Memory effects in the asymptotic diffusive behavior of a classical oscillator described by a generalized Langevin equation

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We investigate the memory effects present in the asymptotic dynamics of a classical harmonic oscillator governed by a generalized Langevin equation. Using Laplace analysis together with Tauberian theorems we derive asymptotic expressions for the mean values, variances, and velocity autocorrelation function in terms of the long-time behavior of the memory kernel and the correlation function of the random force. The internal and external noise cases are analyzed. A simple criterion to determine if the diffusion process is normal or anomalous is established.

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I. INTRODUCTION

The study of diffusive phenomena is a fundamental topic for the analysis of stochastic processes in several areas of natural science. A large number of these processes exhibit a normal diffusion, which is characterized by a mean-square displacement that grows linearly in time [1]. However, a great number of stochastic processes show a more complex behavior, which has been the focus of extensive research during the last years [2]. A force-free stochastic process is said to exhibit anomalous diffusion when the mean-square displacement has the asymptotic form t^{λ}. In this case, the process is called subdiffusive when $\lambda < 1$ and superdiffusive when $\lambda > 1$. One of the dynamical origins of anomalous diffusion is the nonlocality in time. This can be handled if the problem is formulated in terms of a generalized Langevin equation (GLE) [3–6], which takes into account the memory effects through an aftereffect function. This method has been successfully used in the description of diverse anomalous diffusion phenomena, like conformational fluctuations within a single-protein molecule [7,8], reaction kinetics and fluorescence intermittency of single enzymes [9,10], and nuclear fusion reactions [11].

The GLE of a diffusing particle evolving under the influence of an external potential $U(X)$ and a random force $F(t)$ is written as follows:

$$\ddot{X}(t) + \int_0^t dt' \;\gamma(t-t')\dot{X}(t') + U'(X) = F(t), \qquad (1)$$

where $\gamma(t)$ is the dissipative memory kernel and $F(t)$ is a zero-centered and stationary Gaussian random force characterized by a correlation function

$$\langle F(t)F(t')\rangle = C(|t-t'|) = C(\tau). \qquad (2)$$

If the noise $F(t)$ is an "internal noise," the memory kernel $\gamma(t)$ is related to the noise correlation function via the second fluctuation-dissipation theorem [12]

$$C(t) = k_B T \gamma(t), \qquad (3)$$

where T is the absolute temperature and k_B is the Boltzmann constant. Note that when the noise is internal, the relaxation time of the system is essentially the same as the correlation time of the noise.

On the other hand, $F(t)$ is called "external noise" when the dissipation and the noise have different sources. In this case, the fluctuation-dissipation theorem (3) is no longer valid.

Anomalous diffusion has been mostly investigated in the case of a free particle. However, in several systems one encounters a damped harmonic motion under the action of a time-dependent noise. We recently derived exact expressions for the mean values, variances, and velocity autocorrelation function (VACF) corresponding to an oscillator with a power-law noise correlation function and in the case of internal noise [6]. Nevertheless, in several situations one wants only to know the long-time system behavior, when the differences between normal and anomalous diffusion are significant.

In this work we investigate the long-time behavior of a harmonically bounded particle in terms of the asymptotic behavior of the memory kernel and the autocorrelation function of the fluctuating force involved in the GLE. Explicit expressions for the first moments, variances, and velocity autocorrelation function of the diffusing particle in the cases of internal and external noise are given. This analysis allows us to obtain a criterion for determining whether the diffusion process is normal or anomalous. The outline of this paper is as follows. In Sec. II we introduce general results valid for arbitrary memory kernels and noise autocorrelation functions. Section III is devoted to the study of the asymptotic behavior of the first moments, variances, and VACF in the internal noise situation. The case of external noise is analyzed in Sec. IV. In Sec. V we apply the previous results to a specific example. Finally, in Sec. VI we present our conclusions.

II. GENERAL ANALYSIS

Before analyzing the GLE for some specific dissipative memory kernel and correlation function, we recall some

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known results for the first moments, variances, and VACF [6]. In what follows we consider the GLE (1) with an external potential $U(X)=\omega^2X^2/2$ and the deterministic initial conditions $x_0=X(0)$ and $v_0=\dot{X}(0)$. Formal expressions for the displacement $X(t)$ and the velocity $\dot{X}(t)$ can be obtained by means of the Laplace transform. The displacement $X(t)$ satisfies

$$X(t) = \langle X(t)\rangle + \int_0^t dt'\ G(t-t')F(t'), \tag{4}$$

where

$$\langle X(t)\rangle = x_0[1-\omega^2 I(t)] + v_0 G(t). \tag{5}$$

The relaxation function $G(t)$ is the Laplace inversion of

$$\hat{G}(s) = \frac{1}{s^2 + \hat{\gamma}(s)s + \omega^2}, \tag{6}$$

where $\hat{\gamma}(s)$ is the Laplace transform of the memory kernel and

$$I(t) = \int_0^t dt'\ G(t'). \tag{7}$$

On the other hand, the velocity $\dot{X}(t)$ satisfies

$$\dot{X}(t) = \langle \dot{X}(t)\rangle + \int_0^t dt'\ g(t-t')F(t'), \tag{8}$$

where

$$\langle \dot{X}(t)\rangle = v_0 g(t) - \omega^2 x_0 G(t) \tag{9}$$

and the relaxation function $g(t)$ is the derivative of $G(t)$, i.e.,

$$g(t) = \frac{dG(t)}{dt}. \tag{10}$$

Note that from Eqs. (5) and (9) it follows that $I(0)=0$, $G(0)=0$, and $g(0)=1$.

Using Eqs. (4) and (8) and taking into account the symmetry property of the correlation function, the explicit expressions of the variances can be written as [3]

$$\sigma_{xx}(t) = \langle [X(t)-\langle X(t)\rangle]^2\rangle = 2\int_0^t dt_1 G(t_1)\int_0^{t_1} dt_2 G(t_2) C(t_1-t_2), \tag{11}$$

$$\sigma_{vv}(t) = \langle [\dot{X}(t)-\langle \dot{X}(t)\rangle]^2\rangle = 2\int_0^t dt_1 g(t_1)\int_0^{t_1} dt_2 g(t_2) C(t_1-t_2), \tag{12}$$

$$\sigma_{xv}(t) = \langle [X(t)-\langle X(t)\rangle][\dot{X}(t)-\langle \dot{X}(t)\rangle]\rangle = \frac{1}{2}\dot{\sigma}_{xx}(t) = \int_0^t dt_1 G(t_1)\int_0^{t} dt_2 g(t_2) C(t_1-t_2). \tag{13}$$

In the internal noise situation the variances can be conveniently simplified using Eq. (3) as [3]

$$\beta\sigma_{xx}(t) = 2I(t) - G^2(t) - \omega^2 I^2(t), \tag{14}$$

$$\beta\sigma_{vv}(t) = 1 - g^2(t) - \omega^2 G^2(t), \tag{15}$$

$$\beta\sigma_{xv}(t) = G(t)\{1 - g(t) - \omega^2 I(t)\}, \tag{16}$$

where $\beta=1/k_BT$.

Moreover, as in the free particle case [4], the long-time behavior of the normalized VACF is related to the relaxation function $g(t)$ as [6]

$$C_v(t) = \lim_{\tau\to\infty}\frac{\langle \dot{X}(\tau)\dot{X}(\tau+t)\rangle}{\langle \dot{X}(\tau)\dot{X}(\tau)\rangle} = g(t). \tag{17}$$

III. ASYMPTOTIC BEHAVIOR

In order to investigate the asymptotic behavior of the first moments (5) and (9), variances (11)–(13), and VACF (17) we will analyze the behavior of the kernels $I(t)$, $G(t)$, and $g(t)$ starting from its Laplace transforms and using Tauberian theorems. For this purpose we assume that the memory kernel $\gamma(t)$ goes to zero when $t\to\infty$. Then,

$$\lim_{t\to\infty}\gamma(t) = \lim_{s\to 0} s\,\hat{\gamma}(s) = 0, \tag{18}$$

where we used the final value theorem [13]. Taking into account that the integral kernel $I(t)$ defined in Eq. (7) is the Laplace inversion of

$$\hat{I}(s) = \frac{\hat{G}(s)}{s} = \frac{s^{-1}}{s^2 + s\,\hat{\gamma}(s) + \omega^2}, \tag{19}$$

the application of the final value theorem yields

$$\lim_{t\to\infty} I(t) = \lim_{s\to 0} s\hat{I}(s) = \lim_{s\to 0}\frac{1}{s^2 + s\,\hat{\gamma}(s) + \omega^2}. \tag{20}$$

Note that assumption (18) guarantees that the system reaches a stationary state. The strict $t=\infty$ limit in Eq. (20) gives

$$I(\infty) = \frac{1}{\omega^2}, \tag{21}$$

$$G(\infty) = 0, \tag{22}$$

$$g(\infty) = 0, \tag{23}$$

where we use the fact that $\hat{G}(s)=s\hat{I}(s)$ and $\hat{g}(s)=s\hat{G}(s)$. As it is expected for a damped oscillator, the mean values of the position and the velocity and also the VACF tend to zero. This can easily be verified from Eqs. (17), (5), and (9).

In the internal noise situation, the stationary state correspond to the equilibrium state. Substituting (21)–(23) into (14)–(16) yields

$$\sigma_{xx}(\infty) = \frac{k_BT}{\omega^2}, \tag{24}$$

$$\sigma_{vv}(\infty) = k_B T, \tag{25}$$

$$\sigma_{xv}(\infty) = 0, \tag{26}$$

which has been previously obtained through a similar procedure [3].

To proceed with the analysis, we assume that

$$s\hat{\gamma}(s) \gg s^2 \quad (s \to 0), \tag{27}$$

which means that for long times the friction term is more important than the inertial term.

Note that if the memory kernel takes the form $\gamma(t) = \gamma_0 \delta(t)$ (δ-correlated white noise), the GLE (1) reduces to the ordinary Langevin equation [1], corresponding to a system without memory. In this situation the Laplace transform of the memory kernel is $\hat{\gamma}(s) = \gamma_0$ and trivially satisfies the two conditions (18) and (27). It can be verified that in this case the kernels $I(t)$, $G(t)$, and $g(t)$ decay with an exponential behavior. In consequence, the mean values also decay exponentially just like variances and VACF in the internal noise case. In particular, the asymptotic variances can be written as

$$\omega^2 \beta \sigma_{xx}(t) \approx 1 - \left(1 + \frac{\omega^2}{\gamma_0^2}\right) e^{-2\omega^2 t/\gamma_0}, \tag{28}$$

$$\beta \sigma_{vv}(t) \approx 1 - \frac{\omega^2}{\gamma_0^2}\left(1 + \frac{\omega^2}{\gamma_0^2}\right)^{-2\omega^2 t/\gamma_0}, \tag{29}$$

$$\beta \sigma_{xv}(t) \approx \frac{1}{\gamma_0}\left(1 + \frac{\omega^2}{\gamma_0^2}\right) e^{-2\omega^2 t/\gamma_0}. \tag{30}$$

These equations correspond to the usual expressions obtained in the overdamped limit, which is a consequence of condition (27). Note that for a classical oscillator, normal diffusion means that the mean-square displacement decays exponentially.

In what follows we will characterize the memory kernel in the same way as in Refs. [4,14] according to the two possible behaviors of its Laplace transform at $s=0$:

$$\hat{\gamma}(s=0) = \int_0^\infty dt' \, \gamma(t'). \tag{31}$$

(i) If the integral (31) is finite and nonvanishing, the memory kernel satisfies that [4]

$$\hat{\gamma}(s) \sim \gamma_0 + o(1) \quad (s \to 0), \tag{32}$$

where $\gamma_0 = \hat{\gamma}(0)$. Note that this kind of memory kernels satisfies both conditions (18) and (27). Inserting these conditions and (32) into (19) yields

$$\hat{I}(s) \approx \frac{s^{-1}}{s\hat{\gamma}(s) + \omega^2} \approx \frac{s^{-1}}{s\gamma_0 + \omega^2} \quad (s \to 0). \tag{33}$$

Using Tauberian theorems [15], the long-time behavior of $I(t)$ can be obtained from the Laplace inversion of (33). Thus,

$$I(t) \approx \frac{1}{\omega^2}\{1 - e^{-2\omega^2 t/\gamma_0}\}, \tag{34}$$

and, from (7) and (10), one gets

$$G(t) \approx \frac{1}{\gamma_0} e^{-2\omega^2 t/\gamma_0}, \tag{35}$$

$$g(t) \approx -\frac{\omega^2}{\gamma_0^2} e^{-2\omega^2 t/\gamma_0}. \tag{36}$$

Then, the first moments (5) and (9) decay exponentially to zero. Furthermore, it can be demonstrated that in the internal noise case the variances (14)–(16) are given by expressions (28)–(30). Consequently, the memory kernel that satisfies (32) leads to a normal diffusion. Note that, in accordance with Ref. [14], a long-range memory kernel does not imply necessarily an anomalous diffusive behavior.

(ii) The other possible behavior is found when the integral (31) vanishes or diverges. Taking into account conditions (18) and (27), the limit $s \to 0$ of the Laplace transform (19) can be approximated as

$$\hat{I}(s) \approx \frac{s^{-1}}{s\hat{\gamma}(s) + \omega^2} \approx \frac{s^{-1}}{\omega^2}\left\{1 - \frac{s\hat{\gamma}(s)}{\omega^2}\right\} \quad (s \to 0). \tag{37}$$

Applying Tauberian theorems [15] in (37) it can be deduced that the long-time behavior of $I(t)$ becomes

$$I(t) \approx \frac{1}{\omega^2} - \frac{\tilde{\gamma}(t)}{\omega^4}, \tag{38}$$

where $\tilde{\gamma}(t)$ denotes the asymptotic behavior of the memory kernel. Using relations (7) and (10) the kernels $G(t)$ and $g(t)$ can be written as

$$G(t) \approx -\frac{1}{\omega^4}\frac{d\tilde{\gamma}(t)}{dt}, \tag{39}$$

$$g(t) \approx -\frac{1}{\omega^4}\frac{d^2\tilde{\gamma}(t)}{d^2 t}. \tag{40}$$

Substitution of the asymptotic expressions (38)–(40) into Eqs. (5) and (9) allows one to obtain the long-time behavior of the mean displacement and velocity, which can be written as

$$\langle X(t)\rangle \approx \frac{1}{\omega^2}\left\{x_0\tilde{\gamma}(t) - \frac{v_0}{\omega^2}\frac{d\tilde{\gamma}(t)}{dt}\right\}, \tag{41}$$

$$\langle \dot{X}(t)\rangle \approx \frac{1}{\omega^2}\left\{x_0\frac{d\tilde{\gamma}(t)}{dt} - \frac{v_0}{\omega^2}\frac{d^2\tilde{\gamma}(t)}{d^2 t}\right\}. \tag{42}$$

On the other hand, in the internal noise case the variances (14)–(16) can be written as

$$\beta\sigma_{xx}(t) \approx \frac{1}{\omega^2} - \frac{\tilde{\gamma}^2(t)}{\omega^6}, \tag{43}$$

$$\beta\sigma_{vv}(t) \approx 1 - \frac{1}{\omega^6}\left(\frac{d\tilde{\gamma}(t)}{dt}\right)^2, \tag{44}$$

$$\beta\sigma_{xv}(t) \approx -\frac{1}{\omega^6}\tilde{\gamma}(t)\frac{d\tilde{\gamma}(t)}{dt}, \tag{45}$$

and the asymptotic behavior of the VACF (17) is given by

$$C_v(t) \approx -\frac{1}{\omega^4}\frac{d^2\tilde{\gamma}(t)}{d^2t}. \tag{46}$$

It is worth pointing out that knowledge of $\tilde{\gamma}(t)$ fully determines the asymptotic behavior of the first moments and the variances and VACF in the internal noise case. Moreover, note that it is not necessary to know the explicit form of the Laplace transform $\hat{\gamma}(s)$ of the memory kernel to obtain the asymptotic oscillator dynamics. For this purpose, one only needs to know the behavior of the integral (31). It is interesting to observe that it is possible to deduce the form of the asymptotic memory kernel $\tilde{\gamma}(t)$ if one knows the behavior of one of the variances or the VACF. Finally, the previous analysis indicates that the presence of long tails in $\gamma(t)$ is a necessary condition to obtain an anomalous diffusive behavior of the oscillator (but not sufficient, as is shown above).

IV. EXTERNAL NOISE

The kernels $I(t)$, $G(t)$, and $g(t)$ obtained in the previous section are valid for internal or external noises because they are independent of the noise correlation function $C(t)$. In the case of internal noise the variances could have been calculated making use of the fluctuation-dissipation theorem.

In what follows we will investigate the long-time behavior of the variances in the external noise situation. For this purpose, the position variance (11) is written as

$$\sigma_{xx}(t) = 2\int_0^t dt_1 G(t_1)\rho(t_1), \tag{47}$$

where we define

$$\rho(t_1) = \int_0^{t_1} dt_2 G(t_2)C(t_1 - t_2). \tag{48}$$

The derivatives of the position and velocity variances can be written as

$$\frac{d\sigma_{xx}(t)}{dt} = 2\sigma_{xv}(t) = 2G(t)\rho(t), \tag{49}$$

$$\frac{d\sigma_{vv}(t)}{dt} = 2g(t)\frac{d\rho(t)}{dt}. \tag{50}$$

To calculate the long-time behavior of the variances it is sufficient to know the behavior of $\rho(t)$ for long times, which can be done noticing that $\rho(t)$ is the convolution

$$\hat{\rho}(s) = \hat{G}(s)\hat{C}(s). \tag{51}$$

To proceed with the analysis, it is necessary to characterize the noise correlation function in the same way we have done for the memory kernel. In the case of external noise, there are four asymptotic possible behaviors depending on the form of $\hat{\gamma}(s)$ and $\hat{C}(s)$ when $s \to 0$. The Laplace transform of $C(t)$ at $s=0$ is given by

$$\hat{C}(s=0) = \int_0^\infty dt'\, C(t'). \tag{52}$$

(i) A possibility is that the integral (52) is finite, i.e.,

$$\hat{C}(s) \sim C_0 + o(1) \quad (s \to 0), \tag{53}$$

with C_0 a nonvanishing constant. This case can be examined noticing that $\rho(t)$ behaves as

$$\rho(t) \approx C_0 G(t) \quad (t \to \infty), \tag{54}$$

where $G(t)$ is given by (35) or (39), depending on the behavior of $\hat{\gamma}(s)$ when $s \to 0$.

(ia) When also $\hat{\gamma}(0) = \gamma_0$ is a nonvanishing constant one gets

$$\dot{\sigma}_{xx}(t) = \sigma_{xv}(t) \approx 2C_0\frac{1}{\gamma_0^2}e^{-2\omega^2 t/\gamma_0}, \tag{55}$$

$$\dot{\sigma}_{vv}(t) \approx 2C_0\frac{\omega^4}{\gamma_0^4}e^{-2\omega^2 t/\gamma_0}, \tag{56}$$

showing that this case corresponds to a normal diffusion.

(ib) When $\hat{\gamma}(0)$ is zero or diverges, inserting (39) and (54) into (49) and (50) yields

$$\dot{\sigma}_{xx}(t) = 2\sigma_{xv}(t) \approx 2\frac{C_0}{\omega^8}\left(\frac{d\tilde{\gamma}(t)}{dt}\right)^2, \tag{57}$$

$$\dot{\sigma}_{vv}(t) \approx 2\frac{C_0}{\omega^8}\left(\frac{d^2\tilde{\gamma}(t)}{dt^2}\right)^2. \tag{58}$$

Then, in this case the presence of long tails in $\tilde{\gamma}(t)$ indicates that the system diffuses anomalously.

(ii) In what follows, we investigate the other possibility for the behavior of $\hat{C}(s)$ when $s \to 0$. If the integral (52) is zero or divergent, taking the limit $s \to 0$ in the Laplace transform (51) and using (6) one obtains

$$\hat{\rho}(s) = \frac{\hat{C}(s)}{s^2 + s\hat{\gamma}(s) + \omega^2} \approx \frac{\hat{C}(s)}{\omega^2} \quad (s \to 0). \tag{59}$$

Applying Tauberian's theorems [15] in (59), one can deduce that for long times

$$\rho(t) \approx \frac{\tilde{C}(t)}{\omega^2}, \tag{60}$$

where $\tilde{C}(t)$ is the behavior of the noise correlation function for long times.

(iia) If $\hat{\gamma}(0)$ is a nonvanishing constant, the variance behavior is obtained inserting (60) and (35) into (49) and (50). Then,

$$\dot{\sigma}_{xx}(t) = 2\sigma_{xv}(t) \approx \frac{2}{\gamma_0\omega^2}\tilde{C}(t)e^{-2\omega^2 t/\gamma_0}, \tag{61}$$

$$\dot{\sigma}_{vv}(t) \approx -\frac{2}{\gamma_0^2}\frac{d\tilde{C}(t)}{dt}e^{-2\omega^2 t/\gamma_0}. \quad (62)$$

In this case the evolution shows an exponential component, independently of the form of $\tilde{C}(t)$.

(iib) Finally, when both $\hat{\gamma}(0)$ and $\hat{C}(0)$ are zero or diverge, the behavior of the variances is given by

$$\dot{\sigma}_{xx}(t) = 2\sigma_{xv}(t) \approx -\frac{2}{\omega^6}\tilde{C}(t)\frac{d\tilde{\gamma}(t)}{dt}, \quad (63)$$

$$\dot{\sigma}_{vv}(t) \approx -\frac{2}{\omega^6}\frac{d\tilde{C}(t)}{dt}\frac{d^2\tilde{\gamma}(t)}{dt^2}, \quad (64)$$

where we have inserted (39) and (60) into (49) and (50). Then, in this case the presence of long tails in $\gamma(t)$ and $C(t)$ implies that the particle anomalously diffuses. As expected, expressions (63) and (64) agree with the internal noise solutions (43)–(45) when the validity of the fluctuation-dissipation relation (3) is imposed.

V. EXAMPLES

In what follows we will apply the previous results to a specific model. For this purpose we consider a GLE characterized by a memory kernel and a noise correlation function of the form [3,6,16]

$$\gamma(t) = \frac{\gamma_\lambda}{\Gamma(1-\lambda)}t^{-\lambda}, \quad (65)$$

$$C(t) = \frac{C_\alpha}{\Gamma(1-\alpha)}t^{-\alpha}, \quad (66)$$

where $0<\lambda,\alpha<2$ and $\lambda,\alpha\neq 1$. The proportionality coefficients γ_λ and C_α are positive, independent of time, but can be functions of the exponents λ and α. Then, the Laplace transform of the memory kernel is given by

$$\hat{\gamma}(s) = \gamma_\lambda s^{\lambda-1}. \quad (67)$$

Note that condition (18) is satisfied if $\lambda>0$ and condition (27) implies that $\lambda<2$. On the other hand, one can realize that $\hat{\gamma}(0)$ diverges for $0<\lambda<1$ and $\hat{\gamma}(0)=0$ for $1<\lambda<2$.

Substitution of (65) and (66) into (63) and (64) yields

$$\sigma_{xx}(t) \sim -\frac{2\lambda\gamma_\lambda C_\alpha t^{-(\lambda+\alpha)}}{\omega^6(\lambda+\alpha)\Gamma(1-\lambda)\Gamma(1-\alpha)}, \quad (68)$$

$$\sigma_{vv}(t) \sim -\frac{2\lambda(\lambda+1)\alpha\gamma_\lambda C_\alpha t^{-(\lambda+\alpha+2)}}{\omega^6(\lambda+\alpha+2)\Gamma(1-\lambda)\Gamma(1-\alpha)}, \quad (69)$$

$$\sigma_{xv}(t) = \frac{\lambda\gamma_\lambda C_\alpha t^{-(\lambda+\alpha+1)}}{\omega^6\Gamma(1-\lambda)\Gamma(1-\alpha)}. \quad (70)$$

If the noise is internal, then $\alpha=\lambda$ and $\gamma_\lambda=\beta C_\lambda$. Then, inserting (65) into (43)–(45) yields

$$\beta\sigma_{xx}(t) = \frac{1}{\omega^2} - \frac{\gamma_\lambda^2}{\omega^6\Gamma(1-\lambda)^2}t^{-2\lambda}, \quad (71)$$

$$\beta\sigma_{vv}(t) = 1 - \frac{\lambda^2\gamma_\lambda^2}{\omega^6\Gamma(1-\lambda)^2}t^{-2(\lambda+1)}, \quad (72)$$

$$\beta\sigma_{xv}(t) = \frac{\lambda\gamma_\lambda^2}{\omega^6\Gamma(1-\lambda)^2}t^{-2\lambda-1}, \quad (73)$$

and, from (46) and (65), the VACF is written as

$$C_v(t) \approx -\frac{\lambda(\lambda+1)\gamma_\lambda}{\omega^4\Gamma(1-\lambda)}t^{-(\lambda+2)}. \quad (74)$$

Expressions (71)–(74) coincide with the long-time behavior of the exact solutions found in Ref. [6] for the whole range of time. Note that Eq. (74) implies that the VACF decays with a positive power-law tail for $1<\lambda<2$ and with a long negative tail when $0<\lambda<1$. This negative correlation was called "whip-back" effect in the free particle situation [16–18]. This effect is responsible for the slower diffusion of the particle (subdiffusion).

Finally, we want to stress the following issue. The previous analysis is valid for any memory kernel and noise correlation function that they behave like a power law for long times, provided that $\hat{\gamma}(0)$ is zero or diverges. In particular, we recently propose a memory kernel modeled as [19]

$$\gamma(t) = \gamma_\lambda\tau^{-\lambda}E_\lambda[-(t/\tau)^\lambda], \quad (75)$$

where $E_\lambda(y)$ denotes the Mittag-Leffler function [20] defined through the series

$$E_\lambda(y) = \sum_{j=0}^{\infty}\frac{y^j}{\Gamma(\lambda j+1)}, \quad \lambda>0, \quad (76)$$

τ acts as a characteristic memory time, γ_λ is a proportionality coefficient, and $0<\lambda<2$. In this case, the Laplace transform of the memory kernel reads

$$\hat{\gamma}(s) = \frac{\gamma_\lambda s^{\lambda-1}}{1+s^\lambda\tau^\lambda}, \quad (77)$$

which again diverges for $0<\lambda<1$ and vanishes for $1<\lambda<2$.

The memory kernel (75) behaves as a stretched exponential for short times. For long times, using the asymptotic behavior of the Mittag-Leffler function [20]

$$E_\lambda(-y) \approx [y\Gamma(1-\lambda)]^{-1}, \quad y>0, \quad (78)$$

it can be deduced that the Mittag-Leffler memory kernel (75) behaves as an inverse power law like (65) for $\lambda\neq 1$. Therefore, the asymptotic solutions of the generalized Langevin equation with a Mittag-Leffler memory kernel (75) and noise correlation function are the same that those obtained using the power-law functions (65) and (66).

VI. CONCLUSIONS

In this work we have investigated the memory effects in the asymptotic behavior of a particle under the influence of a harmonic potential and governed by a generalized Langevin equation with arbitrary memory kernel and noise correlation function. We have obtained unifying expressions for the asymptotic behavior of the displacement, velocity, variances, and velocity autocorrelation function in the cases of internal and external noise. This behavior depends on a simple form of the long-time behavior of the memory kernel and the noise correlation function, which can be characterized through its Laplace transforms $\hat{\gamma}(s)$ and $\hat{C}(s)$ when $s \to 0$. The simple and quite general results presented here enable us to characterize the long-time behavior of the diffusing oscillator and make its application possible in the analysis of diverse diffusive processes.

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