PHYSICAL REVIEW E **76**, 055201(R) (2007)

Uncovering interaction of coupled oscillators from data

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(Received 21 May 2007; revised manuscript received 10 October 2007; published 9 November 2007)

We develop a technique for reconstructing the phase dynamics equations for weakly coupled oscillators from data. We show how, starting from general scalar observables, one can first reconstruct the dynamics in terms of the corresponding protophases, and then, performing a transformation to the genuine, observable-independent phases, obtain an invariant description of the phase dynamics. We demonstrate that natural frequencies of oscillators can be recovered if several observations of coupled systems at different coupling strengths are available. We apply our theory to numerical examples and to a physical experiment with coupled metronomes.

DOI: 10.1103/PhysRevE.76.055201 PACS number(s): 05.45.Xt, 02.50.Sk, 05.45.Tp

Inferring the laws of interaction between different objects from their observed dynamics is one of the oldest tasks in physics, going back to the deduction of gravitational interaction from Kepler's laws. While for fundamental interactions a whole variety of specially designed experiments is at one's disposal, for particular nonuniversal coupling the determination of the underlying interaction laws relies heavily on the analysis of observations. A number of such techniques have been suggested within the framework of nonlinear analysis of bi- or multivariate data [1]. Here we address the problem of inferring from observations the laws of weak interaction of oscillatory systems, a problem relevant to the analysis of coupled lasers, electronic circuits, chemical reactions, cardiorespiratory interaction, neural oscillators, functional brain connectivity, etc. [2–4]. We propose an approach for reconstructing the phase dynamics equations, which are *invariant* with respect to the chosen observables.

Our analysis is based on the theory of coupled oscillators [5,6], briefly outlined below. An autonomous periodic oscillator is described by a vector of state variables $\mathbf{x}$ and a differential equation $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x})$. Its limit cycle solution can be characterized by a phase ϕ that grows linearly in time, $\dot{\phi}=\omega_0$, where ω_0 is the oscillation frequency. Two oscillators are described by two phases $\phi_{1,2}$. If these oscillators interact (e.g., in the case of lasers due to overlap of their radiation fields), then the basic model is that of coupled ordinary differential equations. For weak coupling, one can neglect its effect on the amplitudes, because they are robust and, in a widely valid approximation, reduce the description to the phase dynamics [5], which leads to

$$\dot{\phi}_1 = \omega_1 + q^{(1)}(\phi_1,\phi_2), \quad \dot{\phi}_2 = \omega_2 + q^{(2)}(\phi_1,\phi_2), \qquad (1)$$

where $q^{(1,2)}$ are 2π-periodic coupling functions. Due to the coupling, the time averages of phase derivatives, called observed frequencies, $\Omega_{1,2}=\langle\dot{\phi}_{1,2}(t)\rangle$, generally differ from the natural frequencies $\omega_{1,2}$. Solutions of this system can be quasiperiodic with incommensurate $\Omega_{1,2}$ or periodic. The latter case is designated as synchronization. The effect of the noise can be described by incorporating random terms into Eqs. (1); in the simplest case they become Langevin-type equations. Such equations also describe the dynamics of weakly coupled chaotic oscillators [6]. For the rest of the paper we restrict our consideration to the case when both coupling and noise or chaos are weak, so that the phase description (1) holds.

In an experiment one typically measures *scalar observables* $y_{1,2}=g_{1,2}(\mathbf{x}_{1,2})$ and analyzes the corresponding scalar time series $Y_{1,2}(t)$, exhibiting more or less regular oscillations (we denote time series of observables by corresponding capital letters). $Y_{1,2}(t)$ can be used to extract "phases" via construction of a two-dimensional embedding, e.g., by using the Hilbert transform $\hat{Y}_{1,2}$ of $Y_{1,2}$ and computing $\Theta_{1,2}=\arctan(\hat{Y}_{1,2}/Y_{1,2})$ [6,8], We call $\theta_{1,2}$ the *protophases* to emphasize that they differ from $\phi_{1,2}$. Protophases heavily depend on the scalar observables and on the embedding. Already for an autonomous oscillator, θ generally does not grow linearly in time but obeys $\dot{\theta}=f(\theta)$. In terms of protophases, the equations of coupled systems are

$$\dot{\theta}_1 = f^{(1)}(\theta_1,\theta_2), \quad \dot{\theta}_2 = f^{(2)}(\theta_1,\theta_2). \qquad (2)$$

Note that, contrary to Eqs. (1), Eqs. (2) do not provide an invariant description of the coupled dynamics since they are unalike for different protophases. The problems we solve in this Rapid Communication are (i) reconstruction of the *genuine phases* ϕ from the protophases θ, and (ii) reconstruction (with a certain accuracy, to be discussed below) of the invariant coupled Eqs. (1) from the observed $\Theta_{1,2}$. Finally, we show that, (iii) if observations for different coupling strengths are available, then natural frequencies $\omega_{1,2}$ may be revealed.

Consider first an autonomous oscillator. The transformation $\theta\to\phi$ can be found from the relation $d\phi/dt=(d\phi/d\theta)d\theta/dt=\omega_0$. In the presence of fluctuations, $dt/d\theta$ should be averaged over θ:

$$\frac{d\phi}{d\theta} = \omega_0\left\langle \frac{dt}{d\theta}(\theta)\right\rangle = 2\pi\sigma(\theta). \qquad (3)$$

Note that $\sigma(\theta)$ is nothing else but the probability distribution density of the protophase. We represent it as a Fourier series $\sigma(\theta)=\Sigma_n S_n e^{in\theta}$ with coefficients $S_n=(2\pi)^{-1}\int_0^{2\pi}\sigma(\theta)e^{-in\theta}d\theta$. Because $\sigma(\theta)$ can be computed as the time average along the observed trajectory $\Theta(t)$, $0\le t\le T$, we write

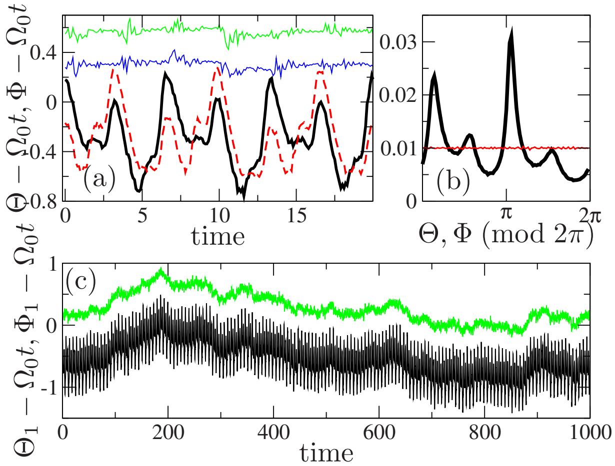

FIG. 1. (Color online) (a) Two protophases $\Theta_{1,2}$ for Eq. (6) (bold and dashed lines) and corresponding phases $\Phi_{1,2}$ (lower and upper solid lines), obtained by transformation (5). Ω_0 is the observed frequency. Regular components of oscillation around linear growth, present in $\Theta_{1,2}$, are removed in $\Phi_{1,2}$. (b) A nonuniform distribution of $\Theta_1 \bmod 2\pi$ (bold line) and a uniform distribution of $\Phi_1 \bmod 2\pi$ (solid line) (the results for Θ_2 and Φ_2 are similar and therefore not shown). (c) The protophase and the phase, plotted on a large time scale, show that the transformation (5) only removes oscillations 2π periodic in θ; other dynamical features, e.g., the phase diffusion, are preserved. In (a), (c) the Φ curves are shifted up for better visibility.

$$\sigma(\theta) = \langle \delta(\Theta(t) - \theta) \rangle = \frac{1}{T}\int_0^T \delta(\Theta(t) - \theta)dt. \qquad (4)$$

Substitution of Eq. (4) in the expression for S_n yields $S_n = (2\pi T)^{-1}\int_0^T e^{-in\Theta(t)}dt$. If the time series consists of N equidistant observations, the latter expression simplifies to $S_n = (1/2\pi N)\Sigma_k e^{-in\Theta_k}$. The final result is the transformation from the protophase θ to the phase:

$$\phi = 2\pi\int_0^\theta \sigma(\theta')d\theta' = \theta + 2\pi\sum_{n\neq 0}\frac{S_n}{in}(e^{in\theta} - 1). \qquad (5)$$

For illustration, we simulate a van der Pol oscillator forced by Gaussian noise ξ, $\langle \xi(t)\xi(t')\rangle = \delta(t-t')$,

$$\ddot{x} + (1 - x^2)\dot{x} + x = 0.05\xi(t). \qquad (6)$$

To demonstrate the effect of different observables, we take an embedding $(x,\dot{x})$ and shift the origin. Our protophases are $\theta_1 = \arctan[(\dot{x} - 0.4 + \eta_1)/(x - 0.2 + \eta_2)]$ and $\theta_2 = \arctan[(\dot{x} + 0.4 + \eta_3)/(x - 0.4 + \eta_4)]$, where η_i are Gaussian random variables with standard deviation 0.05; they imitate observational noise. Applying transformation (5) to time series $\Theta_{1,2}$ we obtain two estimates $\Phi_{1,2}$ of the genuine phase; the results are shown in Fig. 1. They clearly demonstrate that the transformation (5) provides a variable that grows linearly on average, $\Phi_{1,2} \sim \Omega_{1,2}t$, and, taken mod 2π, is distributed uniformly. We emphasize that, while removing the oscillatory observable-dependent components, we preserve components caused by dynamical and observational noises (and, in general, by interactions).

Now we extend our approach to the case of bivariate data $\Theta_1(t), \Theta_2(t)$, $0 \le t \le T$, generated by a coupled system (1), with the aim of finding genuine phases $\phi_{1,2}$ and reconstructing Eqs. (1). As the first step, we reconstruct Eqs. (2). Writing the first of Eqs. (2) as a Fourier series

$$\dot{\theta}_1 = f^{(1)}(\theta_1, \theta_2) = \sum F^{(1)}_{n,m} e^{in\theta_1 + im\theta_2}, \qquad (7)$$

we find the Fourier coefficients $F^{(1)}_{n,m} = (2\pi)^{-2}\int_0^{2\pi}dx\int_0^{2\pi}dy f^{(1)} \times (x,y)e^{-inx-imy}$ by minimizing the approximation error $(\dot{\Theta}_1 - \Sigma_{n,m}F^{(1)}_{nm}e^{in\theta_1+im\theta_2})^2$, averaged with respect to the probability density $\rho(\theta_1,\theta_2)$. This leads to a linear system

$$\int_0^{2\pi}\int_0^{2\pi} d\theta_1 d\theta_2 \dot{\Theta}_1 \rho(\theta_1,\theta_2)e^{in\theta_1+im\theta_2}$$

$$= \sum_{k,l} F^{(1)}_{kl}\int_0^{2\pi}\int_0^{2\pi} d\theta_1 d\theta_2 \rho(\theta_1,\theta_2)e^{i(n+k)\theta_1 + i(m+l)\theta_2}.$$

Writing the probability density $\rho(\theta_1,\theta_2) = \langle \delta(\Theta_1 - \theta_1)\delta(\Theta_2 - \theta_2)\rangle$ on the left side as an average along the trajectory [cf. Eq. (4)], and on the right side as a Fourier series $\rho(\theta_1,\theta_2) = \Sigma_{n,m}A_{n,m}e^{in\theta_1+im\theta_2}$, we obtain $B^{(1)}_{-n,-m} = \Sigma_{k,l}A_{-n-k,-m-l}F^{(1)}_{k,l}$ where $A_{s,r} = T^{-1}\int_0^T dt\, e^{-is\Theta_1(t) - ir\Theta_2(t)}$ and $B^{(1)}_{n,m} = T^{-1}\int_0^{\Theta_1(T)}d\Theta_1 e^{-in\Theta_1 - im\Theta_2}$. Solving these equation for $F^{(1)}_{n,m}$, we obtain the coupling function $f^{(1)}$ in Eq. (2). In the same way, we obtain $F^{(2)}_{n,m}$ and $f^{(2)}$. Since the equations for the Fourier coefficients $F^{(1,2)}_{n,m}$ are convolutions, the formulas for the corresponding functions can be simply written as ratios $f^{(1,2)} = b^{(1,2)}/\rho$, where we introduced $b^{(1,2)}(\theta_1,\theta_2) = \Sigma_{n,m}B^{(1,2)}_{n,m}e^{in\theta_1+im\theta_2}$. This equation shows that our method works if $\rho > 0$, i.e., if the systems are far from synchrony and the observed trajectory covers the whole torus $(0,2\pi) \times (0,2\pi)$ [9].

The next step is to find the transformations $\theta_{1,2} \to \phi_{1,2}$ and to reconstruct the functions $q^{(1,2)}$ in Eqs. (1). We represent these functions as

$$q^{(1,2)}(\phi_1,\phi_2) = \tilde{\omega}_{1,2} + \bar{q}^{(1,2)}(\phi_{1,2}) + \tilde{q}^{(1,2)}(\phi_1,\phi_2). \qquad (8)$$

The phase-independent terms $\tilde{\omega}_{1,2}$ appear if the coupling directly shifts the oscillator frequency; hereafter we denote $\bar{\omega}_{1,2} = \omega_{1,2} + \tilde{\omega}_{1,2}$. Furthermore, we have separated the dependence on the "own phase" $\bar{q}(\phi_{1,2})$ and on the "driving phase" $\tilde{q}(\phi_1,\phi_2)$ by imposing the conditions

$$\int_0^{2\pi}\tilde{q}^{(1)}(\phi_1,\phi_2)d\phi_2 = \int_0^{2\pi}\tilde{q}^{(2)}(\phi_1,\phi_2)d\phi_1 = 0. \qquad (9)$$

Writing the transformations $\theta_{1,2} \to \phi_{1,2}$ as [cf. Eq. (3)]

$$\frac{d\phi_{1,2}}{d\theta_{1,2}} = 2\pi\sigma^{(1,2)}(\theta_{1,2}), \qquad \int_0^{2\pi}\sigma^{(1,2)}(x)dx = 1, \qquad (10)$$

and combining them with Eq. (2), we obtain

$$\dot{\phi}_{1,2} = 2\pi\sigma^{(1,2)}(\theta_{1,2})f^{(1,2)}(\theta_1,\theta_2). \qquad (11)$$

Comparison with Eqs. (1) and taking account of Eqs. (8) yields for the first oscillator

$$\bar{\omega}_1 + \bar{q}^{(1)}(\phi_1) + \tilde{q}^{(1)}(\phi_1,\phi_2) = 2\pi\sigma^{(1)}(\theta_1)f^{(1)}(\theta_1,\theta_2).$$

Multiplying with $d\phi_2 = 2\pi\sigma^{(2)}d\theta_2$, integrating over 2π, and taking account of Eqs. (9), we obtain

$$\bar{\omega}_1 + \bar{q}^{(1)}(\phi_1) = 2\pi\sigma^{(1)}(\theta_1)\int_0^{2\pi} d\theta_2\sigma^{(2)}(\theta_2)f^{(1)}(\theta_1,\theta_2). \tag{12}$$

For weak coupling $|\bar{q}^{(1)}| \ll \omega_1$ holds; hence we can neglect the last term on the left-hand side. Treating the second oscillator in the same way, we obtain finally

$$\bar{\omega}_1 = 2\pi\sigma^{(1)}(\theta_1)\int_0^{2\pi} d\theta_2\sigma^{(2)}(\theta_2)f^{(1)}(\theta_1,\theta_2),$$

$$\bar{\omega}_2 = 2\pi\sigma^{(2)}(\theta_2)\int_0^{2\pi} d\theta_1\sigma^{(1)}(\theta_1)f^{(2)}(\theta_1,\theta_2). \tag{13}$$

Together with normalization conditions (10), Eqs. (13) represent a closed system of equations for unknown $\sigma^{(1,2)}$ and $\bar{\omega}_{1,2}$, which can be easily solved by iterations, starting with $\sigma_1^{(1,2)}(x) = (2\pi)^{-1}$. Numerics shows that these iterations converge rather fast. As a result, we obtain the desired transformation in a form similar to Eq. (5) and recover the genuine phases $\phi_{1,2}$ from the protophases $\theta_{1,2}$. Computation of $\dot{\phi}_{1,2}$ [see Eq. (11)] completes the task of reconstruction of the coupling functions: the dependence of $\dot{\phi}_{1,2}$ on ϕ_1,ϕ_2 can be plotted or presented via the Fourier coefficients [10]. In finding the genuine phase we make a small error by neglecting in (12) the term $\bar{q}$ describing the coupling-induced dependence on the own phase. This is unavoidable, because the dependencies on the own phase due to coupling and due to the observable-dependent protophase are indistinguishable. Nevertheless, by neglecting $\bar{q}$ in (12), we still keep the baby while throwing out the bath water.

Now we demonstrate that the recovered functions are observable independent. Suppose we start the reconstruction with a different pair of protophases $\psi_{1,2}$, related to the old ones $\theta_{1,2}$ via $d\theta_{1,2}/d\psi_{1,2} = \gamma^{(1,2)}$. Then $d\phi_{1,2}/d\psi_{1,2} = (d\phi_{1,2}/d\theta_{1,2})d\theta_{1,2}/d\psi_{1,2} = 2\pi\beta^{(1,2)}$ with $\beta^{(1,2)} = \sigma^{(1,2)}\gamma^{(1,2)}$. Next, $\dot{\psi}_{1,2} = \dot{\theta}_{1,2}d\psi_{1,2}/d\theta_{1,2} = f^{(1,2)}/\gamma^{(1,2)} = g^{(1,2)}$. Similarly to Eqs. (13), we write for the transformations $\psi_{1,2} \to \phi_{1,2}$

$$\bar{\omega}_1 = 2\pi\beta^{(1)}(\psi_1)\int_0^{2\pi} d\psi_2\beta^{(2)}(\psi_2)g^{(1)}(\psi_1,\psi_2),$$

$$\bar{\omega}_2 = 2\pi\beta^{(2)}(\psi_2)\int_0^{2\pi} d\psi_1\beta^{(1)}(\psi_1)g^{(2)}(\psi_1,\psi_2). \tag{14}$$

Substituting here $\beta^{(1,2)}$, $g^{(1,2)}$, and $d\psi_{1,2}$, we prove that Eqs. (14) are equivalent to Eqs. (13).

Together with the coupling functions we obtain the frequencies $\bar{\omega}_{1,2}$. If the coupling does not possess constant terms $\tilde{\omega}_{1,2}$, then $\bar{\omega}_{1,2}$ provide the natural frequencies of the oscillators $\omega_{1,2}$. Generally, $\tilde{\omega}_{1,2} \neq 0$ and we cannot find the natural frequencies just from one observation. However, we can reveal these important parameters of oscillatory systems if (i) at least two observations made for different, though unknown, values of coupling are available, and (ii) we assume that the form of the function for different coupling is the same; only its magnitude varies proportionally. In other words, we assume that the constant frequency shifts $\tilde{\omega}_{1,2}$ are proportional to the norms $\mathcal{N}$ of the oscillatory components of coupling functions, $\tilde{\omega}_{1,2} \sim \mathcal{N}^{(1,2)} = \|\tilde{q}^{(1,2)}\|_2$. This allows us to calculate the natural frequencies as

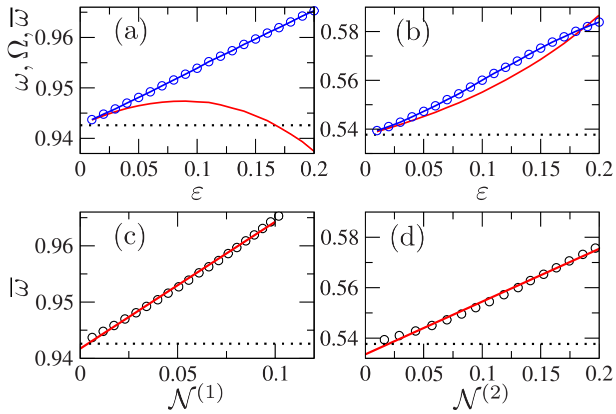

FIG. 2. (Color online) Natural frequencies $\omega_{1,2}$ (dotted lines), observed frequencies $\Omega_{1,2}$ (solid lines), and constant terms of reconstructed phase equations $\bar{\omega}_{1,2}$ (circles) for first (a) and second (b) system (15). Note that $\bar{\omega}_{1,2}(\varepsilon) \sim \varepsilon$. (c), (d) show corresponding dependencies of $\bar{\omega}$ on the norm $\mathcal{N}$ of the recovered coupling function $\tilde{q}$ (solid lines show linear fit). Note that in (c), (d) only reconstructed quantities are shown; extrapolation to $\mathcal{N}^{(1,2)} = 0$ allows one to determine the natural frequencies with a good accuracy.

$$\omega_{1,2} = (\bar{\omega}_{1,2a}\mathcal{N}_b^{(1,2)} - \bar{\omega}_{1,2b}\mathcal{N}_a^{(1,2)})/(\mathcal{N}_b^{(1,2)} - \mathcal{N}_a^{(1,2)}),$$

where a and b correspond to two available observations.

We first test the described method with a system of two coupled noisy van der Pol oscillators:

$$\ddot{x}_{1,2} + (1 - x_{1,2}^2)\dot{x}_{1,2} + \nu_{1,2}^2 x_{1,2} = \varepsilon(\dot{x}_{2,1} - \dot{x}_{1,2}) + 0.05\xi_{1,2}(t). \tag{15}$$

Here $\nu_1 = 1$, $\nu_2 = (\sqrt{5}-1)/2$, and noise is Gaussian with correlations $\langle\xi_1(t)\xi_2(t')\rangle = \delta_{1,2}\delta(t-t')$. To show, that the method indeed provides invariant coupling functions, we reconstruct Eqs. (1) for coupling $\varepsilon = 0.15$, using different protophases $\theta_{1,2} = \arctan[(\dot{x}_{1,2} - u)/(x_{1,2} - v)]$, where u and v are uniformly distributed in $(-0.4, 04)$. We compute the correlation coefficient r for functions, obtained for different u, v, and find that in 75% of trials $r > 0.97$. Finally, we show that we can recover the natural frequencies. Taking $\theta_{1,2} = \arctan[(\dot{x}_{1,2} - 0.2)/(x_{1,2} - 0.2)]$ we reconstruct Eqs. (1) for different values of ε; the results are presented in Fig. 2.

Next, we verify our theory in a version of the classical Huygens pendulum-clock experiment, using two mechanical metronomes, placed on a rigid base (cf. [7]). For two different frequency settings, we record the motion of the pendula for three conditions: (i) metronomes are uncoupled and (ii), (iii) metronomes are coupled by one or two rubber bands, respectively [Fig. 3(a)]. The pendula were filmed by a

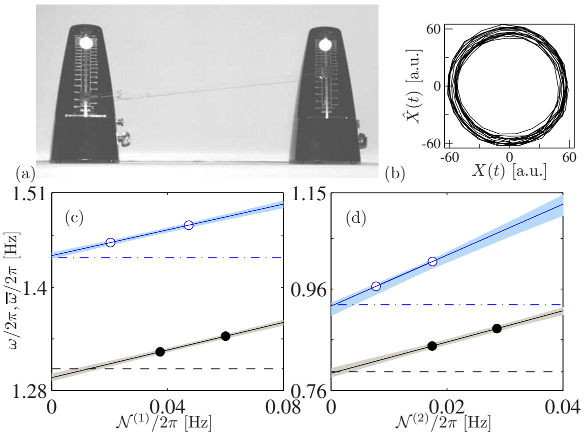

FIG. 3. (Color online) (a) Experimental setup: two metronomes, coupled via one or two rubber bands. (b) Example of an observable $X(t)$ and its Hilbert transform $\hat{X}(t)$. (c), (d) Experimental results for two metronomes. Constant terms of the coupling function $\bar{\omega}_{1,2}$ are plotted vs norm of its oscillatory component $\mathcal{N}^{(1,2)}$ for two settings (filled and empty circles) of autonomous frequencies. The recovered natural frequencies are 1.295±0.006 and 0.795±0.01 Hz (first setting) and 1.437±0.004 and 0.92±0.02 Hz (second setting). They are in good agreement with the frequencies of uncoupled systems 1.305 and 0.797 Hz (dashed lines) and 1.434 and 0.929 Hz (dot-dashed lines). Error bounds for linear regression, estimated from analysis of disjoint segments of the full records, are shown as gray stripes.

standard digital video camera and the x coordinates (horizontal deviation) were extracted offline [11] and used for the processing. Protophases were obtained by means of the Hilbert transform [Fig. 3(b)]. The main result is shown in Figs. 3(c) and 3(d): the natural frequencies are recovered with good precision from observations of the coupled system.

In summary, we have developed a technique to recover the genuine, observable-invariant phases from observable-dependent protophases, both for autonomous and for coupled systems. We emphasize that our technique is not a smoothing filter, but a deterministic transformation that removes only the protophase-periodic components (i.e., the dependence of $\dot{\theta}$ on θ) but preserves all other dynamical features. In this method, the obtained phases, as well as the equations of phase dynamics, are invariant with respect to a large class of observables used for the computation of protophases. The method works if coupled system are far from synchrony [12] and provides the coupling functions up to a term periodic in the own phase; it can be generalized for the case of several interacting systems. Our approach opens a perspective for an observable-invariant characterization of interaction between the systems, allowing one to measure the interaction strength by observable-invariant quantities $c_{1,2} = \mathcal{N}_{1,2}/\bar{\omega}_{1,2}$ and to quantify the directionality of interaction by $(c_1 - c_2)/(c_1 + c_2)$ (cf. [2] and application to brain data in [4]). Moreover, the results obtained in previous studies [3,4] can be straightforwardly recomputed to become observable independent by incorporating our approach. Additionally, our results are beneficial for the quantification of the phase diffusion, because all protophase-periodic components are eradicated by the transformation [see Fig. 1(c)].

We thank J. Schaefer (IIftC Kiel), E. Izhikevich, and H. Kantz for useful discussions and EU (Project BRACCIA), DFG (SFB 555), and Stiftung Ursula Merz, Berlin for financial support.

[1] H. Kantz and T. Schreiber, *Nonlinear Time Series Analysis* (Cambridge University Press, Cambridge, U.K., 2004).
[2] M. G. Rosenblum and A. S. Pikovsky, Phys. Rev. E **64**, 045202(R) (2001); L. Cimponeriu, M. Rosenblum, and A. S. Pikovsky, *ibid.* **70**, 046213 (2004).
[3] A. M. Yacomotti, G. Mindlin, M. Giudici, S. Balle, S. Barland, and J. Tredicce, Phys. Rev. E **66**, 036227 (2002); M. D. Prokhorov and V. I. Ponomarenko, *ibid.* **72**, 016210 (2005); I. Z. Kiss, Y. Zhai, and J. L. Hudson, Phys. Rev. Lett. **94**, 248301 (2005); J. Miyazaki and S. Kinoshita, *ibid.* **96**, 194101 (2006); R. Mrowka *et al.*, Am. J. Physiol. Regulatory Integrative Comp. Physiol. **145**, R1395 (2003); R. F. Galán, G. B. Ermentrout, and N. N. Urban, Phys. Rev. Lett. **94**, 158101 (2005).
[4] J. Gross *et al.*, Proc. Natl. Acad. Sci. U.S.A. **99**, 2299 (2002).
[5] Y. Kuramoto, *Chemical Oscillations, Waves and Turbulence* (Springer, Berlin, 1984).
[6] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization. A Universal Concept in Nonlinear Sciences* (Cambridge University Press, Cambridge, U.K., 2001).
[7] J. Pantaleone, Am. J. Phys. **70**, 992 (2002).
[8] Observables should be "good" so that $\theta_{1,2}$ grow monotonically and $\langle\dot{\theta}_{1,2}\rangle = \langle\dot{\phi}_{1,2}\rangle$; more complex cases will be treated elsewhere.
[9] Note that the coefficients $F_{n,m}^{(1,2)}$ [see Eq. (7)], can be also determined by means of multiple regression. Typically, the methods are equivalent; comparison of their robustness with respect to noise, data length, etc., will be presented elsewhere.
[10] It is computationally efficient to apply the univariate transformations (5) to each protophase prior to the bivariate transformation (10).
[11] Each record provided about 4000 frames at a frame rate 25 Hz. The coordinates of pendula were determined via comparison of each video frame with the reference frame of pendula at rest [cf. T. Greczylo and E. Debowska, Eur. J. Phys. **23**, 441 (2002)].
[12] Note that protophases suffice for characterization of synchrony, where only the ratio of the observed frequencies is important.