Complex network synchronizability: Analysis and control

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In this paper, the investigation is first motivated by showing two examples of simple regular symmetrical graphs, which have the same structural parameters, such as average distance, degree distribution, and node betweenness centrality, but have very different synchronizabilities. For a given network with identical node dynamics, it is further shown that two key factors influencing the network synchronizability are the network inner linking matrix and the eigenvalues of the network topological matrix. Several examples are then provided to show that adding new edges to a network can either increase or decrease the network synchronizability. In searching for conditions under which the network synchronizability may be increased by adding edges, it is found that for networks with disconnected complementary graphs, adding edges never decreases their synchronizability. Moreover, it is found that an unbounded synchronized region is always easier to analyze than a bounded synchronized region. Therefore to effectively enhance the network synchronizability, a design method is finally presented for the inner linking matrix of rank 1 such that the resultant network has an unbounded synchronized region, for the case where the synchronous state is an equilibrium point of the network.

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I. INTRODUCTION AND PROBLEM FORMULATION

The subject of network synchronization has recently attracted increasing attention from various fields (see [1–10] and references therein). Of particular importance is how the synchronizability depends on various structural parameters of the network, such as average distance, clustering coefficient, degree distribution, and weight distribution, among others. Some important results have been established for such concerned problems based on the notions of master stability function and synchronized region [1,5,11–14]. Some interesting relationships between synchronizability and structural parameters of networks have also been reported, e.g., smaller average network distance does not necessarily mean better synchronizability [15], therefore the betweenness centrality was proposed as a good indicator for synchronizability [16]; and two networks with the same degree sequence were constructed in a probabilistic sense to demonstrate that they can have different synchronizabilities [17], showing that synchronizability has no direct relations with degree distributions. Moreover, the effect of perturbations of coupling matrices on the synchronizability was studied in [18]. Motivated by all these research works, this paper attempts to further explore the analysis and control problems of synchronizability for various complex dynamical networks.

Consider a dynamical network consisting of N coupled identical nodes, with each node being an n-dimensional dynamical system, described by

$$\dot{x}_i = f(x_i) - c\sum_{j=1}^{N} a_{ij}H(x_j), \quad i = 1,2, \ldots ,N, \tag{1}$$

where $x_i=(x_{i1},x_{i2},\ldots,x_{in})\in\mathbb{R}^n$ is the state vector of node i, $f(\cdot):\mathbb{R}^n\to\mathbb{R}^n$ is a smooth vector-valued function, constant $c>0$ represents the coupling strength, $H(\cdot):\mathbb{R}^n\to\mathbb{R}^n$ is called the inner linking function, and $A=(a_{ij})_{N\times N}$ is called the outer coupling matrix or topological matrix, which represents the coupling configuration of the entire network. This paper only considers the case that the network is diffusively connected, i.e., its entries satisfy $a_{ii}=-\sum_{j=1,j\neq i}^{N}a_{ij}$, $i=1,2,\ldots,N$. Further, suppose that, if there is an edge between node i and node j, then $a_{ij}=a_{ji}=-1$, i.e., A is a Laplacian matrix. If the graph corresponding to A is connected, i.e., A is irreducible, then 0 is an eigenvalue of A with multiplicity 1, and all the other eigenvalues of A are strictly positive, which are denoted by

$$0 = \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \le \lambda_N. \tag{2}$$

The dynamical network (1) is said to achieve (asymptotical) synchronization if $x_1(t)\to x_2(t)\to\cdots\to x_N(t)\to s(t)$, as $t\to\infty$, where, because of the diffusive coupling configuration, the synchronous state $s(t)\in\mathbb{R}^n$ is a solution of an individual node, i.e., $\dot{s}(t)=f(s(t))$.

As shown in [5,13], the local stability of the synchronized solution $x_1(t)=x_2(t)=\cdots=x_N(t)=s(t)$ can be determined by analyzing the following so-called master stability equation:

$$\dot{\omega} = [Df(s(t)) + \alpha DH(s(t))]\omega, \tag{3}$$

where $\alpha\in\mathbb{R}$, and $Df(s(t))$ and $DH(s(t))$ are the Jacobian matrices of functions f and H at $s(t)$, respectively.

The largest Lyapunov exponent L_{max} of network (1), which can be calculated from system (3) and is a function of α, is referred to as the master stability function. In addition, the region S of negative real α where L_{max} is also negative is called the synchronized region. Based on the results of [5,13], the synchronized solution of dynamical network (1) is locally asymptotically stable if, and only if,

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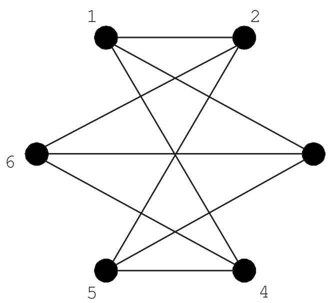

FIG. 1. Graph G_1.

$$-c\lambda_k \in S, \quad k=2,3,\ldots,N. \tag{4}$$

The synchronized region S can be an unbounded region, a bounded region, an empty set, or a union of several such regions.

Obviously, for given node dynamics of a linearly coupled network, two key factors influencing the synchronizability are the inner linking matrix $H(\cdot)=H$ and the eigenvalues of the topological matrix A. The inner linking matrix is directly related to the synchronized region, as studied in [11,13,19]. The larger the synchronized region, the easier the synchronization. The topological matrix, on the other hand, is directly related to condition (4). If S is an unbounded sector $(-\infty,\alpha]$, the eigenvalue λ_2 of A determines the synchronizability [8]; if S is a bounded sector $[\alpha_1,\alpha_2]$, the ratio $r(A)=\frac{\lambda_2}{\lambda_N}$ determines the synchronizability [1]. No matter what the synchronized region is, the larger the λ_2 and $r(A)$ are, the easier the synchronization is. This paper will further study this issue with more careful analysis.

The rest of this paper is organized as follows. In Sec. II, two simple graphs on six nodes are given to show that networks with the same structural parameters, such as average distance, degree distribution, and betweenness centrality, can have different synchronizabilities. If the synchronized region S is unbounded, adding edges never decreases the synchronizability, but this may not be true if S is bounded. In Sec. III, a class of networks with disconnected complementary graphs are discussed. For such networks, adding edges never decreases the synchronizability no matter what type of region S is. In Sec. IV, an inner linking matrix of rank 1 is designed for realizing unbounded synchronized regions in the case that the synchronous state is an equilibrium point. In Sec. V, some network synchronization examples are provided to illustrate the theoretical results. The paper is concluded by the last section.

Throughout this paper, for any given undirected graph G, eigenvalues of G mean eigenvalues of its corresponding Laplacian matrix. Notations for graphs and their corresponding Laplacian matrices are not differentiated, and networks and their corresponding graphs are not distinguished, unless otherwise indicated.

II. TWO SIMPLE GRAPHS THAT TELL THE MAIN IDEA

In this section, the two simple graphs G_1 and G_2 on six nodes, shown in Figs. 1 and 2, are considered, where G_1 is a typical bipartite graph with many interesting properties.

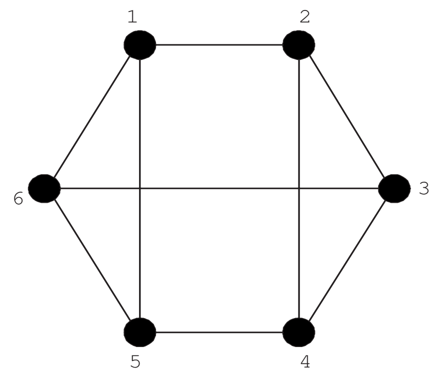

FIG. 2. Graph G_2.

Obviously, graphs G_1 and G_2 have the same degree sequence, where the degree of every node is 3; the same average distance $\frac{7}{5}$; and the same node betweenness centrality 2 [16,20]. Although these two graphs have the same structural characteristics, their corresponding networks have different synchronizabilities, as shown below. Their Laplacian matrices are

$$\begin{pmatrix} -3 & 1 & 1 & 1 & 0 & 0 \\ 1 & -3 & 0 & 0 & 1 & 1 \\ 1 & 0 & -3 & 0 & 1 & 1 \\ 1 & 0 & 0 & -3 & 1 & 1 \\ 0 & 1 & 1 & 1 & -3 & 0 \\ 0 & 1 & 1 & 1 & 0 & -3 \end{pmatrix},$$

$$\begin{pmatrix} -3 & 1 & 0 & 0 & 1 & 1 \\ 1 & -3 & 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 1 & 0 & 1 \\ 0 & 1 & 1 & -3 & 1 & 0 \\ 1 & 0 & 0 & 1 & -3 & 1 \\ 1 & 0 & 1 & 0 & 1 & -3 \end{pmatrix},$$

respectively. The eigenvalues of G_1 are $0,3,3,3,3$, and 6; the eigenvalues of G_2 are $0,2,3,3,5$, and 5. Obviously, $\lambda_2(G_1)=3>\lambda_2(G_2)=2$, and $r(G_1)=0.5>r(G_2)=0.4$. Therefore the synchronizability of network G_1 is better than that of network G_2.

Graphs G_1 and G_2 have the same structural parameters, but it is clear that they have different average clustering coefficients, denoted by $C(G_i)$, $i=1,2$, with $C(G_2)>C(G_1)$. As mentioned above, the clustering coefficient does not have direct relation to synchronization [15]. For example, globally coupled graphs have the largest clustering coefficient, 1, and they have the best synchronizability. However, for the above two graphs, the larger clustering coefficient does not indicate better synchronizability. This is demonstrated by the following process of adding edges.

Consider enhancing λ_2 and r by adding edges to G_2. For this purpose, the following result is needed [21]. For any given connected undirected graph G of size N, its nonzero eigenvalues indexed as in Eq. (2) grow monotonically with the number of added edges, that is, for any added edge e, $\lambda_i(G+e) \geq \lambda_i(G)$, $i=1,\ldots,N$.

By this statement, obviously, if the synchronized region is unbounded, adding edges never decreases the synchronizability. However, for bounded synchronized regions, this is not necessarily true. For example, adding an edge between node 1 and node 3 in graph G_2 (Fig. 2), denoted by $e\{1,3\}$, leads to a new graph $G_2+e\{1,3\}$, whose eigenvalues are $0, 2.2679, 3, 4, 5$, and 5.7321. Thus $r(G_2+e\{1,3\})=0.3956$ is even smaller than $r(G_2)=0.4$. This means that the synchronizability of network $G_2+e\{1,3\}$ is worse than that of network G_2. Adding a new edge between node 1 and node 4 instead, one gets $r(G_2+e\{1,3\}) < r(G_2+e\{1,3\}+e\{1,4\}) = 0.3970 < r(G_2)$. This means that the synchronizability of network $G_2+e\{1,3\}+e\{1,4\}$ is better than $G_2+e\{1,3\}$, but still worse than G_2. Therefore by adding edges, the network synchronizability may increase or decrease, for which no general rule has been found to date.

On the other hand, during the process of adding edges, average distance decreases and average clustering coefficient increases; but this does not indicate better synchronizability, consistent with the conclusion in [15].

It was shown [16] that the synchronizability is always improved as the maximum betweenness centrality is reduced, which is consistent with the conclusion of [15]. In the above two graphs, however, it shows that the same betweenness centrality does not necessarily mean the same synchronizability. On the other hand, adding three edges between nodes 1 and 6, 2 and 3, 3 and 4, respectively, in graph G_1, and then computing their corresponding eigenvalues, it can be verified that the networks built on graphs $G=G_1+e\{1,6\}+e\{2,3\}+e\{3,4\}$ and G_1 have the same synchronizability. However, in this case, the maximum betweenness centrality of G, $\frac{11}{6}$, is smaller than that of G_1, 2. This shows that the smaller betweenness centrality does not necessarily indicate better synchronizability, revealing the complexity in the relationship between synchronizability and network structural parameters.

Note that adding edges in G_1 also increases the clustering coefficient and decreases the average distance, but this does not result in the increase of synchronizability. In the following, it explains why adding three edges in G_1 does not increase the synchronizability.

III. NETWORKS WITH DISCONNECTED COMPLEMENTARY GRAPHS

For a given graph G, the complement of G is the graph containing all the nodes of G, and all the edges that are not in G. The complementary graph of G is denoted by G^c. For example, the complementary graphs of G_1 and G_2 in Figs. 1 and 2 are shown in Figs. 3 and 4, respectively. In the previous section, it shows that adding edges sometimes decreases the synchronizability. However, for a class of graphs with disconnected complementary graphs, this never occurs. In order to discuss such networks, the following results are needed [21].

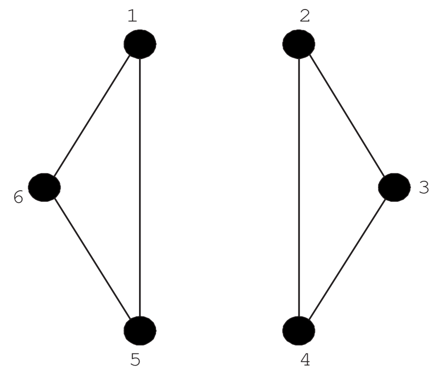

FIG. 3. Graph G_1^c.

For any given graph G, one has:

(i) $\lambda_N(G)$, the largest eigenvalue of G, satisfies $\lambda_N(G) \leq N$.

(ii) $\lambda_N(G)=N$ if, and only if, G^c is disconnected.

(iii) If G^c is disconnected and has (exactly) q connected components, then the multiplicity of $\lambda_N(G)=N$ is $q-1$.

(iv) $\lambda_i(G^c)=N-\lambda_{N-i+2}(G)$, $2 \leq i \leq N$.

The complementary graph of G_1 is shown in Fig. 3, which is disconnected. The largest eigenvalue of G_1 is 6, which remains the same when the graph receives additional edges. Hence combining with the previous discussions, the synchronizability of the networks built on graph G_1 never decrease with adding edges. Although this is true, adding any three edges to graph G_1 does not enhance the synchronizability, since the least nonzero eigenvalue $\lambda_2=3$ of G_1 has multiplicity 4 (the multiplicity of the largest eigenvalue in G_1^c). This is due to the fact that, for any graph G, $\text{rank}(\lambda_i I-(G+e)) \leq \text{rank}(\lambda_i I-G)+1$.

According to the above results, the multiplicity of the largest eigenvalue of a graph G is related to the number of connected components of its complement G^c. In order to reduce the number of edges needed to enhance the synchronizability, the multiplicity of the largest eigenvalue of G^c (i.e., the multiplicity of the least nonzero eigenvalue of G) should be large. Therefore better understanding and careful manipulation of complementary graphs are useful for enhancing the network synchronizability; and, at least for dense networks, the complementary graphs are easier to analyze than the original graphs, e.g., G_1^c is simpler than G_1.

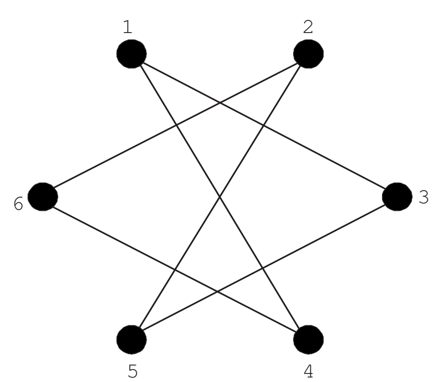

FIG. 4. Graph G_2^c.

The graphs shown in Figs. 1 and 2 can be generalized to graphs of size $N=2n$. Suppose that graph G_1 is bipartite in the sense that it contains two sets of nodes, each set containing n isolated nodes, and each node in one set connects to all the nodes in the other set, i.e., the complementary graph of G_1 is two separated fully connected subgraphs of size n. Graph G_2 is composed of two fully connected subgraphs of size n and n edges connecting each node in one subgraph to the corresponding node in the other subgraph. In this case, the least nonzero and maximum eigenvalues of G_1 are $\frac{N}{2}$ and N, respectively, and $r(G_1)=\frac{1}{2}$. On the other hand, the least nonzero and maximum eigenvalues of G_2 are 2 and $\frac{N}{2}+2$ [21], respectively, with $r(G_2)=\frac{4}{N+4}\to 0$ as $N\to+\infty$. Therefore these two graphs have the same structural parameters but have very different synchronizabilities.

IV. DESIGNING THE INNER LINKING MATRIX

From the above discussions, it can be seen that the bounded synchronized regions are more complicated than the unbounded synchronized regions. Thus the synchronizability is easier to analyze when the synchronized region is unbounded. Hence it is interesting to find out how to design the inner linking matrix such that the network synchronized region is unbounded.

If the synchronous state is an equilibrium point, then both $Df(s(t))$ and $DH(s(t))$ in Eq. (3) reduce to constant matrices, denoted by F and H, respectively. In this case, system (3) becomes

$$\dot{\omega}=[F+\alpha H]\omega. \tag{5}$$

Hence the synchronized region S becomes the stability region of $F+\alpha H$ with respect to parameter α. In this section, consider the design of an H such that $F+\alpha H$ has an unbounded stable region. It is well-known [8] that if H is an antistable matrix (e.g., $H=I_n$), $F+\alpha H$ has an unbounded stable region. However, if H is of full rank, it means that the coupling in the network is a full state coupling among nodes, so the coupling cost may be high. For this reason, consider the design of an H of rank 1 such that the stable region for $F+\alpha H$ is unbounded. In this case, the coupling can be viewed as an input-output coupling as in control systems [22], or an observer-based coupling [23].

Given a matrix $F\in\mathbf{R}^{n\times n}$, there exists a matrix $H\in\mathbf{R}^{n\times n}$ of rank 1 such that the stability region of $F+\alpha H$ with respect to parameter α contains $(-\infty,\alpha_1]$, $\alpha_1<0$, if and only if every unstable eigenvalue of F is corresponding to only one elementary factor.

Without loss of generality, suppose $\alpha_1=-1$. By the canonical control method [24,25], the matrix H can be designed as follows. First, one may take a column vector b such that (F,b) is stabilizable. Then, there exists a matrix $P=P^T$ such that $FP+PF^T-2bb^T<0$, where the superscript means the transpose of the corresponding matrix. Consequently, taking $q=b^TP^{-1}$ leads to the stability of $F+\alpha bq$ for all $\alpha\in(-\infty,-1]$. Therefore $H=bq$ is the matrix to be found.

On the other hand, if $F+\alpha H$ is stable and H is of rank 1, it means that (F,H) is stabilizable, so that every unstable eigenvalue of F must be corresponding to only one elementary factor (i.e., one Jordan block) [25].

Let $z_i=qx_i$ and the inner linking function $H(x_j)=bqx_j$ in network (1). Then z_i can be viewed as the output of node i of Eq. (1) and the linking function bqx_j can be viewed as the influence of the output of node j to the other nodes. Clearly, the above coupling is simpler than full state couplings.

If F is stable, i.e., the node system is locally stable, then there is always a matrix H of rank 1 such that the resulting network has an unbounded synchronized region, as shown by the examples given below.

Moreover, one may also design an H such that the unstable region of $F+\alpha H$ is unbounded, if desired, which is useful for desynchronization problems. Such an H can be obtained as follows. If every eigenvalue of F corresponds to only one elementary factor, one may take a column vector b such that (F,b) is controllable [25]. By control theory method, using the similar transformation, one can transfer F and b into the following standard forms:

$$F=\begin{pmatrix}0 & 1 & \cdots & 0\\ \vdots & \cdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\\ \beta_0 & \beta_1 & \cdots & \beta_{n-1}\end{pmatrix},\quad b=\begin{pmatrix}0\\ \vdots\\ 0\\ 1\end{pmatrix}.$$

A necessary condition for the stability of F is that $\beta_i<0$ for all $0\le i\le n-1$. Based on this point, one can take a row vector $q=(\xi_0,\ldots,\xi_{n-1})$ such that $\beta_0+\alpha\xi_0>0$ for all $\alpha\in(-\infty,-1]$. Then $H=bq$ is the matrix to be found such that $F+\alpha H$ has an unbound unstable region with respect to parameter α.

V. EXAMPLES

Example 1. Consider the network (1) consisting of the third-order smooth Chua's circuits [26], in which each node is described by

$$\dot{x}_{i1}=-k\alpha x_{i1}+k\alpha x_{i2}-k\alpha(ax_{i1}^3+bx_{i1}),$$

$$\dot{x}_{i2}=kx_{i1}-kx_{i2}+kx_{i3},$$

$$\dot{x}_{i3}=-k\beta x_{i2}-k\gamma x_{i3}. \tag{6}$$

The vector x_i in Eq. (1) is $(x_{i1},x_{i2},x_{i3})^T$ here. Linearizing Eq. (6) at its zero equilibrium gives

$$\dot{x}_i=Fx_i,\quad F=\begin{pmatrix}-k\alpha-k\alpha b & k\alpha & 0\\ k & -k & k\\ 0 & -k\beta & -k\gamma\end{pmatrix}. \tag{7}$$

Take $k=1$, $\alpha=-0.1$, $\beta=-1$, $\gamma=1$, $a=1$, and $b=-25$. Then F is stable, i.e., the node system (6) is locally stable about zero. Further, take the inner linking matrix

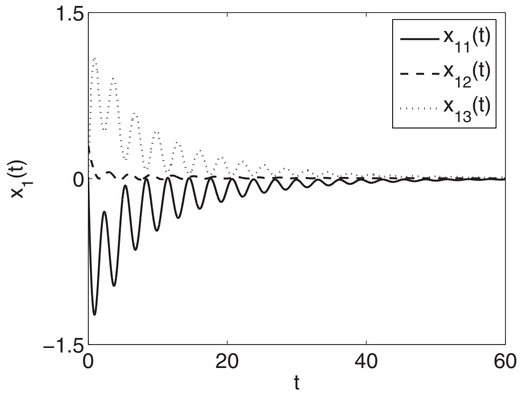

FIG. 5. Network on G_1.

$$H=\begin{pmatrix} 0.8348 & 9.6619 & 2.6591 \\ 0.1002 & 0.0694 & 0.1005 \\ -0.3254 & -8.5837 & -0.9042 \end{pmatrix}.$$

Then, by simple computation, one knows that $F+\alpha H$ has two disconnected stable regions: $S_1=[-0.0099,0]$ and $S_2=[-2.225,-1)$. Therefore the entire synchronized region is $S_1\cup S_2$. Moreover, suppose that the number of nodes is $N=6$, and the outer coupling matrix A is equal to the G_1 in Sec. II. According to the eigenvalues of G_1 given in Sec. II, one may take the coupling strength $c=\frac{1}{2.9}$. Then, for every eigenvalue of G_1, one has $c\lambda_i\in S_2$. By Eq. (4), network (1) specified with the above data achieves synchronization. However, for the outer coupling matrix G_2 given in Sec. II, for any coupling strength $c\in[0.002,+\infty)$, Eq. (4) does not hold. Therefore for the above node equation, inner coupling matrix and coupling strength, the network built on graph G_1 in Fig. 1 achieves synchronization, but the network built on G_2 in Fig. 2 does not synchronize. Figures 5 and 6 show the states of node 1 in two networks, respectively, the other nodes behave similarly.

Example 2. Consider designing the inner coupling matrix such that the corresponding network has an unbounded synchronized region, where the node system is as in example 1. In this case, F is stable, so for any column vector b, (F,b) is

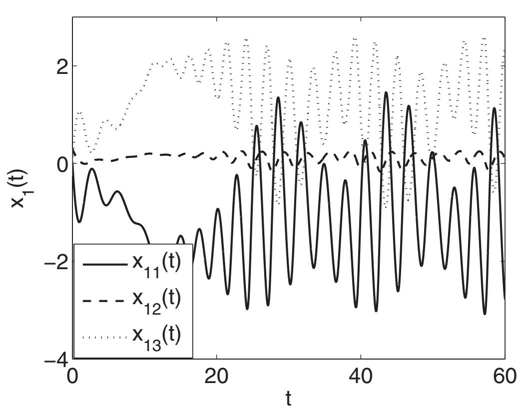

FIG. 6. Network on G_2.

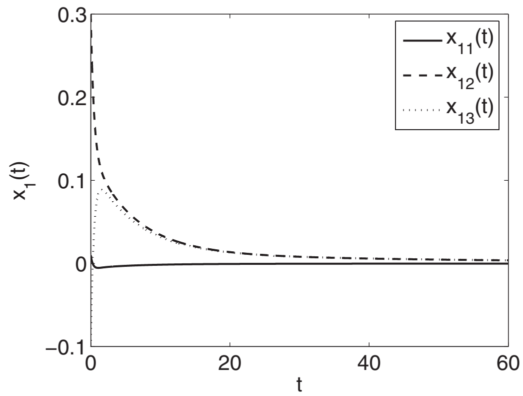

FIG. 7. Network on G_1.

stabilizable, e.g., $b=(0,0,1)^T$. By the above method, one gets $q=(0.0708,-0.155\,90,0.4296)$. Then, change the inner coupling matrix H in example 1 to $H=bq$, while keeping the other parameters unchanged. This $F+\alpha H$ is stable for $\alpha\in(-\infty,-1]$. Consequently, the corresponding network has an unbounded synchronized region. Figures 7 and 8 show their similar states of node 1 in two networks built on graphs G_1 and G_2, respectively, the other nodes behave similarly.

VI. CONCLUSION

In this paper, the synchronizability of complex dynamical networks, which is directly related to the inner linking matrix and the topological matrix, has been carefully discussed. Two simple graphs have been given to show that networks can have different synchronizabilities even when they have the same average distance, node betweenness centrality, and degree distribution. It has also been shown that the larger clustering coefficient, smaller betweenness centrality, and shorter average distance do not necessarily imply better synchronizability. This demonstrates the complexity in the relationship between the synchronizability and network structural parameters. The most significant discovery of this paper is that if the synchronized region is bounded, adding edges can either increase or decrease the network synchronizability; however, for networks with disconnected complementary graphs, adding edges never decreases their synchronizability.

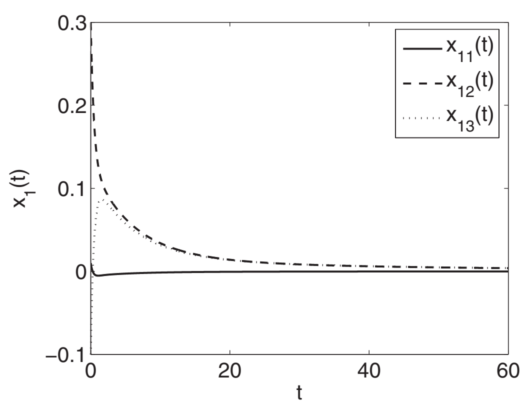

FIG. 8. Network on G_2.

Therefore better understanding and more careful manipulation of complementary graphs are essential for enhancing network synchronizability. Moreover, unbounded synchronized regions are easier to analyze than the bounded ones. Therefore to effectively enhance the network synchronizability, a design method for the inner linking matrix of rank 1 is finally provided such that the resulting network has an unbounded synchronized region for the case where the synchronous state is an equilibrium point of the network.

ACKNOWLEDGMENT

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