Transport phenomena in the asymmetric quantum multibaker map

Leonardo Ermann,[1,2] Gabriel G. Carlo,[1] and Marcos Saraceno[1,3]
[1]*Departamento de Física, CNEA, Libertador 8250, (C1429BNP) Buenos Aires, Argentina*
[2]*Departamento de Física, FCEyN, UBA, Pabellón 1 Ciudad Universitaria, C1428EGA Buenos Aires, Argentina*
[3]*Escuela de Ciencia y Tecnología, UNSAM, Alem 3901, B1653HIM Villa Ballester, Argentina*
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By studying a modified (unbiased) quantum multibaker map, we were able to obtain a finite asymptotic quantum current without a classical analog. This result suggests a general method for the design of *purely* quantum ratchets and sheds light on the investigation of the mechanisms leading to net transport generation by breaking symmetries of quantum systems. Moreover, we propose the multibaker map as a resource to study directed transport phenomena in chaotic systems without bias. In fact, this is a paradigmatic model in classical and quantum chaos, but also in statistical mechanics.

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I. INTRODUCTION

In recent years many works have investigated different kinds of transport phenomena in periodic dynamical systems having no external net force or bias (the so-called ratchet effect) [1]. This interest is of fundamental character, but it is also motivated by the fact that many possible applications exist. For example, they can be useful to understand and develop rectifiers, pumps, particle separation devices, molecular switches, and transistors. Also, there is great interest in biology, since the working principles of molecular motors can be explained in terms of ratchet mechanisms [2]. Finally, we would like to mention cold atoms and Bose-Einstein condensates as promising fields of application, thanks to recent developments of the techniques needed to manipulate them [3,4]. In the following we will have in mind systems that present chaotic features, since ratchets generally behave this way [5–7]. Hence, methods from classical and quantum chaos become extremely useful.

The explanation of the appearance of a net current—i.e., average momentum different from zero—is one of the main topics of this research. In a classical context, this behavior can be understood in terms of broken symmetries. One of the most convincing points of view up to now is that all symmetries of the system leading to momentum inversion (i.e., sign change) should be broken in order to have a net directed current [7]. This amounts to saying that, being only a necessary condition, we can follow Curie's principle and assume that if the current is not forbidden by symmetries, then it should be present. In non-Hamiltonian cases, the asymmetrization of a chaotic attractor leads to a net directed current [5]. In Hamiltonian systems the asymmetrization of a chaotic layer embedded in a mixed phase space, for example, has the same consequences for a set of initial conditions inside of it [8,9]. In this case, a mixed dynamics is in general a necessary condition (for a notable exception see [10,11]).

The vast majority of the previously mentioned papers were focused on the classical aspects, leaving the quantum side less explored. However, there have been several recent publications that deal with this part of the problem. These works regard both the experimental [12] (systems of cold atoms, mainly) and the theoretical sides [8,10,13]. In general the quantum versions share symmetry aspects with their classical counterparts, showing the corresponding current. But sometimes the relation between symmetries and the generated current is less obvious in the quantum case. Tunneling, for example, can modify the direction of the current [14]. Interference, in any of its forms, generates more complex behaviors [15]. In fact, we will see that the quantum and classical behaviors can be very different.

A remarkable situation appears in some cases, when one can find a net quantum current that does not have a classical counterpart. This was essentially studied in Hamiltonian (nondissipative) systems. The first time this phenomenon was found was in the modified kicked rotor (KR) at quantum resonance (i.e., the usual KR with a second harmonic in the kick) [16,17]. This is a classically chaotic system where the time-reversal symmetry changing the sign of the momentum is kept. Later, in the case of the modified kicked Harper model [18], the classical current was found to be exceedingly small in comparison with the quantum one. In all of these cases the current does not reach an asymptotic value, and in fact these systems were called quantum ratchet accelerators. We present here a different behavior by means of a modified multibaker map [19,20]. This system models a particle evolving with free flights and collisions with fixed scatterers. We were able to find a *finite asymptotic* current that is only present in the quantized version. This suggests a general method for obtaining *purely* quantum ratchets (no classical current), without unbounded acceleration. We propose this system as a model to study general quantum transport phenomena in the presence of asymmetries.

II. MODEL

The well-known classical baker's transformation is an area-preserving map defined on the unit square phase space ($0 \leqslant q, p \leqslant 1$). Here we use an asymmetric version which divides the phase space into two regions with different areas and Lyapunov exponents. The general form of this map can be written in terms of one parameter $s \in (0,1)$:

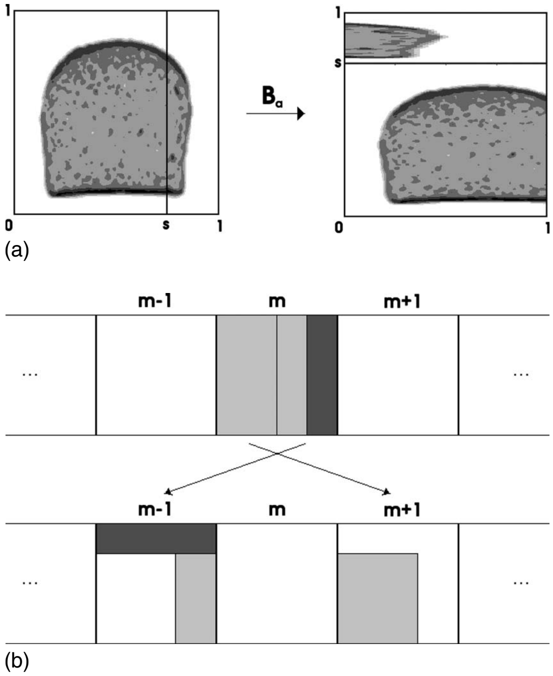

FIG. 1. Geometric action of the asymmetric baker's map on each cell (top) and of the composition with the translation—i.e., the asymmetric multibaker map (bottom).

$$B_s(q,p) \equiv \begin{cases} \left(\frac{1}{s}q, sp\right), & 0 \leqslant q \leqslant s, \\ ((1-s)^{-1}(q-s),(1-s)p+s), & s \leqslant q \leqslant 1, \end{cases} \tag{1}$$

where we can recover the usual symmetrical case by setting $s=1/2$. The geometric action of the map is represented in Fig. 1 (top).

The quantum version of the map is defined in a discrete D-dimensional Hilbert space with $h=1/D$ in terms of the quantum Fourier transform with antiperiodic boundary conditions [21,22]:

$$\hat{B}_s = \hat{G}_D^\dagger \begin{pmatrix} \hat{G}_{D_1} & 0 \\ 0 & \hat{G}_{D_2} \end{pmatrix}, \tag{2}$$

$$(\hat{G}_D)_{kl} \equiv D^{-1/2} e^{-i2\pi(k+1/2)(l+1/2)/D}, \tag{3}$$

where the allowed values of s are such that $D_1=sD$ and $D_2=D-D_1$ are integer numbers. The same procedure can be done to obtain an entire family of asymmetric quantum baker maps (QBMs) [23,24].

The classical multibaker map [19] is defined in a two-dimensional lattice where the phase space at each lattice site is a unit square. The dynamics of the map is a combination of transport to neighboring cells and a local evolution within a cell. The map is defined as $M_s=B_s \circ T$, where the baker term is the asymmetric baker map defined in Eq. (1) applied on each cell m, and the transport term is defined as

$$T = \begin{cases} (m+1,q,p), & 0 \leqslant q \leqslant 1/2, \\ (m-1,q,p), & 1/2 \leqslant q \leqslant 1. \end{cases} \tag{4}$$

The geometric action of the multibaker map in the phase space of a lattice of squares is represented in Fig. 1 (bottom). While the baker's map is asymmetric, the transport term is unbiased as the phase-space volume is transported symmetrically, as can be clearly seen. Notice that the transport is entirely due to translations and therefore there are no tunneling effects from cell to cell.

This transformation is not biased either in the p or in the q coordinate; the reason is that the baker transformation maps the unit square onto itself and the transport term is balanced, as previously explained. This is the equivalent to the zero average net force typical of the dynamical systems addressed in studies of directed transport [1]. In what follows we will focus on the coarse-grained current, which in the classical case is defined by

$$J_{class}(t) = \langle m(t)\rangle - \langle m(t-1)\rangle. \tag{5}$$

In this expression $\langle m(t)\rangle$ is the average value of the cell position m for a given ensemble of initial conditions, at a time t. It is easy to see that this definition does not take care of the fluctuations inside of each cell. However, we underline that there is no bias, making this model completely general.

From the classical point of view, the presence or absence of an asymptotic current follows the general criteria specified in [7] and depends on the symmetries that reverse the sign of the transport. Here there are two such symmetries

$$S_I:\ q \to 1-q, \quad p \to 1-p, \quad T \to T, \tag{6}$$

$$S_{II}:\ q \to p, \quad p \to q, \quad T \to T^{-1}, \quad t \to -t. \tag{7}$$

The first one maps B_s to B_{1-s} and therefore is broken unless $s=1/2$. S_{II} is a time-reversal symmetry and is preserved at all times. Thus, according to the criteria of [7] there cannot be an asymptotic classical current for unbiased initial conditions. Transient effects can be present for biased conditions but will die off very rapidly due to the exponential mixing property of the Baker map. Numerical calculations confirm this expectation [see the inset of Fig. 2(b) and caption].

The quantum multibaker map is the composition of a translation on the lattice site that depends on the value of the projectors acting on the right and left subspaces of the baker map at each cell [20]. This can be written as

$$\hat{M}_s \equiv \hat{B}_s \circ \hat{T} = (\hat{I} \otimes \hat{B}_s)(\hat{U} \otimes \hat{P}_R + \hat{U}^\dagger \otimes \hat{P}_L), \tag{8}$$

where P_R and P_L are the projectors and $\hat{U}$ is a unitary translation operator acting on the lattice subspace $\hat{U}|m\rangle=|m+1\rangle$ (with $\{|m\rangle, m=\ldots,-2,-1,0,1,2,\ldots\}$ the basis set of the lattice).

III. NUMERICAL RESULTS

The discrete time propagation of an initial state $\rho_0=\rho_{\text{lat}} \otimes \rho_{\text{QBM}}$ is

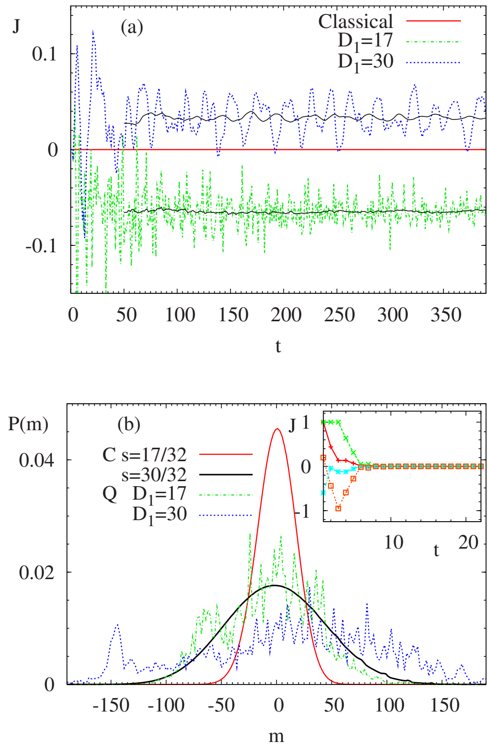

FIG. 2. (Color online) (a) Coarse-grained current J as a function of time for the QBM with $D=32$ and $D_1=17$ (dot-dashed line) and $D_1=30$ (dotted line). As solid lines the smoothed current over 20 steps is shown. The solid line at $J=0$ corresponds to the classical current calculated for 10^8 initial conditions. (b) The probability distribution $P(m,t)$ at time $t=200$ for the same cases as before, $s=17/32$ and $s=30/32$ in the classical and quantum versions with $D=32$. In the inset we show how the net current becomes null in the classical case for different initial conditions, calculated for 10^8 points in phase space, in particular for two squares of area $1/16$ centered in $(q,p)=(1/8,1/2)$ [$s=0.6$ (+) and $s=0.8$ (×)] and $(3/8,1/2)$ [$s=0.6$ (★) and $s=0.8$ (□)].

$$\rho(t) = (\hat{M}_s)^t \rho_0 (\hat{M}_s^{\dagger})^t. \tag{9}$$

Here we will focus on initial states localized in one site of the lattice $\rho_{\text{lat}}=|0\rangle\langle 0|$. A word regarding the analogy with quantum walks is in order here. In fact, this system can be thought as the D-dimensional quantum baker map coupled to a quantum walker in an infinite-dimensional lattice [25,26]. The role of the *quantum coin* in the quantum walk is performed here by the quantum baker map. The fact that the *coin* is classically ergodic is, however, an important difference and is a fundamental reason for having no classical current.

The transport properties of the system can be computed with the coarse-grained density of probabilities of the lattice. This distribution is obtained by tracing out the internal degrees of freedom inside each cell and projecting on the lattice basis. Since

$$P(m,t) = \text{Tr}[(|m\rangle\langle m| \otimes I_{\text{QBM}})\rho(t)], \tag{10}$$

the mean values of the coarse-grained position and the quantum current become

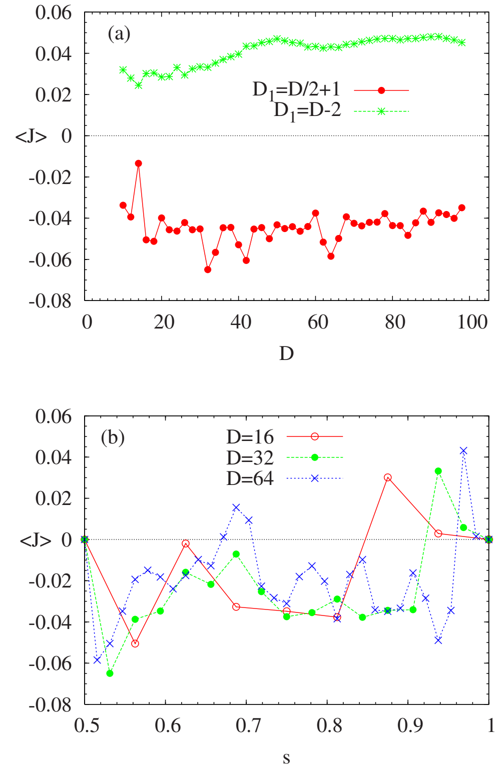

FIG. 3. (Color online) Averaged coarse-grained current $\langle J\rangle$ (a) for different values of the dimension of the Hilbert space (D) of the QBM with fixed $D_1=D/2+1$ and $D_1=D-2$ and (b) for the rational values of $s=D_1/D\in[0.5;1)$ with $D=16,32,64$. The average of J is computed up to 450 iterations of the different maps beginning with $t=100$ (where the current approximately reaches its asymptotic behavior).

$$\langle x(t)\rangle = \sum_{m=-\infty}^{\infty} mP(m,t), \tag{11}$$

$$J(t) = \langle x(t)\rangle - \langle x(t-1)\rangle. \tag{12}$$

We now turn to analyze the current behavior of this system by means of numerical simulations. First, we point out that if we take a mixed superposition of centered eigenstates of momentum as the initial condition for the QBM, we find that there is no current (J) for the symmetric case ($s=1/2$). This confirms the need to break the S_I symmetry in order to have a net directed transport. Throughout the following calculations we consider as initial condition the mixed state corresponding to the incoherent superposition of the two central momentum eigenstates—i.e.,

$$\rho_{\text{QBM}} = \frac{1}{2}\hat{G}_D\left(\left|\frac{D}{2}-1\right\rangle\left\langle\frac{D}{2}-1\right| + \left|\frac{D}{2}\right\rangle\left\langle\frac{D}{2}\right|\right)\hat{G}_D^{\dagger}. \tag{13}$$

Figure 2(a) shows the coarse-grained current as a function of time with Hilbert space dimension $D=32$. We have plotted the results corresponding to its maximum negative and positive asymptotic values, for which $D_1=17$ and $D_1=30$, respectively ($s=D_1/D\in[0.5,1)$). It is worth mentioning that J

rapidly reaches a stationary behavior (at about $t=50$) with fluctuations centered around a finite nonzero value as is illustrated by the smoothed current in Fig. 2 (solid lines). This shows that no unbounded acceleration is present, so there is a rectification of transport rather than an effective force in this purely quantum ratchet. The classical current for analog initial conditions is also displayed, being zero at all times. In Fig. 2(b) we show the probability distribution $P(m,t)$ and its classical version.

In Fig. 3(a) we can see the averaged (asymptotic) coarse grained current $\langle J \rangle$ as a function of the Hilbert space dimension D for $D_1=D-2$ and $D_1=D/2-1$ (in order to make this averages we have taken the values of the current from $t=100$ up to $t=450$). These cases are those at which J approximately reaches its maxima in absolute value (positive and negative current, respectively). We observe a smooth dependence of this quantity on D (apart from small fluctuations), showing that the effect is generic. In fact, this situation is completely different from previous cases [16,17]. In these works a purely quantum current was found only at resonant values of a modified KR.

In Fig. 3(b) we show the averaged current versus s for different fixed values of D with $s \geqslant 0.5$. We do not show values for $s<0.5$ since J turns out to be an odd function of s around $s=0.5$ ($\langle J_s \rangle = -\langle J_{1-s} \rangle$). In fact, if we apply the symmetry transformation S_I to Eq. (9) and then trace out the internal degrees of freedom inside of each cell, we obtain that $P_s(m,t)=P_{1-s}(-m,t)$ for all t. This result is valid for any initial ρ_{QBM} symmetrical under S_I. Therefore, the symmetry of the current becomes clear (details will be presented elsewhere [28]). As a consequence, there is a simple way to obtain current inversion in our system.

IV. CONCLUSIONS

In summary, we have found a finite asymptotic current in a modified and unbiased quantum multibaker map; this current has no classical counterpart. Moreover, we can invert its direction by exploiting a symmetry with respect to the S_I transformation. This is much in the same spirit as the corresponding current inversions found in the literature [6,11].

The behavior of our system is rather similar to what happens in quantum walks. Different combinations of the initial states and/or the unitary operators associated with the quantum coin produce a different bias in the final distribution of probabilities [27]. We believe that this property, which is of pure quantum origin, is the reason for finding a net transport in our system. In fact, we think that the existence of a net current is due exclusively to interference effects which persists in the $D \to \infty$ limit. To obtain the classical behavior—i.e., a vanishing current—some noise has to be introduced, just as in the case of a quantum walk [26].

We underline that this is a completely different case than that explained in [13], where the quantum current appearance is explained by means of the desymmetrization of Floquet states, a fact that can be directly related to the corresponding classical properties of the system. A detailed study of this will be presented elsewhere [28]. Finally, we consider that the features of this map can be exploited to design generic Hamiltonian systems that behave in a similar way. In that case, there will be the possibility to implement them in cold atom experiments.

ACKNOWLEDGMENTS

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