Feigel effect: Extraction of momentum from vacuum?

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The Green-function formalism for the electromagnetic field in a magnetoelectric (ME) medium is constructed as a generalization of conventional Casimir theory. Zero temperature is assumed. It is shown how the formalism predicts electromagnetic momentum to be extracted from the vacuum field, just analogous to how energy is extracted in the Casimir case. The possibility of extracting momentum from vacuum was discussed recently by Feigel [Phys. Rev. Lett. **92**, 020404 (2004)]. In contrast to Feigel's approach, we assume that the ME coupling occurs naturally, rather than being produced by external strong fields. We also find the same effect qualitatively via another route by considering one single electromagnetic mode.

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I. INTRODUCTION

Consider a magnetoelectric (ME) medium whose constitutive relations can be written on compact form as

$$\mathbf{D} = \varepsilon_0 \varepsilon \cdot \mathbf{E} + \frac{1}{\mu_0 c}\chi \cdot \mathbf{B}, \tag{1}$$

$$\mathbf{H} = -\frac{1}{\mu_0 c}\chi^T \cdot \mathbf{E} + \frac{1}{\mu_0}\mu^{-1} \cdot \mathbf{B}. \tag{2}$$

Here χ is the ME coupling parameter, assumed in general to be a pseudotensor, with $(\chi^T)_{ik} \equiv \chi_{ki}$. We employ *Système International* units, so that the relation $\varepsilon_0\mu_0 = 1/c^2$ refers to a vacuum, and we let the permittivity tensor ε_{ik} and permeability tensor μ_{ik} be nondimensional (i.e., relative), so that the relations $D_i = \varepsilon_0\varepsilon_{ik}E_k$ and $B_i = \mu_0\mu_{ik}H_k$ apply in the nonchiral case when $\chi = 0$. We shall take all material quantities ε_{ik}, μ_{ik}, and χ_{ik} to be real and frequency independent. The tensors ε_{ik} and μ_{ik} are symmetric; this being a general electrodynamic property following from the symmetry of the kinetic coefficients [1]. No such symmetry condition exists for χ, however. In some materials χ_{ik} is symmetric, $\chi_{ik} = g\,\delta_{ik}$ with g a pseudoscalar function, or, what is of more interest in the present context, χ_{ik} can be antisymmetric. An anisotropic crystal is called biaxial if the diagonal permittivity components $\varepsilon_x \neq \varepsilon_y \neq \varepsilon_z$ along the principal axes, and is called uniaxial if $\varepsilon_x = \varepsilon_y \neq \varepsilon_z$.

In the following we will focus attention on the situation where the anisotropy in χ_{ik} occurs *naturally*. Cases where the anisotropy is created artificially, by means of strong electric and magnetic fields perpendicular to the direction of light propagation (cf., for instance, Ref. [2]), are for the most part outside the scope of the present paper.

The macroscopic theory of ME media has been known for a long time. The reader may consult the book of O'Dell, for instance [3], as well as classic papers [4,5]. A recent review is given by Fiebig [6]; other relatively recent papers are Refs. [7–10]. As explained in the Fiebig paper, two major sources for "large" ME effects can be identified. (i) In composite materials the ME effect is generated as a product property of a magnetostrictive and a piezoelectric compound. A linear ME polarization is induced by a weak ac magnetic field oscillating in the presence of a strong dc bias field. (ii) In multiferroics the internal magnetic and/or electric fields are enhanced by multiple long-range ordering. The ME effect can be strong enough to trigger magnetic or electrical phase transitions.

The recent paper of Feigel [11]—cf. also the comments [12–15]—sharpened the interest in this special kind of materials. The main idea of this paper was to suggest a new quantum mechanical effect, namely the extraction of material *momentum* from the electromagnetic vacuum oscillations. The suggested effect is thus analogous to the well-known Casimir effect [16], in which case it is an energy, not a momentum, that is extracted from the vacuum field. The Feigel effect thus belongs to a very active area in modern physics. Its main theme is the observability and the interpretation of vacuum-induced phenomena in macroscopic media. The effect has, moreover, a bearing on the famous Abraham-Minkowski energy-momentum problem in dielectric matter [17,18].

This brings us to the main topic of the present paper, which is to investigate how the Green function approach, frequently used in Casimir-related problems, can be applied to a ME medium. To our knowledge, such a general approach has not been developed before. We follow the same basic field theoretical method as in the recent paper of Ellingsen and Brevik [19], dealing with the Casimir effect. We will show that, even in the presence of the complexity in formalism caused by the ME effect, the theory leads to a right-left asymmetry in a medium-filled cavity enclosed within conducting walls placed at positions $z = 0$ and a, and thus permits the extraction of momentum from the vacuum field, in principle. Our field theoretical formalism thus supports earlier results that were based upon consideration of particular modes only. We will also have the opportunity to comment occasionally on some of the papers that followed the Feigel paper [20–22].

In Secs. II and III we establish the governing equation for the Green functions, relate this to the two-point functions for the electromagnetic fields, and give explicit solutions in the presence of the two conducting plates. In Sec. IV we digress to consider the momentum conservation equation for a ME

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medium, and show how the right-left momentum asymmetry occurs for one single mode. In Sec. V we return to the Green-function approach, and show how the momentum asymmetry occurs also for the vacuum field, when summing over all modes propagating in the $\pm x$ directions.

We thus discuss the momentum asymmetry via two different approaches. A more detailed overview of the outline of the paper is given in Sec. VI.

Readers interested in recent reviews on the Casimir effect may consult Refs. [23–28]. Much information can also be found in Refs. [29,30].

We emphasize that the formalism below is constructed from the same main standpoint as in conventional Casimir theory: we calculate the change in field momentum caused by the geometric boundaries, i.e., the plates. The undisturbed system with respect to which we regularize Green-function expressions is an infinite medium (without plates), made up of the same material. It is thus clear that in the limit when the separation between the plates goes to infinity, the effect that we calculate has to go to zero.

II. GOVERNING EQUATIONS FOR GREEN'S FUNCTION

In this section we will establish the governing equations for the retarded Green function in the chiral medium. When this function is known, one can find the electromagnetic two-point functions and thus construct expressions for energy and momentum in the field. From now on, we assume the material to be isotropic, so that $\varepsilon_{ik}=\varepsilon\delta_{ik}$, $\mu_{ik}=\mu\delta_{ik}$. Important in our context is that the coupling parameter χ_{ik} will still be permitted to be anisotropic. As already mentioned we take all material parameters ε, μ, and χ_{ik} to be real and frequency independent. They will, moreover, be assumed to be independent of the spatial coordinates. Our medium is thus assumed to be spatially homogeneous but chiral. (If the anisotropy of χ_{ik} is created artificially, by means of strong crossed electric and magnetic fields, the anisotropy property of χ_{ik} holds of course only in the constant field region between the condenser plates.)

Let us first invert the constitutive relations (1) and (2) to get

$$\mathbf{E}=\frac{1}{\varepsilon\varepsilon_0}\left(\mathbf{D}-\frac{\mu}{c}\chi\cdot\mathbf{H}\right), \tag{3}$$

$$\mathbf{B}=\mu\mu_0\left(\mathbf{H}+\frac{c}{\varepsilon}\chi^T\cdot\mathbf{D}\right). \tag{4}$$

These expressions hold when the EM effect is small, $|\chi_{ik}|\ll 1$, what in practice is always the case. Terms of order χ^2 are neglected.

Consider now Maxwell's equations in conventional form,

$$\nabla\cdot\mathbf{D}=\rho,\quad \nabla\cdot\mathbf{B}=0, \tag{5}$$

$$\nabla\times\mathbf{E}=-\dot{\mathbf{B}},\quad \nabla\times\mathbf{H}=\mathbf{J}+\dot{\mathbf{D}}, \tag{6}$$

and take the curl of the first member of Eq. (6). Observing Eq. (4) we then get, when neglecting terms of order χ^2 throughout, the following coupled vector equation for the basic fields $\mathbf{E}$ and $\mathbf{B}$:

$$\nabla\times\nabla\times\mathbf{E}+\frac{\varepsilon\mu}{c^2}\ddot{\mathbf{E}}+\frac{\mu}{c}\chi\cdot\ddot{\mathbf{B}}+\frac{\mu}{c}\nabla\times(\chi^T\cdot\dot{\mathbf{E}})=-\mu\mu_0\dot{\mathbf{J}}. \tag{7}$$

If $\chi=0$, the coupling between the fields is absent. In component form the equation can be written

$$\nabla^2E_i-\partial_i(\nabla\cdot\mathbf{E})-\frac{\varepsilon\mu}{c^2}\ddot{E}_i-\frac{\mu}{c}\chi_{ik}\ddot{B}_k-\frac{\mu}{c}\chi_{lk}\,\mathrm{curl}_{ik}\,\dot{E}_l=\mu\mu_0\dot{J}_i. \tag{8}$$

We have here defined $\mathrm{curl}_{ik}\equiv\epsilon_{ijk}\partial_j$, where ϵ_{ijk} is the antisymmetric Levi-Civita symbol with $\epsilon_{123}=1$.

In Eq. (8), the magnetic field B_k can actually be replaced by electric field components in view of one of Maxwell's equations, $\dot{B}_k=-\mathrm{curl}_{kl}\,E_l$. We obtain

$$\left[\delta_{il}\nabla^2-\partial_i\partial_l-\delta_{il}\frac{\varepsilon\mu}{c^2}\partial_t^2+\frac{\mu}{c}\chi_{ik}\,\mathrm{curl}_{kl}\,\partial_t-\frac{\mu}{c}\chi_{lk}\,\mathrm{curl}_{ik}\,\partial_t\right]E_l=\mu\mu_0\dot{J}_i, \tag{9}$$

with $\partial_t=\partial/\partial t$.

We now turn to the Green-function approach. According to the source theory of Schwinger *et al.* (see, for instance, Refs. [24] or [31]), we make the correspondence $\mathbf{J}\rightarrow\dot{\mathbf{P}}$, $\rho\rightarrow-\nabla\cdot\mathbf{P}$. We introduce a dyad $\mathbf{\Gamma}(x,x')$ such that

$$\mathbf{E}(x)=\frac{1}{\varepsilon_0}\int d^4x'\,\mathbf{\Gamma}(x,x')\cdot\mathbf{P}(x'), \tag{10}$$

where $x=(\mathbf{r},t)$. Due to causality, t' is only integrated over the region $t'\leq t$. The dyad $\mathbf{\Gamma}$ is the retarded Green function; also called the generalized susceptibility. We take the Fourier transform of $\mathbf{\Gamma}$,

$$\mathbf{\Gamma}(x,x')=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega\tau}\mathbf{\Gamma}(\mathbf{r},\mathbf{r}',\omega),\qquad \tau=t-t', \tag{11}$$

exploiting the stationarity of the system. We transform also the electric field,

$$\mathbf{E}(x)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega t}\mathbf{E}(\mathbf{r},\omega), \tag{12}$$

with a similar expression for $\mathbf{P}(x)$. The governing equation for the Green function then becomes

$$\nabla\times\nabla\times\mathbf{\Gamma}(\mathbf{r},\mathbf{r}',\omega)-\frac{\varepsilon\mu\omega^2}{c^2}\mathbf{\Gamma}(\mathbf{r},\mathbf{r}',\omega)+\frac{i\mu\omega}{c}\chi\cdot[\nabla\times\mathbf{\Gamma}(\mathbf{r},\mathbf{r}',\omega)]-\frac{i\mu\omega}{c}\nabla\times[\chi^T\cdot\mathbf{\Gamma}(\mathbf{r},\mathbf{r}',\omega)]$$

$$= \frac{\mu\omega^2}{c^2}\delta(\mathbf{r}-\mathbf{r}')\mathbf{1}, \tag{13}$$

or, on component form,

$$\left[\partial_i\partial_j - \delta_{ij}\nabla^2 - \frac{\varepsilon\mu\omega^2}{c^2}\delta_{ij} + \frac{i\mu\omega}{c}\chi_{il}\,\mathrm{curl}_{lj} - \frac{i\mu\omega}{c}\chi_{jl}\,\mathrm{curl}_{il}\right]\Gamma_{jk}(\mathbf{r},\mathbf{r}',\omega) = \frac{\mu\omega^2}{c^2}\delta(\mathbf{r}-\mathbf{r}')\delta_{ik}. \tag{14}$$

If $\chi_{ik}=0$ and $\mu=1$, this equation reduces to Eq. (75.16) in Ref. [32] (their symbol D_{ik} is the same as our $-\hbar c^2\Gamma_{ik}/\omega^2$).

If $\Gamma_{ik}(\mathbf{r},\mathbf{r}',\omega)$ is known, we can make use of the fluctuation-dissipation theorem (which has a meaning both classically and quantum mechanically; cf. Refs. [32,33]), to calculate the two-point functions:

$$i\langle E_i(\mathbf{r})E_k(\mathbf{r}')\rangle_\omega = \frac{\hbar}{\varepsilon_0}\,\mathrm{Im}\{\Gamma_{ik}(\mathbf{r},\mathbf{r}',\omega)\}, \tag{15}$$

$$i\langle B_i(\mathbf{r})B_k(\mathbf{r}')\rangle_\omega = \frac{\hbar}{\varepsilon_0}\frac{1}{\omega^2}\mathrm{curl}_{ij}\,\mathrm{curl}'_{kl}\,\mathrm{Im}\{\Gamma_{jl}(\mathbf{r},\mathbf{r}',\omega)\}, \tag{16}$$

$$\langle E_i(\mathbf{r})B_k(\mathbf{r}')\rangle_\omega = \frac{\hbar}{\varepsilon_0}\frac{1}{\omega}\mathrm{curl}'_{kl}\,\mathrm{Im}\{\Gamma_{il}(\mathbf{r},\mathbf{r}',\omega)\}. \tag{17}$$

Here $\mathrm{curl}'_{ik}=\epsilon_{ijk}\partial'_j$, where ∂'_j is the derivative with respect to component j of $\mathbf{r}'$. The expressions (15)–(17) refer to zero temperature; a factor $\mathrm{sgn}(\omega)$ is omitted throughout. The spectral correlation tensor $\langle E_i(\mathbf{r})E_k(\mathbf{r}')\rangle_\omega$ is defined according to

$$\langle E_i(x)E_k(x')\rangle = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega\tau}\langle E_i(\mathbf{r})E_k(\mathbf{r}')\rangle_\omega. \tag{18}$$

[Note the meaning of the formalism here: the spectral correlation tensor is related to the Fourier transform $\langle E_i(\mathbf{r},\omega)E_k(\mathbf{r}',\omega')\rangle$ of the two-point function $\langle E_i(x)E_k(x')\rangle$ via

$$\langle E_i(\mathbf{r},\omega)E_k(\mathbf{r}',\omega')\rangle = 2\pi\langle E_i(\mathbf{r})E_k(\mathbf{r}')\rangle_\omega\delta(\omega+\omega'); \tag{19}$$

cf. Eq. (122.12) in Ref. [33] or also Appendix B in Ref. [34].] Before going on to solve these equations, we will specify the geometry to be assumed in the rest of this paper.

III. SPECIFICATION OF THE GEOMETRY. SOLUTIONS FOR THE GREEN FUNCTIONS

Let us assume the same setup as in conventional Casimir theory, namely two perfectly conducting parallel plates separated by a gap a. The geometry is sketched in Fig. 1.

As mentioned earlier, we will mainly be considering the case where the ME effect occurs naturally. We assume accordingly that χ_{ik} is given initially and is constant everywhere in the fluid, on the inside as well as on the outside of the plates. Because of the translational invariance in the x and y directions we can transform the Green function once more to obtain

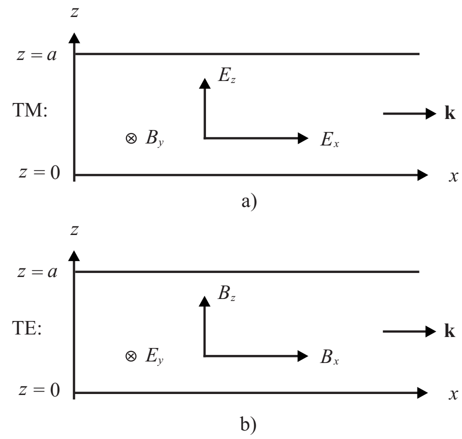

FIG. 1. Sketch of the geometry. The TM and TE modes are shown. The tensor χ_{ik}, constant everywhere in the fluid, is given by Eq. (20). The wave vector $\mathbf{k}$ is directed along the x axis.

$$\Gamma(\mathbf{r},\mathbf{r}',\omega) = \int\frac{d^2k}{(2\pi)^2}e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\mathbf{g}(z,z',\mathbf{k},\omega). \tag{20}$$

We also transform the delta function:

$$\delta(\mathbf{r}-\mathbf{r}') = \int\frac{d^2k}{(2\pi)^2}e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\delta(z-z'), \tag{21}$$

and assume that χ_{ik} has the following form:

$$\chi_{ik} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \chi_{yz} \\ 0 & \chi_{zy} & 0 \end{pmatrix}. \tag{22}$$

Our coordinate system is thus henceforth fixed, relative to the material. We focus attention on only one particular wave number $\mathbf{k}$ in the following, namely $\mathbf{k}=k_x\mathbf{e}_x$, directed along the x axis. In Figs. 1(a) and 1(b) the transverse magnetic (TM) and the transverse electric (TE) modes in the cavity corresponding to this $\mathbf{k}$ vector are indicated [35].

Our conventions above mean that we can let $\nabla^2\to\partial_z^2-k_x^2$. We can now write down the governing equations for the Fourier components g_{ik} from Eqs. (14). The simplest equation follows by setting $(ik)=(yy)$:

$$\partial_z^2 g_{yy} - \left(\kappa^2 - \frac{2\mu k_x\omega}{c}\chi_{yz}\right)g_{yy} = -\frac{\mu\omega^2}{c^2}\delta(z-z'), \tag{23}$$

where we have defined

$$\kappa^2 = k_x^2 - \varepsilon\mu\omega^2/c^2. \tag{24}$$

Equation (23) is uncoupled; this being a consequence of our choice for $\mathbf{k}$ implying that $\partial_y\to 0$.

Setting $(ik)=(xx)$ we obtain

$$i\left(k_x+\frac{\mu\omega}{c}\chi_{zy}\right)\partial_z g_{zx}-\left(\partial_z^2+\frac{\varepsilon\mu\omega^2}{c^2}\right)g_{xx}=\frac{\mu\omega^2}{c^2}\delta(z-z'),\tag{25}$$

and with $(ik)=(zz)$,

$$i\left(k_x+\frac{\mu\omega}{c}\chi_{zy}\right)\partial_z g_{xz}+\left(\kappa^2+\frac{2\mu k_x\omega}{c}\chi_{zy}\right)g_{zz}=\frac{\mu\omega^2}{c^2}\delta(z-z').\tag{26}$$

The last two equations are coupled. Consider finally the nondiagonal components: with $(ik)=(zx)$ we obtain

$$i\left(k_x+\frac{\mu\omega}{c}\chi_{zy}\right)\partial_z g_{xx}+\left(\kappa^2+\frac{2\mu k_x\omega}{c}\chi_{zy}\right)g_{zx}=0,\tag{27}$$

and with $(ik)=(xz)$,

$$i\left(k_x+\frac{\mu\omega}{c}\chi_{zy}\right)\partial_z g_{zz}-\left(\partial_z^2+\frac{\varepsilon\mu\omega^2}{c^2}\right)g_{xz}=0.\tag{28}$$

The coupling in the differential equation (25) for g_{xx} can be removed if we make use of Eq. (27) differentiated with respect to x. Some manipulations, again observing that χ_{ik} is small, yield

$$\partial_z^2 g_{xx}-K^2 g_{xx}=\frac{K^2}{\varepsilon}\delta(z-z'),\quad K=\kappa\left(1+\frac{\mu k_x\omega}{\kappa^2 c}\chi_{zy}\right).\tag{29}$$

Equation (23) can be rewritten similarly:

$$\partial_z^2 g_{yy}-L^2 g_{yy}=-\frac{\mu\omega^2}{c^2}\delta(z-z'),\quad L=\kappa\left(1-\frac{\mu k_x\omega}{\kappa^2 c}\chi_{yz}\right).\tag{30}$$

The differential equations (29) and (30) for the diagonal components are convenient for further manipulation. Note that the values of K and L are dependent on whether the direction of propagation of the wave is to the right or to the left. If $\chi_{ik}=0$, the expressions agree with those of Refs. [19,36].

We now proceed to solve the equations, beginning with Eq. (30). As $E_y=0$ at $z=0$ and $z=a$ because of the boundary conditions, we have $g_{yy}(0,z',\mathbf{k},\omega)=g_{yy}(a,z',\mathbf{k},\omega)=0$. The solution of Eq. (30) can then be written

$$g_{yy}=\frac{\mu\omega^2}{2Lc^2}\left\{e^{-L|z-z'|}-e^{-L(z+z')}+\frac{2[\cosh L(z-z')-\cosh L(z+z')]}{\exp(2La)-1}\right\}.\tag{31}$$

When $\chi_{ik}=0$, this expression agrees with that given in Appendix C of Ref. [37] in the limit of perfectly conducting plates.

Next considering g_{xx}, we must analogously have $g_{xx}(0,z',\mathbf{k},\omega)=g_{xx}(a,z',\mathbf{k},\omega)=0$ in view of the boundary conditions. The solution of Eq. (29) becomes

$$g_{xx}=-\frac{K}{2\varepsilon}\left\{e^{-K|z-z'|}-e^{-K(z+z')}+\frac{2[\cosh K(z-z')-\cosh K(z+z')]}{\exp(2Ka)-1}\right\}.\tag{32}$$

The expressions (31) and (32) are fairly complicated. For practical purposes it is possible to simplify the expressions considerably, by omitting terms containing $(z+z')$. The reason is that these terms do not contribute to physical quantities like the Casimir force on the plates or to the field momentum in the gap. This can be seen in two different ways. The simplest way is to argue, as in Sec. 81 in [32], that by putting $z=z'$ in solutions having the argument $(z+z')$ one would obtain physical quantities like field momentum in the gap varying with the position z. This would contradict the law of conservation of momentum. Another way of examining this rather subtle point is to include the $(z+z')$ terms everywhere in the formalism, and to verify that they really do not contribute in the end. In addition to the discussion in [32], one can find more mathematical details about this point in the paper [19] and in the thesis [38].

We can, moreover, omit the source-dependent inhomogeneous $|z-z'|$ term in each of the Green functions. This term represents the solution pertaining to the delta function source inside a homogeneous medium filling all space. Being geometry independent, it cannot contribute to any physical quantity related to the geometry. All in all, we shall in the following use the "effective" Green functions

$$g_{xx}=-\frac{K}{\varepsilon}\frac{\cosh K(z-z')}{\exp(2Ka)-1},\tag{33}$$

$$g_{yy}=\frac{\mu\omega^2}{Lc^2}\frac{\cosh L(z-z')}{\exp(2La)-1}.\tag{34}$$

Consider finally the remaining diagonal component, g_{zz}. To this end we first observe the symmetry property

$$g_{xz}(z,z',\mathbf{k},\omega)=g_{zx}(z',z,-\mathbf{k},\omega),\tag{35}$$

which is an example of the general relation

$$\Gamma_{ik}(\mathbf{r},\mathbf{r}',\tau)=\Gamma_{ki}(\mathbf{r}',\mathbf{r},-\tau),\tag{36}$$

when expressed in Fourier space (cf. Sec. 81 in Ref. [32]). From Eq. (27) we have, when inserting the expression (33),

$$g_{zx}(z,z',\mathbf{k},\omega)=\frac{i}{\varepsilon}\left(k_x+\frac{\mu\omega}{c}\chi_{zy}\right)\frac{\sinh K(z-z')}{\exp(2Ka)-1}.\tag{37}$$

Now, to the required order $K(-k_x)=\kappa^2/K(k_x)$, according to Eq. (29). Thus we get from Eq. (35)

$$g_{xz}(z,z',\mathbf{k},\omega)=\frac{i}{\varepsilon}\left(k_x-\frac{\mu\omega}{c}\chi_{zy}\right)\frac{\sinh[\kappa^2(z-z')/K]}{\exp(2\kappa^2 a/K)-1},\tag{38}$$

where here and henceforth $K=K(k_x)$ as defined in Eq. (29). From Eq. (26) we then finally get (delta-function omitted)

$$g_{zz} = \frac{\kappa^2 k_x^2}{K^3 \varepsilon} \frac{\cosh[\kappa^2(z-z')/K]}{\exp(2\kappa^2 a/K) - 1}. \tag{39}$$

Before applying these Green-function expressions to the Feigel effect, we shall in the next section follow a more simplistic approach and consider the right-left asymmetry in the field momentum considering one single mode only.

IV. ENERGY-MOMENTUM FORMALISM: RIGHT-LEFT FIELD MOMENTUM ASYMMETRY

A. Energy-momentum formalism

Before considering the momentum asymmetry for one single chosen direction of propagation, we need to develop the formalism related to the electromagnetic energy-momentum tensor. In this section we take a general approach, allowing for external charges ρ and currents $\mathbf{J}$. The coupling tensor χ_{ik} is allowed to be general [not necessarily of the form given in Eq. (22)], though constant, and we allow also for optical anisotropy by letting $\varepsilon\delta_{ik} \to \varepsilon_{ik}$, $\mu\delta_{ik} \to \mu_{ik}$ with the material parameters constant.

It is convenient to write the constitutive relations (1) and (2) in tensor form,

$$D_i = \varepsilon_0 \varepsilon_{ik} E_k + \frac{1}{\mu_0 c} \chi_{ik} B_k, \tag{40}$$

$$H_i = -\frac{1}{\mu_0 c} \chi_{ki} E_k + \frac{1}{\mu_0} \mu^{-1}{}_{ik} B_k. \tag{41}$$

In view of Maxwell's equations (5) and (6) we obtain the conservation equation for energy,

$$\boldsymbol{\nabla} \cdot \mathbf{S} + \dot{w} = -\mathbf{E} \cdot \mathbf{J}, \tag{42}$$

where $\mathbf{E}\cdot\mathbf{J}$ is the energy dissipation,

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} \tag{43}$$

the Poynting vector, and

$$w = \frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B}) \tag{44}$$

the energy density.

As for the momentum conservation, it is convenient to start from the equation

$$\partial_t(\mathbf{D} \times \mathbf{B})_i = -\rho E_i - \epsilon_{ijk} J_j B_k + \partial_k(E_i D_k + H_i B_k) - \varepsilon_{kl} E_{k,i} E_l - \mu^{-1}{}_{kl} B_{l,i} B_k, \tag{45}$$

which follows from Maxwell's equations (here $E_{i,k} \equiv \partial_k E_i$, etc.). Introducing the Lorentz force density

$$\mathbf{f}^L = \rho \mathbf{E} + \mathbf{J} \times \mathbf{B}, \tag{46}$$

as well as the Minkowski stress tensor [17],

$$T_{ik}^M = E_i D_k + H_i B_k - \frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B})\delta_{ik}, \tag{47}$$

we can write the momentum conservation equation as

$$\partial_k T_{ik}^M - \dot{g}_i^M = f_i^L, \tag{48}$$

where

$$\mathbf{g}^M = \mathbf{D} \times \mathbf{B} \tag{49}$$

is the Minkowski momentum density. (The symbol $\mathbf{g}$ for momentum is not to be confused with the Green functions.) It is generally known that the above expressions hold when the medium is optically anisotropic. It is, however, somewhat remarkable that they hold when $\chi_{ik} \neq 0$ also; there seems to be no simple physical reason why χ_{ik} should drop out from the formalism.

In the case of high frequency fields, in particular optical fields, the Minkowski theory appears to be both simple and capable of describing all experiments (cf. the analysis of one of the present authors on this point some years ago [39]; some more recent papers are listed in Ref. [19]). However, at low frequencies where the effect of the oscillations are themselves observable—notably in the Lahoz-Walker experiment [40]—the experiments agree not with the Minkowski but rather with the Abraham force, which accordingly can be taken to be the most "physical" alternative at these frequencies. The Abraham theory [18] consists in symmetrizing the stress tensor,

$$T_{ik}^A = \frac{1}{2}(E_i D_k + E_k D_i) + \frac{1}{2}(H_i B_k + H_k B_i) - \frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B})\delta_{ik}, \tag{50}$$

and taking the momentum density to be

$$\mathbf{g}^A = \frac{1}{c^2}\mathbf{E} \times \mathbf{H}, \tag{51}$$

the latter satisfying the relation $\mathbf{g} = \mathbf{S}/c^2$, the so-called Planck's principle of inertia of energy.

We assume henceforth optical anisotropy so that ε and μ are scalars, and also that $\rho = 0$, $\mathbf{J} = 0$. The Minkowski and Abraham stress tensors become thereby equal, $T_{ik}^M = T_{ik}^A$. The Abraham conservation equation for momentum can be written as

$$\partial_k T_{ik}^A - \dot{g}_i^A = [(\varepsilon\mu - 1)/c^2]\partial_t(\mathbf{E} \times \mathbf{H})_i, \tag{52}$$

where the term on the right-hand side is the "Abraham term." It was precisely this term that was measured by Walker and Lahoz [40]. In a high-frequency field, it fluctuates out. We shall return to the Abraham force in Sec. VI.

B. Momentum asymmetry

Referring to Fig. 1, we consider to begin with only the right-moving TE wave corresponding to the field components

$$E_y = \sqrt{\frac{2}{a}} \sin k_n z e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}, \tag{53}$$

$$B_x = \sqrt{\frac{2}{a}} \frac{ik_n}{\omega} \cos k_n z e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}, \tag{54}$$

$$B_z = \sqrt{\frac{2}{a}}\frac{k_x}{\omega}\sin k_n z e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}, \tag{55}$$

the other components being zero. (We use the same normalization of the fields as Tiggelen *et al.* [22].) Here $\mathbf{k}\cdot\mathbf{x}=k_x x$, and $k_n=\pi n/a$ with $n=1,2,3\ldots$ the transverse wave number. For a given value of k_x, the eigenfrequencies ω are thus discrete. We can derive the dispersion equation by going back to the field equation (9) for $E_i=E_y$ in the source-free case, observing that $\partial_l E_l=\partial_y E_y=0$, inserting the form (22) for χ_{ik}. We obtain

$$\left(k_x^2+k_n^2-\frac{\varepsilon\mu}{c^2}\omega^2-\frac{2\mu k_x\omega}{c}\chi_{yz}\right)E_y=0, \tag{56}$$

which implies to the lowest order in χ_{yz}

$$\omega=\frac{c}{\sqrt{\varepsilon\mu}}\sqrt{k_x^2+k_n^2}\left[1-\sqrt{\frac{\mu}{\varepsilon}}\frac{k_x}{\sqrt{k_x^2+k_n^2}}\chi_{yz}\right]. \tag{57}$$

The right-left asymmetry is manifest. A left-moving wave is described by the substitution $k_x\to -k_x$.

Let us now calculate the field energy density, w, for the TE mode. We get

$$\begin{aligned} w &= \frac{1}{4}(\mathbf{E}\cdot\mathbf{D}^*+\mathbf{H}\cdot\mathbf{B}^*) \\ &= \frac{\varepsilon_0\varepsilon}{2a}\left[1+\frac{c^2}{\varepsilon\mu}\frac{k_x^2}{\omega^2}\right]\sin^2 k_n z+\frac{1}{2\mu_0\mu a}\frac{k_n^2}{\omega^2}\cos^2 k_n z; \end{aligned} \tag{58}$$

the χ_{yz} terms drop out when w is written in this way. It is convenient to consider the expression integrated from $z=0$ to a, thereby getting the energy W per unit length and width,

$$W=\int_0^a w\,dz=\frac{\varepsilon_0\varepsilon}{4}\left[1+\frac{c^2}{\varepsilon\mu}\frac{k_x^2+k_n^2}{\omega^2}\right]. \tag{59}$$

Using Eq. (57) we can write this in terms of the wave-number components,

$$W=\frac{\varepsilon_0\varepsilon}{2}\left[1+\frac{1}{2}\sqrt{\frac{\mu}{\varepsilon}}\frac{k_x}{\sqrt{k_x^2+k_n^2}}\chi_{yz}\right]. \tag{60}$$

The Poynting vector in the x direction, S_x, may be calculated as

$$S_x=\frac{1}{2}(\mathbf{E}\times\mathbf{H}^*)_x=\left(\frac{1}{\mu_0\mu a}\frac{k_x}{\omega}-\frac{\chi_{yz}}{\mu_0 c a}\right)\sin^2 k_n z, \tag{61}$$

which means that the integrated energy flux when expressed in terms of wave number becomes

$$q_x=\int_0^a S_x dz=\frac{\varepsilon_0 c}{2}\left[\sqrt{\frac{\varepsilon}{\mu}}\frac{k_x}{\sqrt{k_x^2+k_n^2}}-\frac{k_n^2}{k_x^2+k_n^2}\chi_{yz}\right]. \tag{62}$$

Alternatively, we might calculate the energy flux as $q_x=Wu_x$, where u_x is the group velocity

$$u_x=\frac{\partial\omega}{\partial k_x}=\frac{c}{\sqrt{\varepsilon\mu}}\frac{k_x}{\sqrt{k_x^2+k_n^2}}-\frac{c}{\varepsilon}\chi_{yz}. \tag{63}$$

This agreement is as we should expect, since we are dealing with the propagation of low-amplitude waves. The kinematic group velocity concept and the dynamic energy flow velocity concept should be the same.

Consider finally the Minkowski momentum density g_x^M:

$$g_x^M=\frac{1}{2}(\mathbf{D}\times\mathbf{B}^*)_x=\frac{\varepsilon_0\varepsilon}{a}\frac{k_x}{\omega}\left(1+\frac{k_x c}{\varepsilon\omega}\chi_{yz}\right)\sin^2 k_n z. \tag{64}$$

Comparison between Eqs. (61) and (64) shows that the relationship $g_x^M=(\varepsilon\mu/c^2)S_x$, known from conventional optics, does *not* hold when χ_{yz} is different from zero. We also give the expression (64) when integrated over z:

$$G_x^M=\int_0^a g_x^M dz=\frac{\varepsilon_0\varepsilon}{2}\frac{k_x}{\omega}\left(1+\frac{k_x c}{\varepsilon\omega}\chi_{yz}\right). \tag{65}$$

Again, the right-left asymmetry is manifest.

V. GREEN-FUNCTION APPROACH TO THE FEIGEL EFFECT

Our intention now is to calculate the Minkowski momentum asymmetry in the chiral medium using the Green-function approach from Sec. III. We start from the following general expression, reverting to real representation for the fields,

$$\mathbf{g}^M=\lim_{x'\to x}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega\tau}\int\frac{d^2k}{(2\pi)^2}e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\langle\mathbf{D}(\mathbf{r})\times\mathbf{B}(\mathbf{r}')\rangle_{\omega\mathbf{k}}. \tag{66}$$

We assume zero temperature, so that the brackets $\langle\,\rangle$ mean purely quantum mechanical average. As no thermal fields are excited, the field momentum as well as the field energy stem exclusively from the vacuum zero-point oscillations. Whereas in the previous section we considered the contribution from one single selected mode only, we shall now consider the effect of summing over all available vacuum modes. We shall impose one restriction, however: the wave number $\mathbf{k}$ will be required to lie either in the positive or the negative x direction. This corresponds to our Green-function approach in Sec. III. Mathematically, it means that we can let $\int d^2k/(2\pi)^2\to\int dk_x/2\pi$. As the distribution of fields does not vary in the transverse y direction, we can effectively let $\partial_y\Rightarrow 0$ when applied to the fields. Evidently, the x component of field momentum has to be zero in the case of a nonchiral medium; if there is an asymmetry following from the formalism this has to be caused by the presence of χ_{ik}. As before, we assume the particular form (22) for χ_{ik}.

The x component of Eq. (66) becomes (we omit the "lim" from now on)

$$\begin{aligned} g_x^M=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega\tau}\int_{-\infty}^{\infty}\frac{dk_x}{2\pi}e^{ik_x(x-x')}\langle D_y(\mathbf{r})B_z(\mathbf{r}') \\ -D_z(\mathbf{r})B_y(\mathbf{r}')\rangle_{\omega k}. \end{aligned} \tag{67}$$

We insert from Eqs. (1) and (2)

$$D_y = \varepsilon_0 \varepsilon E_y + \frac{\chi_{yz}}{\mu_0 c} B_z, \tag{68}$$

$$D_z = \varepsilon_0 \varepsilon E_z + \frac{\chi_{zy}}{\mu_0 c} B_y, \tag{69}$$

and get

$$g_x^M = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega\tau} \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} e^{ik_x(x-x')} \Big[\varepsilon_0 \varepsilon \langle E_y(\mathbf{r}) B_z(\mathbf{r}') \rangle_{\omega k} - \varepsilon_0 \varepsilon \langle E_z(\mathbf{r}) B_y(\mathbf{r}') \rangle_{\omega k} - \frac{\chi_{zy}}{\mu_0 c} \langle B_y(\mathbf{r}) B_y(\mathbf{r}') \rangle_{\omega k} + \frac{\chi_{yz}}{\mu_0 c} \langle B_z(\mathbf{r}) B_z(\mathbf{r}') \rangle_{\omega k} \Big]. \tag{70}$$

We have thus so far expressed g_x^M in terms of the two-point functions for the fundamental fields. Using Eqs. (16) and (17) we calculate

$$\langle E_y(\mathbf{r}) B_z(\mathbf{r}') \rangle_{\omega k} = \frac{\hbar}{\varepsilon_0} \frac{-ik_x}{\omega} \operatorname{Im} g_{yy}, \tag{71}$$

$$\langle E_z(\mathbf{r}) B_y(\mathbf{r}') \rangle_{\omega k} = \frac{\hbar}{\varepsilon_0} \frac{1}{\omega} (\partial_z' \operatorname{Im} g_{zx} + ik_x \operatorname{Im} g_{zz}), \tag{72}$$

$$\langle B_y(\mathbf{r}) B_y(\mathbf{r}') \rangle_{\omega k} = \frac{\hbar}{\varepsilon_0} \frac{i}{\omega^2} (\partial_z^2 \operatorname{Im} g_{xx} - ik_x \operatorname{Im} \partial_z g_{zx} - ik_x \partial_z \operatorname{Im} g_{xz} - k_x^2 \operatorname{Im} g_{zz}), \tag{73}$$

$$\langle B_z(\mathbf{r}) B_z(\mathbf{r}') \rangle_{\omega k} = \frac{\hbar}{\varepsilon_0} \frac{-ik_x^2}{\omega^2} \operatorname{Im} g_{yy}. \tag{74}$$

We have here, as above, naturally defined $\langle\,\rangle_{\omega k}$ via the relation

$$\langle E_y(\mathbf{r}) B_z(\mathbf{r}') \rangle_{\omega} = \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} e^{ik_x(x-x')} \langle E_y(\mathbf{r}) B_z(\mathbf{r}') \rangle_{\omega k}, \tag{75}$$

etc. We can thus express g_x^M as

$$g_x^M = \hbar \int_{-\infty}^{\infty} \frac{d\omega}{2\pi\omega} e^{-i\omega\tau} \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} e^{ik_x(x-x')} \langle\,\rangle, \tag{76}$$

where

$$\langle\,\rangle = -ik_x \varepsilon \operatorname{Im} g_{yy} - \varepsilon(\partial_z' \operatorname{Im} g_{zx} + ik_x \operatorname{Im} g_{zz}) + \frac{ic}{\omega} \chi_{zy} (-\partial_z^2 \operatorname{Im} g_{xx} + ik_x \operatorname{Im} \partial_z g_{zx} + ik_x \partial_z \operatorname{Im} g_{xz} + k_x^2 \operatorname{Im} g_{zz}) - \frac{ik_x^2 c}{\omega} \chi_{yz} \operatorname{Im} g_{yy}. \tag{77}$$

This expression shows that it is necessary to calculate g_{yy}, g_{zx}, and g_{zz} to order χ_{ik}. From Eqs. (34), (37), and (39), we get

$$g_{yy} = \frac{\mu\omega^2}{c^2} \frac{1}{\kappa d} \left[1 + \frac{\mu k_x \omega}{\kappa^2 c} \left(1 + \frac{2\kappa a}{d} e^{2\kappa a} \right) \chi_{yz} \right], \tag{78}$$

$$\partial_z' g_{zx} = -\partial_z g_{zx} = -\frac{i\kappa k_x}{\varepsilon d} \left\{ 1 + \frac{\mu\omega}{k_x c} \left[1 + \frac{k_x^2}{\kappa^2} \left(1 - \frac{2\kappa a}{d} e^{2\kappa a} \right) \right] \chi_{zy} \right\}, \tag{79}$$

$$g_{zz} = \frac{k_x^2}{\varepsilon\kappa d} \left[1 - \frac{3\mu k_x \omega}{\kappa^2 c} \left(1 - \frac{2\kappa a}{3} \frac{e^{2\kappa a}}{d} \right) \chi_{zy} \right], \tag{80}$$

where d is defined as

$$d = e^{2\kappa a} - 1. \tag{81}$$

The remaining terms in Eq. (77) are, however, multiplying χ_{zy} or χ_{yz}, and so need not to be expanded in χ_{ik}. Thus to sufficient accuracy

$$\partial_z^2 g_{xx} = -\frac{\kappa^3}{\varepsilon} \frac{1}{d}, \tag{82}$$

$$\partial_z g_{zx} = \partial_z g_{xz} = \frac{i\kappa k_x}{\varepsilon} \frac{1}{d}, \tag{83}$$

$$g_{yy} = \frac{\mu\omega^2}{\kappa c^2} \frac{1}{d}, \tag{84}$$

$$g_{zz} = \frac{k_x^2}{\kappa\varepsilon} \frac{1}{d}. \tag{85}$$

We now put $\tau=0$, $x-x'=0$ in Eq. (76), and perform a standard complex frequency rotation whereby $\omega \to i\zeta$, with ζ real [31]. As $d\omega/\omega \to d\zeta/\zeta$, it follows from Eq. (76) that of physical importance are only those terms in $\langle\,\rangle$ that are real after the rotation (g_x^M has to be real). Thus the first terms in Eqs. (78)–(80) do not contribute. This is what we should expect: the asymmetry in momentum is caused by χ_{ik}. After some calculation we obtain, by letting $\int_{-\infty}^{\infty} d\zeta \to 2\int_0^{\infty} d\zeta$, $\int_{-\infty}^{\infty} dk_x \to 2\int_0^{\infty} dk_x$ because of symmetry of the integrand about the origin,

$$g_x^M = \frac{4\hbar\mu}{c} \int_0^{\infty} \frac{d\zeta}{2\pi} \int_0^{\infty} \frac{dk_x}{2\pi} \frac{k_x^4}{\kappa^3 d} \left\{ \left[1 - \frac{\varepsilon\mu\zeta^2}{k_x^2 c^2} \frac{2\kappa a}{d} e^{2\kappa a} \right] \chi_{yz} - \frac{2\kappa^2}{k_x^2} \left[1 + \frac{3k_x^2}{2\kappa^2} - \frac{\kappa a}{d} \left(1 + \frac{k_x^2}{\kappa^2} \right) e^{2\kappa a} \right] \chi_{zy} \right\}. \tag{86}$$

Recall that d is given by Eq. (81), where now $\kappa^2 = k_x^2 + \varepsilon\mu\zeta^2/c^2$. The integrals are seen to be finite. This is so because we have already performed the regularization by omitting those parts in the Green function that refer to the infinite undisturbed system (cf. also the remarks at the end of Sec. I). If the separation becomes infinite, then $d \to e^{2\kappa a} \to \infty$, and $g_x^M \to 0$ as it must; all plate-induced physical effects have to go away in this limit.

The expression (86) may be conveniently rewritten in terms of polar coordinates. Introduce $X = k_x = \kappa\cos\theta$, $Y = (\sqrt{\varepsilon\mu}/c)\zeta = \kappa\sin\theta$, so that

$$X^2 + Y^2 = \kappa^2. \tag{87}$$

The area element in the XY plane is $\kappa d\kappa d\theta = (\sqrt{\varepsilon\mu}/c) dk_x d\zeta$. Then

$$g_x^M = \frac{\hbar}{\pi^2}\sqrt{\frac{\mu}{\varepsilon}}\int_0^{\pi/2}\cos^4\theta d\theta\int_0^\infty \frac{\kappa^2 d\kappa}{d} \times\left\{\left[1-\tan^2\theta\frac{2\kappa a}{d}e^{2\kappa a}\right]\chi_{yz} - \left[5+2\tan^2\theta-(2+\tan^2\theta)\frac{2\kappa a}{d}e^{2\kappa a}\right]\chi_{zy}\right\}. \tag{88}$$

The integrals can be evaluated to give

$$g_x^M = \frac{\hbar\zeta(3)}{16\pi a^3}\sqrt{\frac{\mu}{\varepsilon}}\chi_{zy}, \tag{89}$$

where $\zeta(3)$ is the Riemann zeta function with argument 3. It is noteworthy that only one of the ME coefficients, χ_{zy}, appears in this expression. The factor multiplying χ_{yz} in Eq. (88) turns out to be zero. There seems to be no simple reason for this, although the behavior is obviously related to the complicated structure of Eq. (70) and the need to expand g_{yy}, g_{zx}, and g_{zz} in χ_{ik}; cf. Eqs. (78)–(80). Recall that the x direction has been singled out as special, and also that we have taken the variation of the fields in the transverse y direction to be equal to zero.

The quantity g_x^M is measurable, in principle. Before any measurement can be done, the expression (89) has of course to be augmented by contributions from all the other values of $\mathbf{k}$. Aspects connected with real experiments lie outside the scope of the present paper.

VI. SUMMARY AND DISCUSSION

Let us first recall the assumptions on which the above calculation is based.

(1) The tensor χ_{ik} characterizing the magnetoelectric medium is given naturally over all space in the fluid, on the inside as well as on the outside of the conducting plates. The constitutive relations are Eqs. (1) and (2), or, in inverse form, Eqs. (3) and (4) when the magnitude $|\chi_{ik}|$ of the coupling is small. In the example that we calculated in detail, χ_{ik} is given by Eq. (22). The tensor χ_{ik} may be asymmetric, in contrast to the permittivity ε_{ik} and permeability μ_{ik} which are always symmetric.

(2) We have followed two different approaches, giving the most weight to the Green-function approach since this does not seem to be treated very much in the literature. We took the temperature to be zero. The governing equation for the dyad Γ_{ik} is Eq. (14). The full solution of two of the diagonal components, g_{yy} and g_{xx}, introduced as Fourier components of the Γ's via Eq. (20), are given by Eqs. (31) and (32). We have here assumed that there is no variation of the fields in the transverse y direction. The derivation of Eqs. (31) and (32) generalizes conventional Green-function Casimir theory [24,37] to the case of ME media. For practical purposes it turns out to be possible to simplify the expressions considerably, by omitting terms that do not contribute to physical quantities in the end. The arguments for proceeding in this way are spelled out, for instance, in Ref. [32]. The relevant reduced components of g_{ik} in our case are given at the end of Sec. III.

(3) In Sec. IV we deviated to follow a different, and more simple, approach. After having established the momentum conservation equation for a ME medium, we calculated the right-left asymmetry for one single mode only (choosing one of the modes considered in Ref. [22]). The results are given by Eqs. (64) and (65). Adding two similar modes, one propagating in the $+x$ direction and one in the $-x$ direction, we obtain a net flow of momentum, caused by the coupling χ_{yz}.

(4) In Sec. V we returned to the Green-function approach, calculating the net x component of momentum arising now not from one single mode, but from all modes propagating in the $\pm x$ directions in the *vacuum* field. The main result is given by Eq. (89). All terms independent of χ_{ik} drop automatically out of the formalism, in accordance with what we should expect beforehand.

(5) On physical grounds one may ask: where does the net electromagnetic momentum come from? Obviously, it cannot come from "nothing." We are actually comparing two different physical situations here. The first is when the conducting plates are infinitely far separated. This is our initial "vacuum" state. The final state is when the plates have been brought close to each other, infinitely slowly. The calculated quantity g_x^M is the Minkowski momentum density extracted during this process of change of the plate separation. The coupling parameter χ_{ik} in the fluid is the same, all the time. The process is thus conceptually quite close to the process encountered in usual Casimir theory; the main difference being that it is now momentum, not energy, that is extracted.

(6) The setting of our thought experiment is similar, but not exactly the same, as that envisaged in Feigel's paper [11]. Feigel assumed the coupling χ_{ik} in the fluid to be the result of applying strong electric and magnetic fields. We have deliberately avoided this picture since it complicates the situation in the sense that one has to deal with two sets of fields, both the external fields, and the wave modes. When assuming naturally occurring χ_{ik} instead, as we have done, the interpretation of the effect becomes more transparent.

Before leaving this idea, let us, however, note the following point: Assume that strong crossed fields $\mathbf{E}_0$ and $\mathbf{H}_0$ are applied between the conducting plates at the instant $t=0$. Then, during the time when the external fields increase in strength, there acts an Abraham force in the fluid in the interior. The force density is given by the expression on the right in Eq. (52). Integrating over time, from $t=0$ until the external fields have become constant, we see that the following mechanical momentum density is imparted to the fluid:

$$\mathbf{g}^A = \frac{\varepsilon\mu - 1}{c^2}(\mathbf{E}_0 \times \mathbf{H}_0). \tag{90}$$

This is the dominant momentum given to the fluid between the plates. In addition comes the momentum transferred from the wave modes; these are connected with χ_{ik}. The momentum (89) is actually very similar to the momentum (or more strictly the angular momentum) transferred to the suspended dielectric cylindrical shell in the Walker-Lahoz experiment in ordinary electrodynamics [40,39].

(7) It might appear surprising that in Feigel's paper a high-frequency cutoff ω_{cut} is introduced, whereas in the present treatment there is no need of a cutoff. The reason for this behavior is that the two formalisms are constructed differently: Feigel considers the total contribution, including that of the infinite unconstrained system, whereas in our case we have regularized the infinite contribution away. The case of high frequencies leads in Feigel's case to infinities, whereas in our case it leads to *zero*. Again, this is the same point as was emphasized at the end of Sec. I. We are generally looking at the present problem as a sort of Casimir-type problem.

(8) What is the connection between the Feigel effect and relativity? In this context it might be of interest to recall how the relativistic formulation of electrodynamics in continuous media is formulated. There is always one particular inertial system S^0 here, namely the one where the medium is at rest—this was emphasized already in the classic papers of Jauch and Watson [41]. The relativistic formulation is obtained by introducing two electromagnetic field tensors $F_{\mu\nu}$ and $H_{\mu\nu}$ such that the covariant Maxwell equations

$$\partial_\rho F_{\mu\nu} + \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} = 0, \quad \partial_\nu H_{\mu\nu} = 0 \tag{91}$$

agree with the standard Maxwell equations in S^0 (we assume no external charges or currents). The electromagnetic energy-momentum tensor $S_{\mu\nu}$, assuming Minkowski's expression for the momentum density, is divergence-free,

$$\partial_\nu S_{\mu\nu} = 0, \tag{92}$$

meaning that the energy and momentum of the total field constitute a four-vector. Moreover, this four-vector is *spacelike*, so that it is possible to find inertial systems where the radiation energy becomes negative. A striking demonstration of this property is found in connection with the Cherenkov effect, in the frame where the emitting particle is at rest. A clear introduction to this kind of theory is found in Møller's book [42], and the theory is discussed also in papers of one of the present authors [39,43].

In our opinion there is no strong connection between the Feigel effect and relativity. The force on the fluid, or the momentum transferred to it, are calculated assuming the fluid to be *at rest*. Relativity is as little involved here as it is involved in the description of the Walker-Lahoz experiment. An exceptional case is, however, if the Euler-Heisenberg Lagrangian is drawn into consideration as a model to describe the ME effect (cf. for instance, van Tiggelen *et al.* [22]).

(9) It is of interest to have an idea about the magnitude of the effect that we have considered. Magnetoelectric birefringence is actually found even in a vacuum, when there are strong crossed external fields $\mathbf{E}_0$ and $\mathbf{H}_0$ present. The effect is, however, extremely small. Let $\Delta n = n_B - n_E$ denote the difference in the refractive index between the magnetic and electric directions. Even with a strong magnetic field of 30 T and an electric field of 10^8 V/m the birefringence is only $\Delta n \approx 8 \times 10^{-23}$ [7].

A more promising case is when one applies strong orthogonal fields to a linear isotropic liquid. Thus Roth and Rikken [2] performed an experiment in which molecular liquids were placed in such a strong field region. By passing laser light through the liquid, perpendicular to the fields, they obtained a linear relationship between the field strength and the MR birefringence. With a magnetic field strength up to 17 T and an electric field of 2.5×10^5 V/m the ME birefringence was found to be of order $\Delta n \sim 10^{-11}$. Thus the ME effect is much larger in a liquid than in a vacuum.

Naturally occurring anisotropies, the case that we have been considering, seem actually to be stronger. Thus the crystal $FeGaO_3$ is known to be magnetoelectrically active with ME coefficients about 3×10^{-4} at low frequencies. In this crystal, as well as in analogous crystals like $FeAlO_3$, anisotropies of order 10^{-4} are expected over a wide frequency range from dc to x rays [44].

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