Regularization of the displacement moments for asymmetric Lévy flights

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Although the second displacement moments for Lévy flights are not defined in their usual sense, a few years ago it was shown that nonextensive statistical mechanics can be used to define them for symmetric flights. Here it is shown that the displacement moments for long-jump *asymmetric* Lévy flights can also be regularized by calculating the averages in the form prescribed by nonextensive statistical mechanics. The dependence of the generalized diffusion coefficient on the asymmetry strength is investigated. It is also shown that no extremum q-entropy principle can be associated with the asymmetric Lévy attractors.

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I. INTRODUCTION

Originally the province of mathematicians, work on Lévy distributions has found in the past two decades numerous applications in fields as diverse as fluid mechanics, ecology, economy, and even cancer growth [1–9]. This activity has generated a renewed effort to understand and describe the properties of the Lévy distributions and their relation to those distributions that belong to their attraction basins. For instance, Weeks and Swinney combined strongly asymmetric Lévy flights and sticking probabilities to find a rich collection of anomalous diffusion results [10], while Mantegna and Stanley studied the slow convergence of truncated Lévy flights finding a well-defined crossover between the Lévy and Gaussian regimes [11]. In this connection, it is worth mentioning that, for a symmetric random walk whose jump probabilities have a finite third moment, the Berry-Esséen theorem applies [12]. If a truncated Lévy flight with a probability distribution $p(x) \sim x^{-(1+\alpha)}$ and a cutoff length L is considered, Shlesinger remarked that the number of steps needed to observe Gaussian behavior is $N \sim L^{\alpha}$ [13]. In their review, Metzler and Klafter discuss numerous results for Lévy processes in the framework of a fractional dynamics approach to anomalous diffusion [14]. The first passage times and leapover properties of Lévy motions have also been investigated in detail [15,16].

A crucial result in the theory is Gnedenko's theorem defining the attraction basins of the Lévy distributions [17]. However, the theorem only identifies what the basins are; it does not tell us how fast the probability distributions converge to their attractors. Since experiments are performed over finite times, we cannot in general observe the attractors directly; therefore, it would be useful to characterize the evolution of the distributions toward their attractors. In the case of distributions satisfying the usual central limit theorem (CLT), the answer is provided by Chebyshev's theorem (see Chap. 8 in Ref. [17]). Although, as far as we know, there is no analog theorem for distributions located in the basins of the Lévy attractors, in the context of nonextensive statistical mechanics (NSM) [18] Abe and Rajagopal investigated the rates of convergence of nonextensive statistical distributions to Lévy distributions using a suitable approximation of the corresponding characteristic function [19]. Hopcraft, Jakeman, and Matthews studied the properties of the discrete counterparts of the Lévy densities, calculating the rates of convergence to the corresponding limit distributions [20].

The displacement moments for Lévy flights are not defined in the usual sense, but work by Zanette and Alemany [21,22] and by Tsallis and co-workers [23,24] has shown that it is possible to use NSM to consistently define the second moment for distributions located in the basins of *symmetric* Lévy attractors. It was also shown [23,25] that the symmetric Lévy functions maximize the q entropy just as the Gaussian distribution maximizes the usual Boltzmann-Gibbs entropy. The short and intermediate time properties of the generalized mean square displacement were also discussed and the conditions under which the asymptotic analytical formulas could be applied to finite-time experiments were investigated in Ref. [26].

Much less work has been done on *asymmetric* Lévy functions. Again, we should mention the paper of Abe and Rajagopal [19], who found that functions belonging to a subset of the one-sided Lévy function basins optimize the nonadditive entropy in half space provided the first generalized moment is used as a constraint. There are several questions in this area whose answers remain unknown. The first is whether we can use NSM to define the second moment for distributions located in the basins of *asymmetric* Lévy attractors. A second question is whether there is a maximum entropy principle associated with asymmetric distributions. A final question is whether we can numerically ascertain the rate of convergence toward asymmetric attractors. The purposes of this work are therefore (a) to determine the conditions under which we can define generalized displacement moments, (b) to ascertain if the NSM can be used to buttress these definitions, (c) to examine the influence of single-step asymmetries on the displacement moments, (d) to investigate the dependence of the moments on step number, and (e) to analyze their speed of convergence toward the Lévy-controlled asymptotic forms.

In Sec. II we review the main properties of asymmetric Lévy flights and the associated central limit theorems. In Sec. III we apply the NSM prescriptions to these flights, while in Sec. IV we present and discuss simulational and numerical results for the regularized displacement moments. The nonexistence of a maximum entropy principle for the asymmetric Lévy distributions is proved in the Appendix.

II. ASYMMETRIC LONG JUMP DISTRIBUTIONS

We are interested in jump distributions whose asymptotic decay is algebraic, i.e., distributions of the form

$$p(x) \simeq \begin{cases} c_-|x|^{-(1+\eta)} & \text{if } x \to -\infty, \\ c_+x^{-(1+\kappa)} & \text{if } x \to \infty, \end{cases} \quad (1)$$

where c_- and c_+ are non-negative constants and $\eta,\kappa > 0$. Considering the asymptotic form of these distributions, we may have three types of asymmetry.

(a) Weak asymmetry,

$$\eta = \kappa, \quad c_- \neq c_+, \quad c_-, c_+ \neq 0,$$

(b) strong asymmetry,

$$\eta \neq \kappa,$$

and (c) unilateral distributions,

$$c_- = 0 \quad \text{or} \quad c_+ = 0.$$

It is relevant to determine whether these distributions belong to the attraction basin of a stable law. As mentioned in the Introduction, the answer to this question is provided by a powerful theorem due to Gnedenko [17], which states that a distribution $p(x)$ belongs to the attraction basin of the Lévy function $L_{\alpha,\beta}$ if and only if

$$p(x) \simeq \begin{cases} c_-|x|^{-(1+\alpha)} & \text{if } x \to -\infty, \\ c_+x^{-(1+\alpha)} & \text{if } x \to \infty, \end{cases} \quad (2)$$

with $0 < \alpha < 2$, and

$$\beta = \frac{c_+ - c_-}{c_+ + c_-}. \quad (3)$$

This result is usually known as the generalized central limit theorem (GCLT). The Lévy function $L_{\alpha,\beta}(\mu;x)$ is the Fourier transform of the characteristic function $\varphi_\mu(k)$, where

$$\ln \varphi_\mu(k) = i\mu k - a|k|^\alpha\left[1 + i\beta\frac{k}{|k|}\tan\left(\frac{\pi\alpha}{2}\right)\right]. \quad (4)$$

The tangent must be replaced by $(2/\pi)\ln k$ if $\alpha=1$. Here $a > 0$ and μ are real constants, defining the distribution scale and location, respectively, while the skewness parameter β is directly related to the asymmetry. According to the GCLT, weakly asymmetric power-law distributions belong to the attraction basin of $L_{\alpha,\beta}$, with $-1 < \beta < 1$. One-sided distributions belong to the attraction basin of $L_{\alpha,1}$ (if $c_-=0$) or $L_{\alpha,-1}$ (if $c_+=0$). Strongly asymmetric distributions belong to the attraction basin of $L_{\alpha,1}$ (or $L_{\alpha,-1}$), where we must choose $\alpha = \min(\eta,\kappa)$, as can be seen from a study of the repartition function using the Gnedenko-Doeblin theorem [7,17].

Note that the function $L_{\alpha,1}$ ($L_{\alpha,-1}$) has support on $\mathbb{R}^+$ ($\mathbb{R}^-$) for $\alpha \in (0,1)$, and on the whole real line for $\alpha \in (1,2)$ [27]. The cases $\alpha \in (0,1)$ and $\alpha \in (1,2)$ have been called, respectively, one-sided and two-sided extremal Lévy stable probability distribution functions (PDFs) [16]. A two-sided extremal Lévy PDF decays as a power law on the "favored" side and as $\exp[-g(\alpha)|x|^{\alpha/(\alpha-1)}]$, where $g(\alpha)$ is a function of α alone, on the "wrong" side [27]. The decay of this tail is slower for higher values of α. Of particular interest are the marginal behaviors: If $\alpha \to 1^+$, the tail disappears very fast [as $\exp(-\exp|x|)$ if $\alpha=1$ [27]], while if $\alpha \to 2^-$ it becomes a Gaussian. Since a unilateral jump distribution may belong to the basin of a two-sided extremal Lévy function, we must exercise some care to avoid confusion. Thus we reserve the name "unilateral" for the one-jump distributions with either $c_-=0$ or $c_+=0$. The Gnedenko-Doeblin theorem ensures that the attractors for the unilateral sums are the genuinely one-sided Lévy distributions for $\alpha \in (0,1)$ and the two-sided extremal Lévy functions for $\alpha \in (1,2)$.

If η and κ are both larger than two, the usual CLT applies and the stable law is a Gaussian.

III. NONEXTENSIVE STATISTICAL MECHANICS PREDICTIONS

In this section we consider N-step distributions built through the convolution of the one-step distributions characterized by Eq. (1). Although, if either η or $\kappa < 2$, the conventional second moment does not exist, it has been shown that the q moments of second order are well defined for symmetric distributions with $\alpha \in (0,2)$ [25]. Next, we apply the arguments of Ref. [25] to asymmetric PDFs. The one-step q average of a function $f(x)$ is defined as

$$\langle f(x)\rangle_q = \frac{\int_{-\infty}^{\infty} f(x)[p(x)]^q dx}{\int_{-\infty}^{\infty} [p(x)]^q dx}, \quad (5)$$

where q is a real number. If $P(N,x)$ is the N-step PDF, the N-step Rth order q moment (indicated by the double angular brackets) must be calculated using the formula

$$\langle\langle x^R(N)\rangle\rangle_q = \frac{\int_{-\infty}^{\infty} x^R[P(N,x)]^q dx}{\int_{-\infty}^{\infty} [P(N,x)]^q dx}. \quad (6)$$

In the case of symmetric distributions ($\eta=\kappa=\alpha$ and $c_+=c_-$), the GCLT says that, in the large N limit, $P(N,x) \sim N^{-1/\alpha}L_{\alpha,0}(N^{-1/\alpha}x)$. A few years ago [26], we investigated numerically the behavior of $\langle\langle x^2(N)\rangle\rangle_q$ using the original (unnormalized) averages [23], finding that $\langle\langle x^2(N)\rangle\rangle_q \sim N^{2/(\alpha+1)}$ for large N. This result must be modified if the (now preferred) normalized average is used. We find that, asymptotically,

$$\langle\langle x^R(N)\rangle\rangle_q = N^{R/\alpha}\frac{\int_{-\infty}^{\infty} y^R[L_{\alpha,0}(y)]^q dy}{\int_{-\infty}^{\infty} [L_{\alpha,0}(y)]^q dy} \equiv K_{\alpha,R,0}(q)N^{R/\alpha}. \quad (7)$$

Here $D_q(\alpha)=K_{\alpha,2,0}(q)$ is a generalized (or q-) diffusion coefficient.

The q entropy is defined as

$$S_q(p) = \frac{k_B}{q-1}\left\{1 - \int_{-\infty}^{\infty}[\sigma p(x)]^q \frac{dx}{\sigma}\right\}, \quad (8)$$

where k_B is Boltzmann's constant and σ is a characteristic length. It was found that the optimization of $S_q(p)$, subject to normalization and to the constraint $\langle x^2\rangle_q = \langle\langle x^2(1)\rangle\rangle_q = \sigma^2$ yielded one-step distributions that behaved as $L_{\alpha,0}(x)$ for large $|x|$, provided that [25]

$$q = \frac{\alpha+3}{\alpha+1}. \quad (9)$$

It has been proposed [23,25] that $\langle\langle x^2(N)\rangle\rangle_q$ should be regularized using the value of q given by Eq. (9). Abe and Rajagopal, on the other hand, optimized the q entropy in half-space for $\alpha \in (0,1)$ using as constraints the normalization condition and the first displacement moment $\langle x\rangle_q$. They found that the q entropy is maximized by a Zipf-Mandelbrot distribution with $q=(\alpha+2)/(\alpha+1)$ [19]. Consequently, after many iterations, the optimized PDF must converge to the stable one-sided Lévy distribution $L_{\alpha,1}(N^{-1/\eta}x)$.

Given that, according to the GCLT, the Lévy function $L_{\alpha,\beta}$ is the attractor in the weak asymmetry class, it is tempting to propose that the distribution $L_{\alpha,\beta}(N^{-1/\eta}x)$, with β given by Eq. (3), should optimize the q entropy under suitable constraints. To test this idea we use Lagrange multipliers to find the distributions $p(x)$ that optimize the q entropy subject to the conditions (see also Ref. [28])

$$\int_{-\infty}^{\infty} p(x)dx = 1, \quad (10)$$

$$\langle x\rangle_q = v, \quad (11)$$

and

$$\langle x^2\rangle_q - \langle x\rangle_q^2 = \sigma^2. \quad (12)$$

These constraints are the natural generalizations of the conditions imposed by Prato and Tsallis to find the symmetric functions that optimize the entropy [25]. The result of the optimization process depends on the value of q. If $q<1$, we obtain a distribution with compact support,

$$p(x) = \frac{\Gamma\left(\frac{5-3q}{2-2q}\right)}{\pi^{1/2}u_0\Gamma\left(\frac{3-q}{1-q}\right)}\left[1 - \left(\frac{x-v}{u_0}\right)^2\right]^{1/(1-q)}$$
$$\text{if } |x-v| < u_0, \quad (13)$$

and $p(x)=0$ elsewhere. Here $u_0=\sigma[(3-q)/(1-q)]^{1/2}$ and $\Gamma(x)$ is the usual gamma function.

If $q>1$, the optimal distribution is

$$p(x) = \frac{\Gamma\left(\frac{1}{q-1}\right)}{\pi^{1/2}u_0\Gamma\left(\frac{3-q}{2q-2}\right)}\left[1 + \left(\frac{x-v}{u_0}\right)^2\right]^{1/(1-q)}. \quad (14)$$

The Lagrange multipliers related to the conditions (11) and (12), λ_1 and λ_2, are, respectively,

$$\lambda_1 = \frac{-v\sigma^{q-3}}{\Pi_q} \quad \text{and} \quad \lambda_2 = \frac{\sigma^{q-3}}{2\Pi_q},$$

where $\Pi_q = \int_{-\infty}^{\infty}[p(x)]^q dx$. The limit for $q\to 1$ corresponds to Boltzmann-Gibbs thermostatistics [23]. In this case $\lambda_1 = -v/\sigma^2$, which measures the relative importance of advection and diffusion, and $\lambda_2 = 1/2\sigma^2$.

These optimal distributions reduce to those corresponding to the symmetric problem [25], except for a displacement of magnitude v. This result implies that the asymptotic form of $p(x)$ for $q>1$ is symmetric,

$$p(|x|\to\infty) \sim |x|^{-2/(q-1)}, \quad (15)$$

with not only the same exponent, but also the same coefficient in both directions. This has an important consequence: According to the GCLT, if $q>5/3$, the N-step distribution has the large N form,

$$P_q(N,x) = \frac{1}{N^{1/(\alpha+1)}} L_{\alpha+1,0}\left(\frac{x-\mu}{N^{1/(\alpha+1)}}\right), \quad (16)$$

where the relation between q and α is given by Eq. (9) and we have added the subscript q to the PDF to indicate that it emerges from the optimization of the q entropy. If $q<5/3$, the attractor is a Gaussian.

The parameter μ in Eq. (16) corresponds to a translation of the Lévy function; for $\alpha>0$, it is given by

$$\mu = \left[\frac{2\Gamma(\alpha+1)\sin\left(\frac{\pi\alpha}{2}\right)}{\pi(c_+ + c_-)}\right]^{1/\alpha} N^{(\alpha-1)/\alpha} v, \quad (17)$$

where v is the first moment of the one-step distribution appearing in Eq. (14) and, according to the remark following Eq. (15), we must choose $c_+ = c_-$. Therefore, we conclude that the distributions that optimize the entropy with the "natural" constraints of Eqs. (10)–(12) have, at most, an asymmetry that can be removed through a translation. They never evolve toward the genuinely asymmetric Lévy distributions with $\beta \neq 0$. This result falls short of our hopes: With the chosen conditions, the attractor for the weak asymmetry case does not optimize the q entropy. In the Appendix we generalize this result, showing that the asymmetric Lévy distributions cannot optimize the entropy for any arbitrary conditions on the integer displacement moments (see, however, Ref. [19] for the one-sided case).

A further question arises. Even in the absence of an optimization principle, is it possible to use the NSM to regularize the displacement moments? To answer this question we use the GCLT to generalize Eq. (7) to the case of asymmetric attractors,

$$\langle\langle x^R(N)\rangle\rangle_q = N^{R/\alpha}\frac{\int_{-\infty}^{\infty} y^R[L_{\alpha,\beta}(y)]^q dy}{\int_{-\infty}^{\infty}[L_{\alpha,\beta}(y)]^q dy} \equiv K_{\alpha,R,\beta}(q)N^{R/\alpha}. \tag{18}$$

This equation is valid in the large N limit. It follows because the stable distributions are the asymmetric Lévy functions $L_{\alpha,\beta}$, where in the strong asymmetry case we must choose $\alpha=\min(\eta,\kappa)$, $\beta=\pm 1$. In the one-sided case, $\alpha\in(0,1)$, the integrations must be restricted to the corresponding half-line: $(-\infty,0]$ for $\beta=-1$ and $[0,\infty)$ for $\beta=1$. It is remarkable that, for strong asymmetry, the asymptotic form of $\langle\langle x^R(N)\rangle\rangle_q$ is completely independent from the largest exponent, $\alpha_{\max}=\max(\eta,\kappa)$.

A caveat is necessary here: Since $L_{\alpha,\beta}(y)\sim|y|^{-(\alpha+1)}$, the moments in Eq. (18) are well defined whenever

$$q>\frac{R+1}{\alpha+1}.$$

Therefore, we could regularize the second moment by choosing any $q>3/(\alpha+1)$. This would also be valid for the symmetric problem, of course. It seems reasonable to *prescribe* that the value of q to be used should be the same as that obtained for symmetric PDFs, i.e., Eq. (9).

In particular, a generalized diffusion coefficient $D_q(\alpha,\beta)$ can be defined as

$$D_q(\alpha,\beta)=\lim_{N\to\infty}\left[\frac{1}{N^{2/\alpha}}(\Delta x)_q^2\right], \tag{19}$$

with the q variance $(\Delta x)_q^2=\langle\langle x^2\rangle\rangle_q-\langle\langle x\rangle\rangle_q^2$ characterizing the distribution width. We evaluate $D_q(\alpha,\beta)$ in the next section.

IV. NUMERICAL RESULTS

In this section we report the results of numerical simulations of various N-jump asymmetric Lévy processes. We performed Monte Carlo simulations of the Lévy flights using a generalization of the procedure introduced in Ref. [26]. We consider a particle that performs jumps of length j with a probability

$$p(j)=c_-j^{-(1+\eta)} \tag{20}$$

for jumps to the left and

$$p(j)=c_+j^{-(1+\kappa)} \tag{21}$$

for jumps to the right (j is a natural number).

First, we decide whether the particle will jump to the left (with a probability c_-) or to the right (with a probability c_+). Second, we divide the interval (0,1) in juxtaposed windows whose widths $W_L(l)$ or $W_R(l)$ (depending on the direction of

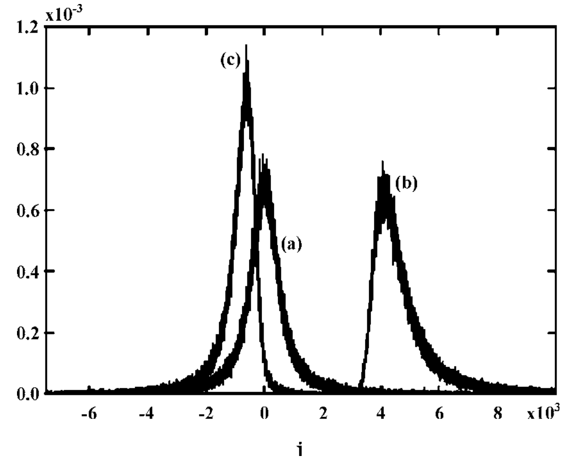

FIG. 1. Histograms for some probability distributions obtained from Monte Carlo simulations. (a) Symmetric distribution with $\eta=\kappa=1.2$, (b) one-sided ($c_-=0$) distribution with $\kappa=1.2$, and (c) strongly asymmetric distribution with $\eta=1.2$ and $\kappa=1.5$. Here $N=1500$.

the jump) are proportional to the jump probabilities for jumps of length l,

$$W_L(l)=\frac{1}{l^{\eta+1}}\left(\sum_{j=1}^{\pounds}\frac{1}{j^{\eta+1}}\right)^{-1} \tag{22}$$

and

$$W_R(l)=\frac{1}{l^{\kappa+1}}\left(\sum_{j=1}^{\pounds}\frac{1}{j^{\kappa+1}}\right)^{-1}. \tag{23}$$

Here £ is the maximum flight length. We then choose a number at random in the interval (0,1). The length of the resulting jump depends on the window the selected number falls into. The next jump starts from the new position, and we repeat the procedure N times, obtaining the final position of the particle in a given particular experiment. The experiment is then performed a large number of times; in this way, we obtain a histogram that provides us with the N-jump distribution $P(N,l)$. Figure 1 illustrates the PDF histograms. At the represented number of steps, the one-sided PDF (b) has a maximum whose height is similar to that of the symmetric PDF (a), but, as expected, it has a long tail in the direction of propagation and vanishes very fast in the backward direction. Asymptotically, it will evolve toward the two-sided extremal Lévy function $L_{1.2,1}$. The strongly asymmetric PDF (c) exhibits a long tail in the direction corresponding to the smallest exponent and a steep decline in the direction corresponding to the largest exponent. At very long times, the strongly asymmetric PDF will also evolve toward the Lévy function $L_{1.2,1}$, but its evolution will be slower than that of (b) due to the compensations generated by the (generally shorter) jumps in the "wrong" direction.

Once the probability distribution $P(N,l)$ is known, the regularized displacement moments are calculated as

$$\langle\langle x^R(N)\rangle\rangle_q = \frac{\sum_{l=-2M_L}^{2M_R} [P(N,l)]^q l^R}{\sum_{l=-2M_L}^{2M_R} [P(N,l)]^q}. \tag{24}$$

The lattice runs from $l=-2M_L$ to $l=2M_R$. The number of operations required by each experiment performed on a lattice of size M is proportional to M^2, which makes simulations on very large M lattices extremely time-consuming. However, since the likelihood of a given particle performing two very long jumps is extremely low, we have optimized the accuracy of the simulation by considering the outer halves of the lattice, i.e., the region $\Omega=[-2M_L,-M_L)\cup(M_R,2M_R]$ as a "particle cemetery": if a jump ends at a lattice point inside Ω, we immediately stop the flight and put the particle in the corresponding histogram bin. This procedure permits us to effectively extend the lattice, markedly increasing the simulation accuracy. For the one-sided case we chose $M_L=0$ (or $M_R=0$) and $M_R=2£$ (or $M_L=2£$). For the other cases (symmetric and weakly and strongly asymmetric flights), we took $£=\min(M_L,M_R)$. In this way we made sure that we did not introduce any spurious anisotropies in the one-step PDF.

The error due to finite-size effects may be easily estimated. If $p(j)\sim j^{-(1+\kappa)}$, the probability that the particle jumps beyond a distance d is approximately $d^{-\alpha}$, which indicates that very large lattices are needed to obtain reliable results for distributions with $\alpha\lesssim 1$. Using Eq. (16) it can be shown that the corresponding error when calculating the q moment $\langle\langle x^R\rangle\rangle_q$ on a d-site lattice is approximately $d^{-(\alpha+2-R)}/(\alpha+2-R)$. This estimate also sets an upper bound to moment regularization: The q moment can be used provided that

$$\alpha < R < \alpha+2. \tag{25}$$

If $R<\alpha$, the moments are defined in the usual fashion.

Figure 2(a) exhibits the q variances for several one-sided distributions as functions of the ratio $(N/N_0)^{1/\alpha}$, with $N_0=1600$ being the maximum number of steps considered. In all cases the asymptotic distribution appears to have been reached rapidly, as evidenced by the alignment of the data, although a more careful study shows that, as is the case with the symmetric distributions [19,26], convergence toward the attractor is faster for the broader (lower α) distributions. This is more clearly seen in Fig. 2(b), where we plot the variance divided by $N^{2/\alpha}$ against the number of steps. Although the asymptotic regime has already been reached for $\alpha=1.2$ when $N\approx 800$, we need $N\approx 1200$ for $\alpha=1.35$ and $\alpha=1.5$ and $N>1600$ for $\alpha=1.8$. The slight increase in the value for $\alpha=1.2$ when $N=1600$ is due to end-of-lattice leakage. The inset in panel (a) shows the curvature of the variance corresponding to the PDF for $\alpha=1.8$. For $\alpha=1$ (not shown), the convergence is even faster than for $\alpha=1.2$.

Next we investigate the convergence for $\alpha>2$. In this region, the first two moments are, with their customary definitions, finite and should be calculated by taking $q=1$ in Eq. (6). The distributions belong to the basin of the Gaussian attractor,

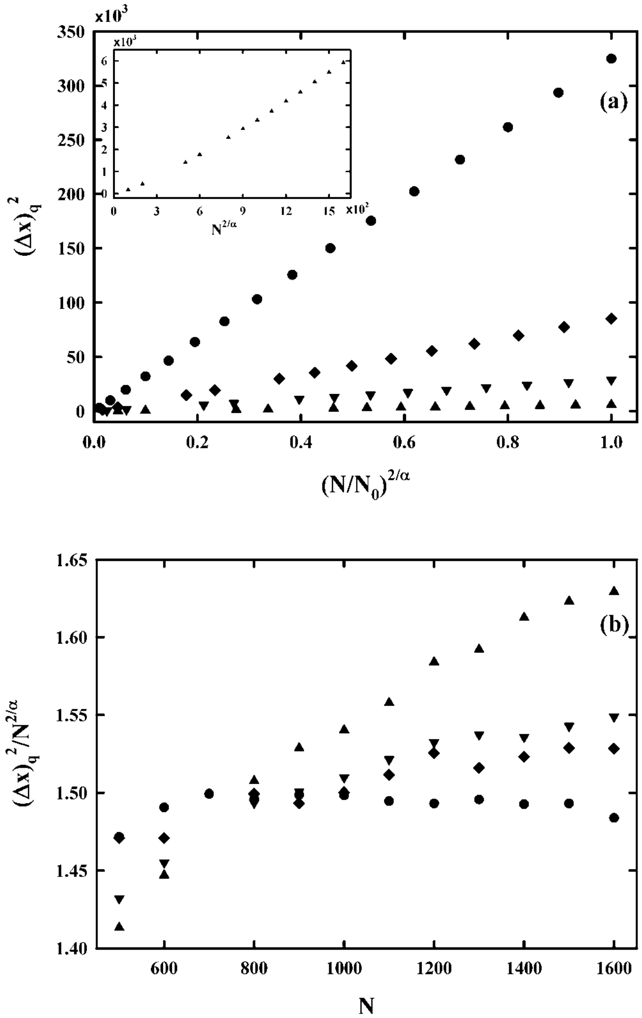

FIG. 2. (a) q variances for one-sided distributions as functions of $(N/N_0)^{2/\alpha}$ and the values of α indicated below. Here $N_0=1600$. Note the convergence toward the asymptotic form. Inset: The case $\alpha=1.8$ reported as a function of $N^{2/\alpha}$ shows a noticeable curvature. (b) q variances multiplied by $N^{-2/\alpha}$ for the cases reported in (a). Only for $\alpha=1.2$ we have a well-defined q-diffusion coefficient if $N\approx 800$. In both panels, ●, ◆, ▼, and ▲ correspond, respectively, to $\alpha=1.2$, $\alpha=1.35$, $\alpha=1.5$, and $\alpha=1.8$.

$$P(N,x) = (4\pi DN)^{-1/2} \exp\left[-\frac{(x-\langle x\rangle)^2}{4DN}\right], \tag{26}$$

where $2DN=\langle x^2\rangle-\langle x\rangle^2$. The moments may be expressed in terms of Riemann's zeta function $\zeta(\alpha)=\Sigma_{k=1}k^{-\alpha}$; we obtain $\langle x\rangle=\zeta(\alpha)/\zeta(\alpha+1)$ and $\langle x^2\rangle=\zeta(\alpha-1)/\zeta(\alpha+1)$. The results shown in Fig. 3 indicate that the convergence to the Gauss attractor slows down as α decreases toward the marginal case $\alpha=2$. Note that the simulated value must be strictly zero for $x<N=1600$; consequently, the backward tails are steeper in the simulated results. As expected, the simulations exhibit long tails to the right, which are barely noticeable for $\alpha=3.2$, but prominent for $\alpha=2.1$.

The weakly asymmetric case was investigated by performing simulations with a fixed value of α and varying over β. In Fig. 4 we present the generalized diffusion coefficient

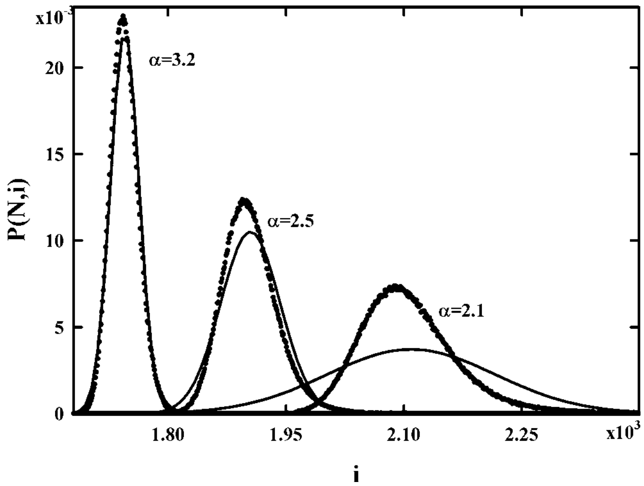

FIG. 3. Histograms for one-sided, $\alpha > 2$, distributions (dotted lines) with the values of α indicated in the figure and $N = 1600$. Note the slower convergence toward the corresponding Gaussian attractors (solid lines) with decreasing α. While the "distance" between the simulation and the attractor is small for $\alpha = 3.2$, the simulated PDF width for $\alpha = 2.1$ is much smaller than that of its Gaussian attractor.

$D_q(\alpha, \beta)$, obtained from the slope of the large (quasilinear) N region of the $(\Delta x)_q^2$ vs $N^{2/\alpha}$ curve, for $\alpha = 1.2$ and various values of β. Note that this slope is a monotonically increasing function of β, which shows that the more asymmetric functions widen faster. The solid circles correspond to the Monte Carlo simulations, while the empty circles were obtained by numerical integration of the asymptotic form, Eq. (18). Note that the simulation yields a slope that is quite close to that predicted by the asymptotic theory. In the case of large values of β, it is likely that the values obtained from the simulation are underestimates due to the error introduced by the lattice finiteness.

For strongly asymmetric flights, the one-sided (or extremal two-sided) Lévy function $L_{\alpha,1}$ (or $L_{\alpha,-1}$) with

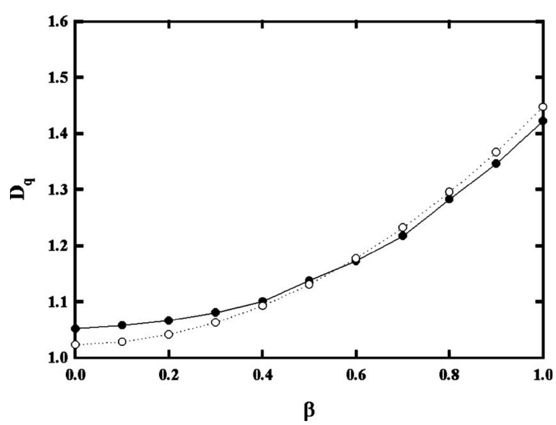

FIG. 4. q-diffusion coefficient for weakly asymmetric distributions for $\alpha = 1.2$ and the values of β indicated in the figure. Solid circles correspond to the simulations, while empty circles correspond to the (numeric) asymptotic calculation. The cases $\beta = 0$ and $\beta = 1$ correspond, respectively, to the symmetric and one-sided distributions.

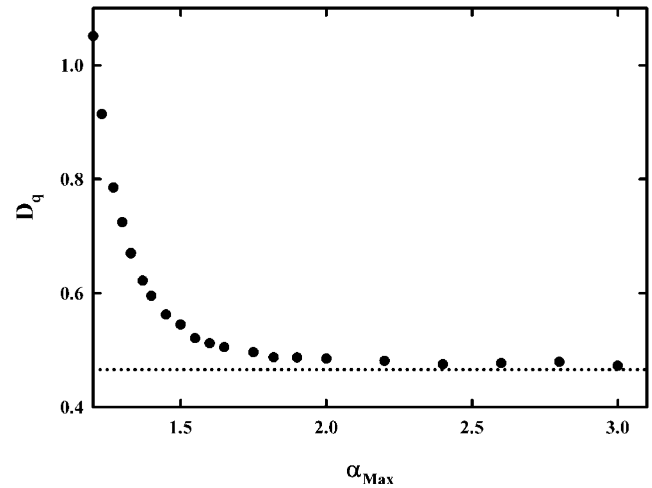

FIG. 5. *Apparent* q-diffusion coefficient for strongly asymmetric distributions as a function of the largest exponent between η and κ. The dotted line corresponds to the true q-diffusion coefficient. The $\alpha_{\max} = 1.2$ limit corresponds to the symmetric PDF, while the large $\alpha_{\max}$ limit gives the *true* q-diffusion coefficient for the strongly asymmetric PDF. The maximum value of N we considered is 1600.

$\alpha = \min(\eta, \kappa)$ is the attractor for *all* values of the largest exponent, $\alpha_{\max}$. Therefore, the generalized diffusion coefficient must be independent of the value of this exponent. We expect, however, a rapid convergence to the corresponding attractor if the exponents are substantially different, but a much slower approximation if the exponents are close, due to strong compensations. When we plot $(\Delta x)_q^2$ vs $N^{2/\alpha}$, we obtain, for moderate and large values of N, lines of very low curvature that yield an *apparent* generalized diffusion coefficient D_q^{app}. In Fig. 5 we plot the dependence of D_q^{app} on the value of the largest exponent between η and κ. We observe that D_q^{app} decreases monotonically as the difference between η and κ increases. The asymptotic form corresponds to the true generalized diffusion coefficient D_q^{app} (approximately 0.48). This q-diffusion coefficient agrees with that obtained for the $\beta = 1$ distribution in Fig. 4, provided that we multiply this value for the factor $2^{2/1.2}$. It is this last result that we represent through a dotted line. (In the genuinely unilateral problem all flights are performed in the chosen direction, while in the strongly asymmetric problem only half of the jumps go in the preferred direction.) If $\alpha_{\max} \lesssim 1.7$, we are still far from the asymptotic regime: The closest the value of $\alpha_{\max}$ is to $\alpha = \min(\eta, \kappa)$, the slowest the convergence to the attractor. The value obtained for the symmetric distribution, $\alpha_{\max} = 1.2$, agrees with that corresponding to $\beta = 0$ in Fig. 4.

The faster convergence found for values of α closer to unity is not surprising in view of the results for symmetric Lévy distributions [26]. The slowdown of the convergence as the marginal value $\alpha = 2$ is approached on both sides is also consistent with the analytical predictions of Ref. [20] for the discrete analog of the Lévy distributions. These authors predict that for their marginal case of the discrete distributions, which corresponds to $\alpha = 1$, there is a very slow logarithmic convergence. If $\alpha \geq 1$, the Poisson distribution plays a similar role for the discrete distributions as that played by the Gaussian PDF for their continuum counterparts when $\alpha \geq 2$

[29]. For the Lévy attractors, we can also argue that, since the backward tail grows when α is increased from one to two, it will take more steps for a unilaterally built sum to resemble its extremal two-sided attractor when the value of α approaches two from below. As discussed before, a similar slowdown is observed in the Gaussian regime as α approaches two from above.

As a concluding point, we note that, according to the Berry-Esséen theorem, a necessary (but not sufficient) condition for our simulations to yield genuine Lévy processes is that $N \ll L^{\alpha}$ [13]. The maximum value of N we have chosen is $N_0 = 1600$ and, in the most unfavorable case ($\alpha = 1.2$), $L^{\alpha} \approx 27500$.

V. CONCLUSIONS

Two important results underpin the regularization of the symmetric Lévy flight displacement moments: The generalized central limit theorem and the optimization of the q entropy. The optimization of the q entropy by functions whose attractors are the symmetric Lévy functions $L_{\alpha,0}$ has been taken as a proof of the ubiquity of Lévy flights in nature [23]. In this work we have tried to formulate and answer some questions regarding the regularization of the displacement moments for asymmetric Levy flights. We have shown that the NSM can be used to conveniently define these moments, but we have also shown that there is no appropriate optimization principle for asymmetric flights, except when these are constrained to a half-space.

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APPENDIX: NONEXISTENCE OF AN OPTIMUM *q* ENTROPY PRINCIPLE FOR ASYMMETRIC LÉVY DISTRIBUTIONS

In Ref. [23] it was shown that the optimization of the q entropy subject to the constraints of normalization and finiteness of the second q moment leads to a symmetric jump distribution with power-law tails that correspond to the same asymptotic behavior as the symmetric Lévy functions. In this appendix we show that it is impossible to generalize this result to asymmetric distributions. The following theorem does not apply to distributions defined on a semi-infinite interval. Such distributions were shown in Ref. [15] to be optimized by the one-sided Lévy distributions.

Theorem 1. Assuming arbitrary constraints on the integer distribution moments, two-sided asymmetric Lévy distributions cannot optimize the q entropy.

Proof. We attempt to optimize the q entropy defined by Eq. (8) subject to N constraints on the integer moments of the distribution,

$$\sum_{j=0}^{R} \alpha_{ij}\langle x^j\rangle = f_i \quad (i=1,\ldots,N), \tag{A1}$$

in addition to the normalization condition, Eq. (10). Here $\langle x^R\rangle$ is the highest moment in the constraints. The optimization procedure is carried out using a Lagrange multiplier Λ_i for each constraint, Λ_0 corresponding to normalization. A cumbersome but straightforward calculation shows that the optimized $p(x)$ must have the form

$$p(x) = \frac{\left(1 - \sum_{j=0}^{R}\sum_{k=1}^{N} a_{kj}\Lambda_k x^j\right)^{1/(1-q)}}{\int_{-\infty}^{\infty}\left(1 - \sum_{j=0}^{R}\sum_{k=1}^{N} a_{kj}\Lambda_k x^j\right)^{1/(1-q)} dx}, \tag{A2}$$

where the coefficients a_{kj} are suitable linear combinations of the α_{ij}'s. Since R is the highest power appearing in the sums of Eq. (A2), it is convenient to define

$$A = \sum_k a_{kR}\Lambda_k \tag{A3}$$

(A is nonzero by definition) and extract Ax^R from the sums:

$$p(x) = \frac{\left\{1 - Ax^R[1 + A^{-1}\sum_{j=0}^{R-1}\sum_{k=1}^{N} a_{kj}\Lambda_k x^{j-R}]\right\}^{1/(1-q)}}{\int_{-\infty}^{\infty}\left\{1 - Ax^R\left[1 + A^{-1}\sum_{j=0}^{R-1}\sum_{k=1}^{N} a_{kj}\Lambda_k x^{j-R}\right]\right\}^{1/(1-q)} dx}. \tag{A4}$$

Since we are only interested in the asymptotic behavior of $p(x)$, we can restrict the class of the functions $\{p(x)\}$ that we take into account. Because the asymmetric (but not unilateral) Lévy distributions are defined on the whole real line, we need to consider only functions $p(x)$ that remain real and non-negative as $x \to \pm\infty$.

Asymptotically only the unity survives inside the square brackets. Thus we only need to analyze the properties of the function

$$G(x) = (1 - Ax^R)^{1/(1-q)}. \tag{A5}$$

We first assume that R is even. If $A > 0$ the distribution has compact support and corresponds to our Eqs. (13) and (14). If $A < 0$, $G(x)$ is defined everywhere and $p(x)$ is asymptotically symmetric. In the particular case of $R=2$, this is the problem investigated in Refs. [21,23].

Next assume R is odd. There are two cases.

(A) If $q \neq 1 - 1/n$, where n is an integer, the exponent in Eq. (A5) is a noninteger. Thus we immediately find that $G(x)$ cannot be simultaneously real for $x \to \infty$ and $x \to -\infty$, whatever the sign of A.

(B) If $q = 1 - 1/n$, where n is an integer, there are two possibilities. n even: Thus $1/(1-q)$ is an even integer and the optimal distribution is again symmetric. n odd:

Thus $1/(1-q)$ is an odd integer and $\mathrm{sgn}[G(x\to\infty)] \neq \mathrm{sgn}[G(x\to-\infty)]$. Therefore, $G(x)$ is asymptotically negative in one direction and cannot be used as a probability.

This proves the theorem. ■

Remark 1. Special cases of Eq. (A1) are constraints imposed on the distribution cumulants.

Remark 2. As we have seen, for symmetric distributions there are infinitely many possible optimization choices, because we need only to require the highest moment in Eqs. (A1) to be even. It is only this highest moment that determines the relation between q and the Lévy index α. The generalization of Eq. (9) is

$$q = \frac{1+\alpha+R}{1+\alpha}. \tag{A6}$$

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