moments. Indeed, the fact that very long free flights may occur with small probability could be the reason why the moments diverge from the Gaussian moments.

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