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Wave excitation in inhomogeneous dielectric media

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The equation describing the propagation of a mode driven by external currents in an inhomogeneous dielectric is derived from the principle of the conservation of wave energy density and wave momentum density. The wave amplitude in steady state is obtained in terms of a simple spatial integration of the driving current. The contribution from the spatial derivative of the dielectric response is found to be important. The analytical predictions are verified through comparison with δf particle-in-cell computations of electron Bernstein wave propagation, thus showing applicability to kinetic systems.

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Waves in dielectric media can be driven by a number of effects, including noise (e.g., due to discreteness in plasma), nonlinearity (currents from the beating of one or more waves), and embedded antennas, such as ac probes. The ability to compute the wave resulting from a driving current is thus important to understanding the generation of waves due to noise, nonlinear wave-wave interactions, and probe excitation. Such computations at present suffer from ambiguity. Homogeneous dielectric theory provides an equation for the amplitude of a mode driven by a current, but, in the inhomogeneous dielectric context, it is unclear how to generalize this equation, as the coefficients vary on the same scale as the mode amplitude, and the degree to which the derivative operator should act on the background dielectric response function is unknown.

In this Rapid Communication, we show that this ambiguity can be resolved by simple arguments relying on the conservation of wave energy and wave momentum. Because wave energy and wave momentum are known for homogeneous media, they are known to lowest order in inhomogeneous media, and, hence, the differential conservation law is known. Thus, the propagation of wave energy and momentum in the presence of a driving current can be found. However, this is insufficient for deriving the full mode propagation as needed, e.g., in mode coupling, as one needs the variation of the phase as well as the amplitude, and the conservation equation gives only the variation of the amplitude. To obtain the phase variation, we start from the equation for propagation in homogeneous media, and we note that in going to inhomogeneous media there is an arbitrariness as to how the derivatives are applied to the local dielectric response function. We postulate that this application is to the entire dielectric function, not in any separate way to its internal factors, such as local densities, temperatures, or lattice constants. This leads us to introduce an arbitrary factor, which we can then determine from the conservation of wave energy and momentum.

We are able to confirm our results in two ways. First, we show that our results can be straightforwardly derived for cold plasma. Second, we perform a series of simulations for electron Bernstein mode propagation. The latter are chosen as they are exemplary warm plasma waves. We show that correctly including the spatial derivatives of the dielectric function in the slow-amplitude equations leads to reproduction of the results from a full-scale particle simulation.

The basic ansatz of this development is that the wave can be represented as the product of a slowly varying amplitude and a rapidly varying phase,

$$\mathbf{E}(\mathbf{x},t) = a(\mathbf{x},t)\hat{u}e^{i\psi(\mathbf{x},t)} + \text{c.c.}, \tag{1}$$

where a is the complex amplitude, $\hat{u}$ is the wave polarization, and ψ is the rapidly varying phase, such that

$$\mathbf{k}_0 = \nabla\psi \quad \text{and} \quad \omega_0 = -\partial\psi/\partial t. \tag{2}$$

For homogeneous, stationary media, the dynamics derives from the fact that the Fourier transform $\widetilde{\mathbf{E}}(\mathbf{k},\omega)$ of the wave satisfies

$$i\omega\epsilon_0\mathbf{D}\cdot\widetilde{\mathbf{E}} = \widetilde{\mathbf{J}}_e, \tag{3}$$

where

$$\mathbf{D} = \mathbf{k}\mathbf{k}(c/\omega)^2 + [1-(kc/\omega)^2]\mathbf{I} + \chi(\mathbf{k},\omega), \tag{4}$$

$\chi(\mathbf{k},\omega)$ is the polarizability of the medium, and $\widetilde{\mathbf{J}}_e$ is the Fourier transform of the *external* driving current $\mathbf{J}_e$, a current not included in the dielectric response, such as noise, nonlinear currents, or probe currents. Here we are concerned with the study of normal modes of the medium having small damping, and so we separate the electromagnetic response tensor into Hermitian (reactive) and anti-Hermitian (dissipative) parts,

$$\mathbf{D} = \mathbf{D}_h + i\mathbf{D}_a, \tag{5}$$

and it is assumed that the dynamics is dominantly determined by the Hermitian part, so that the anti-Hermitian part can be accounted for in the next order. For inhomogeneous, nonstationary media, where $\mathbf{D}_h(\mathbf{x},t,\mathbf{k},\omega)$ is a function of space and time as well, Eq. (3) gives the lowest-order equation, implying that for undriven modes, $\hat{u}$ is an eigenvector of the tensor $\mathbf{D}_h$, and that the corresponding eigenvalue D_0 of the diagonalization,

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$$\mathbf{D}_h = \sum D_\ell \hat{u}_\ell \hat{u}_\ell^\dagger, \tag{6}$$

($\hat{u}_\ell^\dagger$ is the conjugate of the eigenvector $\hat{u}_\ell$) vanishes,

$$D_0(\mathbf{k}_0, \omega_0) = 0, \tag{7}$$

at the local wave vector and frequency.

Our goal is to understand the evolution of the wave amplitude a with the driving current, which now drives the propagating mode. (This methodology is particularly useful in the study of nonlinear mode-mode coupling [1–4].) For homogeneous media in the absence of a driving current, we have that the wave packet propagates at that group velocity, and that it disperses and diffracts upon taking into account higher-order terms. For homogeneous, stationary media, we further know [5,6], from the usual slow-fast expansion, that the amplitude obeys the equation

$$\omega_0 D_{a,0} a + \frac{\partial(\omega D_0)}{\partial \omega}\frac{\partial a}{\partial t} - \frac{\partial(\omega D_0)}{\partial \mathbf{k}} \cdot \frac{\partial a}{\partial \mathbf{x}} = -\langle e^{-i\psi(\mathbf{x},t)} \hat{u}^\dagger \cdot \mathbf{J}_e/\epsilon_0 \rangle, \tag{8}$$

where the angular brackets indicate a local averaging of the rapidly varying phase, and $D_{a,0} \equiv \hat{u}_0^\dagger \cdot \mathbf{D}_a \cdot \hat{u}_0$ gives the effect of dissipation (or amplification). All quantities are differentiated with respect to wave vector and frequency, and then the values $(\mathbf{k}, \omega)$ are inserted.

The difficulty is to determine how the above equation is modified for inhomogeneous, nonstationary media. In that case, the scale on which the dielectric response varies is the same as the scale on which the amplitude varies. Hence, the temporal and spatial derivatives might also act on the dielectric response terms, but precisely how cannot be determined without further analysis. To resolve this, we assume that such effect applies to the dielectric terms as a whole (rather than, say, to just the density of the medium but not the background magnetic field), and that the effect is the same in time and space. From this we deduce that the equation must be of the form,

$$\omega D_{a,0} a + \frac{\partial(\omega D_0)}{\partial \omega}\frac{\partial a}{\partial t} + \alpha \frac{\partial^2(\omega D_0)}{\partial \omega\, \partial t} a - \frac{\partial(\omega D_0)}{\partial \mathbf{k}} \cdot \frac{\partial a}{\partial \mathbf{x}} - \alpha \frac{\partial^2(\omega D_0)}{\partial \mathbf{k} \cdot \partial \mathbf{x}} a = -\langle e^{-i\psi(\mathbf{x},t)} \hat{u}^\dagger \cdot \mathbf{J}_e/\epsilon_0 \rangle, \tag{9}$$

where α is some constant.

To determine the constant α, we consider energy conservation, which in a homogeneous, stationary medium is

$$\frac{\partial}{\partial t}\left(\epsilon_0 \frac{\partial(\omega D_0)}{\partial \omega}|a|^2\right) - \boldsymbol{\nabla} \cdot \left(\epsilon_0 \frac{\partial(\omega D_0)}{\partial \mathbf{k}}|a|^2\right) + 2\omega_0 D_{0,i}|a|^2 = -\langle a e^{i\psi(\mathbf{x},t)} \hat{u}^\dagger \cdot \mathbf{J}_e \rangle + \text{c.c.} \tag{10}$$

(More generally, one could consider action conservation.) This formula is accurate for homogeneous media, and so it must be accurate to lowest order for inhomogeneous media. Expansion of the derivatives and subtraction from Eq. (9) then gives a unique answer, $\alpha = 1/2$. Therefore, the complex amplitude (which includes phase information) obeys the equation

$$\frac{\partial(\omega D_0)}{\partial \omega}\frac{\partial a}{\partial t} + \frac{1}{2}\frac{\partial^2(\omega D_0)}{\partial \omega\, \partial t} a - \frac{\partial(\omega D_0)}{\partial \mathbf{k}} \cdot \frac{\partial a}{\partial \mathbf{x}} - \frac{1}{2}\frac{\partial^2(\omega D_0)}{\partial \mathbf{k} \cdot \partial \mathbf{x}} a + \omega_0 D_{a,0} a = -\langle e^{-i\psi(\mathbf{x},t)} \hat{u}^\dagger \cdot \mathbf{J}_e/\epsilon_0 \rangle. \tag{11}$$

This equation can now be used as the basis for further studies of modes driven by nondielectric currents, such as nonlinearly driven currents, probe currents, or noise-induced currents. We note that this result is nearly present in Refs. [7–9], but, regardless, here we have shown that it follows from a simple argument involving energy conservation.

We have verified the result (11) from a direct derivation for cold plasma and from particle simulation for kinetic plasma. The latter further shows the utility of this approach for finite ratios of wavelength to scale length.

For the cold-fluid case, we assumed inhomogeneity in the $\mathbf{x}$ direction AND used the local, cold-plasma dispersion tensor,

$$\chi = \sum_s \omega_s^2 \left(-\frac{\hat{b}\hat{b}}{\omega^2} + \frac{\mathbf{I} - \hat{b}\hat{b}}{\Omega_s^2 - \omega^2} + \frac{i\hat{b}\mathbf{x}}{\omega(\Omega_s - \omega)} \right), \tag{12}$$

a sum over species label s, with $\hat{b}$ being the unit vector along the magnetic field, ω_s being the plasma frequency for species s, and Ω_s being the signed gyrofrequency for species s. In this case, the only dependence on the wave vector is explicit; the polarization χ has no dependence on $\mathbf{k}$. Hence, the expansion is straightforward and allows precise derivation of the spatial derivatives of Eq. (11).

The second case studied is that of electron Bernstein wave (EBW) propagation, for which the kinetic effects are intrinsic; Bernstein waves do not exist in their absence. To simplify the comparison, we look at the time-independent answer, applicable at long times, and we restrict the problem to one dimension, perpendicular to the magnetic field, so that we can neglect damping as well. In this case, Eq. (11) reduces to

$$\frac{\partial D_0}{\partial k_x}\frac{\partial a}{\partial x} + \frac{\partial^2 D_0}{\partial k_x\, \partial x} a = \left(\frac{\partial D_0}{\partial k_x}\right)^{1/2} \frac{\partial}{\partial x}\left[\left(\frac{\partial D_0}{\partial k_x}\right)^{1/2} a\right] = -\langle e^{-i\psi(\mathbf{x},t)} \hat{u}^\dagger \cdot \mathbf{J}_e/(\omega\epsilon_0) \rangle. \tag{13}$$

This equation can be directly integrated to obtain

$$\left(\frac{\partial D_0}{\partial k_x}\right)^{1/2} a \Bigg|_x - \left(\frac{\partial D_0}{\partial k_x}\right)^{1/2} a \Bigg|_{x_0} = -\int_{x_0}^{x} \left(\frac{\partial D_0}{\partial k_x}\right)^{-1/2} \langle e^{-i\psi(\mathbf{x},t)} \hat{u}^\dagger \cdot \mathbf{J}_e/(\omega\epsilon_0) \rangle. \tag{14}$$

This equation thus gives the amplitude of the excited wave as a quadrature.

Had we ignored the term in Eq. (13) involving the spatial derivative of the dielectric response function, which would be equivalent to setting $\alpha = 0$, the result would have been the absence of the terms in $\partial D_0/\partial k_x$ in that equation. Then, of course, the amplitude equation would not satisfy conservation of the usual dielectric energy.

In the WKB approximation Cairns and Fuchs showed that the second-order equation Eq. (11) yielded good agreement with fourth-order theory of the fast-wave–Bernstein mode coupling problem [10]. To illustrate the accuracy of these equations, we compare results from them as well as from a full kinetic simulation. For the full kinetic simulation we use the linear δf method [11]. For the reduced equations presented here, the dispersion relation

$$D_0(k,\omega) = 1 - \frac{2\beta_d}{\beta} e^{-\beta} \sum_{n=0}^{\infty} \frac{I_n(\beta)}{(\omega/n\Omega_e)^2 - 1} \tag{15}$$

of the EBW is used. In this equation, I_n is the modified Bessel function, $\beta = k^2\rho_e^2$, $\beta_d = \omega_e^2/\Omega_e^2$, ω_e and Ω_e are the electron plasma- and gyrofrequency, and ρ_e is the electron gyroradius.

Our simulations take the direction of inhomogeneity to be along **x** and the magnetic field to be along **y**. The magnetic field has constant strength, $B_0 = 0.02T$. The plasma extends from 0 to $l_x = 0.54$ m, and its density $n(x) = n_0 + 0.5(n_1 - n_0)[1 + 2/\pi \arctan(x - x_{jp})/L_n]$, where $n_0 = 1.47 \cdot 10^{16}$ m^{-3}, $n_1 = 0.1 n_0$, $x_{jp} = 0.214$ m, and $L_n = 0.027$ m. The minimum density n_1 is taken to be twice the density at the upper hybrid resonance layer, so EBWs can propagate in the whole plasma. The driving current source has the form $\mathbf{J}_x(x,t) = J(x)\sin(k_0 x - \omega t)\vec{e}_x$, and the asymmetric distribution $J(x) = J_0/\{1 + \sigma(x - x_s)^2 + \exp[\xi(x_s - x)]\}$ with the peak location $x_s = 0.28$ m and width $\sqrt{\sigma} = 0.02 l_x$. The driving frequency $\omega = 1.7\Omega_e$, and $J_0 = 300$A. The EBW at this frequency is a backward wave. To increase the wave propagation distance for a given l_x, we set $\xi = 80$ so that the current distribution on the left-hand side of the point x_s decays rapidly. Finally, to compute the result from Eq. (14), we actually need not perform the implied local average on the right side of this equation, as nonresonant terms end up integrating to near zero.

We first compare a case with exact resonance, i.e., the wave number of the driving current at its peak equals that of the EBW, i.e., $k_0 = k_{\mathrm{EBW}}(x_s)$, for which the EBW dispersion relation gives $k_{\mathrm{EBW}}(x_s) \approx 0.61/\rho_e$. The comparison of the simulation result with the theoretical predictions is shown in Fig. 1. The plasma density drops rapidly in the vicinity of $x = 0.25$ m. To confirm that the driven wave is indeed an EBW, the $k\rho_e$-x curve obtained from the simulation is compared with the linear EBW dispersion relation. At $t \sim 574/\omega$, we find that the excited EBW reaches the steady state in the region $x \geq 0.15$ m. The amplitudes found from Eq. (14) and also from neglecting the factors of $\partial D_0/\partial k_x$ in that equation are shown. The final panel of this figure shows excellent agreement between the full kinetic simulation and the reduced equation (14), while neglecting the factors of $\partial D_0/\partial k_x$ leads to significant error.

In an inhomogeneous plasma, a nonresonant current [i.e., $k_0 \neq k_{\mathrm{EBW}}(x_s)$] can also drive the wave, as inhomogeneity imparts a finite width in k space to the perturbation. The results in this case can also be captured by our reduced equation. Figure 2 shows the results of a nonresonant current driving the mode. In Figs. 2(a) and 2(c) the parameters where the same as those of Fig. 1, except for the change, $k_0 = 0.3/\rho_e$, which is about half of k_{EBW} at $x_s = 0.28$ m, where the peak of the current is located. In this case, the EBW is

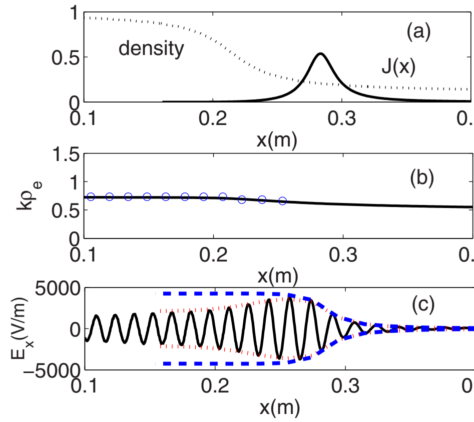

FIG. 1. (Color online) Comparison of the simulation and theoretical prediction for a resonant driving with $J_0 = 200$ A. (a) Normalized density (dotted line) and driving current. (b) $k\rho_e$ versus **x** from linear EBW dispersion relation (solid line) and from the simulation (circles). (c) Spatial distribution of E_x at $t = 574/\omega$: the solid line is from the δf simulation, the red dotted line is from Eq. (14), and the blue dashed line is what would be obtained by neglecting the factors of $\partial D_0/\partial k_x$ in Eq. (14).

still excited, and the amplitude is about one-third of that in the resonant driving case.

To show the difference of the two solutions in the source region, we take $x_s = 0.32$ m, $k_0 = 0.6/\rho_e$, and $\sqrt{\sigma} = 0.032 l_x$, while keeping the other parameters the same as in the resonant driving case. In this case, $k_{\mathrm{EBW}}(x_s) \approx 0.72/\rho_e$, so again the excitation wave number is unmatched. The comparison is shown in Figs. 2(b) and 2(d). In the vicinity of x_s, the apparent discrepancy is caused by the imprecision in amplitude due to the phase variations. However, outside the narrow region near x_s, good agreement is found between the inhomogeneous solution for Eq. (14) and the simulation result, while neglecting the factors of $\partial D_0/\partial k_x$ again leads to significant error.

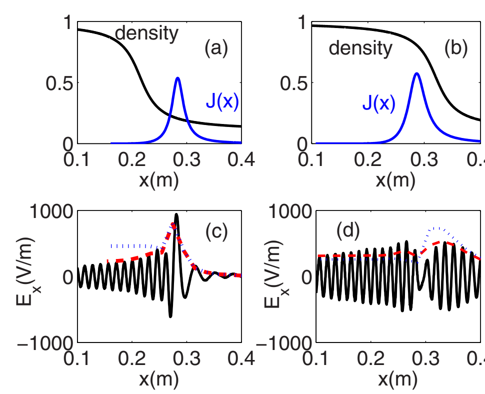

FIG. 2. (Color online) Normalized density and driving current, $k_0 =$ (a) $0.3/\rho_e$ and (c) $0.72/\rho_e$, for two different cases of nonresonant current excitation. (b) and (d) show the corresponding results for the electric field response, with the solid line the results from the δf simulation at $t = 574/\omega$, the red dashed line showing the results from Eq. (14), and the blue dotted line showing the results obtained upon neglecting the factors of $\partial D_0/\partial k_x$.

In summary, we have shown how to obtain the amplitude equation for a wave driven by an external (noise, nonlinear, antenna) current from a simple argument based on conservation of energy. We further reduced this result for the long-time, constant-amplitude limit to a quadrature. Comparison of the resulting formulas with fully kinetic simulations of electron Bernstein modes in inhomogeneous plasmas show that these reduced formulas accurately model the wave amplitude, even in the case where the driving (external) current is nonresonant. Future directions include using these results to compute the dynamics of nonlinearly coupled waves in inhomogeneous media.

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