Gaussian point processes and two-by-two random matrix theory

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The statistics of the multidimensional Gaussian point process are discussed in connection with the spacing statistics of eigenvalues of 2×2 random matrices. We consider the three-dimensional Gaussian point process when two of the coordinates of a point are randomly chosen from a Gaussian distribution having a mean of zero and a variance of $\sigma^2=1$ but the third coordinate is chosen from a Gaussian distribution having a variance in the range of $0\leqslant\sigma_3^2\leqslant 1$. The probability density function associated with a random point being at a distance r from the origin is shown to be closely related to the nearest-neighbor spacing distribution of eigenvalues coming from an ensemble of 2×2 matrices defined by the French-Kota–Pandey-Mehta two-matrix model of random matrix theory. An elementary explanation of this result is given.

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The spacing statistics of the homogeneous Poisson point process in $\mathbb{R}^2$ (denoted as $\mathbf{P}_2$) have recently been shown to be connected to the spacing statistics of eigenvalues of 2×2 random matrices [1,2]. More specifically, among other interesting relations, the nearest-neighbor spacing distribution (NNSD) of $\mathbf{P}_2$, when appropriately normalized, is known to be formally equal to the NNSD of eigenvalues from the Gaussian ensemble of real symmetric 2×2 random matrices [also known as the Wigner distribution of random matrix theory (RMT)]. Furthermore, the second-NNSD of $\mathbf{P}_2$ is known to be formally equal to the NNSD of complex eigenvalues from Ginibre's ensemble of 2×2 complex non-Hermitian random matrices [3].

The above relations immediately prompt the question of whether or not $\mathbf{P}_2$ is unique in this respect. That is, are there any other point processes whose statistics (not necessarily spacing statistics) can somehow be described by the spacing statistics of eigenvalues of random matrices? In the present study we consider another frequently encountered point process, the zero-centered d-dimensional Gaussian point processes (denoted presently as $\mathbf{G}_d$), and show that indeed a connection to RMT does again exist if one examines the probability of a random point being at a distance r from the origin. The probability density function (PDF) associated with this [denoted as $F(r;\sigma,d)$] is of course well known when all dimensions have the same variance σ^2 [4–8]. Incidentally, when $F(r;\sigma,d)$ is normalized such that its mean is unity it is formally equal to the classical Wigner surmises ($\beta=1$, 2, and 4) of RMT, for $d=2$, 3, and 5.

We next study $\mathbf{G}_3$ when two of the coordinates of a point are randomly chosen from a Gaussian distribution having a mean of zero and a variance of $\sigma^2=1$ but the third coordinate is chosen from a Gaussian distribution having a variance in the range of $0\leqslant\sigma_3^2\leqslant 1$. It is shown that the PDF $F(r;\sigma_3,3)$ is closely related to the NNSD of eigenvalues coming from an ensemble of 2×2 matrices defined by the French-Kota–Pandey-Mehta two-matrix model of RMT [9–12]. A continuous transition from the Wigner surmise of the Gaussian orthogonal ensemble (Wigner-GOE) to the Wigner surmise of the Gaussian unitary ensemble (Wigner-GUE) is thus seen to occur for $F(r;\sigma_3,3)$ as σ_3^2 is varied from 0 to 1.

We begin with the well-known multidimensional Gaussian distribution with equal $\sigma_d^2=\sigma^2$ in each dimension:

$$f(x_1,x_2,\ldots,x_d;\sigma,d)=\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^d\times\exp\left[-\frac{1}{2\sigma^2}(x_1^2+x_2^2+\cdots+x_d^2)\right]. \tag{1}$$

$\mathbf{G}_d$ then involves randomly sampling the d-dimensional space using the PDF $f(x_1,x_2,\ldots,x_d;\sigma,d)$, and the probability of finding a point at a distance $r=\sqrt{x_1^2+x_2^2+\cdots+x_d^2}$ from the origin, in a shell of thickness dr, is of course

$$F(r;\sigma,d)dr=\frac{2r^{d-1}}{(\sqrt{2}\sigma)^d\Gamma\left(\frac{d}{2}\right)}\exp\left(-\frac{1}{2\sigma^2}r^2\right)dr, \tag{2}$$

where the mean distance

$$\bar{r}=\int_0^\infty rF(r;\sigma,d)dr=\sqrt{2}\sigma\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}. \tag{3}$$

Upon introduction of the new variable $R=r/\bar{r}$ we obtain the rescaled (normalized) PDF:

$$F(R;d)=\mathcal{A}(d)R^{d-1}\exp(-\mathcal{B}(d)R^2), \tag{4a}$$

where

$$\mathcal{A}(d)=2\frac{\left[\Gamma\left(\frac{d+1}{2}\right)\right]^d}{\left[\Gamma\left(\frac{d}{2}\right)\right]^{d+1}}\quad\text{and } \mathcal{B}(d)=\frac{\left[\Gamma\left(\frac{d+1}{2}\right)\right]^2}{\left[\Gamma\left(\frac{d}{2}\right)\right]^2}. \tag{4b}$$

Let us now present the Wigner surmises for the NNSDs of eigenvalues from the Gaussian orthogonal ($\beta=1$), unitary ($\beta=2$), and symplectic ($\beta=4$) ensembles of RMT (see Ref. [13]):

$$P_W(S;\beta) = \mathcal{A}(\beta) S^\beta \exp(-\mathcal{B}(\beta) S^2), \tag{5a}$$

where

$$\mathcal{A}(\beta) = 2\frac{\left[\Gamma\left(\frac{\beta+2}{2}\right)\right]^{\beta+1}}{\left[\Gamma\left(\frac{\beta+1}{2}\right)\right]^{\beta+2}} \quad \text{and } \mathcal{B}(\beta) = \frac{\left[\Gamma\left(\frac{\beta+2}{2}\right)\right]^{2}}{\left[\Gamma\left(\frac{\beta+1}{2}\right)\right]^{2}}. \tag{5b}$$

Although these distributions are exact for Gaussian ensembles of 2×2 random matrices only, they serve as excellent analytical approximations for Gaussian ensembles of arbitrarily large random matrices [1,14,15]. The parameter β is known as the "level repulsion" parameter—for small values of S the behavior $P_W(S;\beta)\sim S^\beta$ results.

Comparison of Eqs. (4) and (5) reveals that, formally,

$$F(R;\beta+1) = P_W(S;\beta), \quad \beta \in \mathbb{N}. \tag{6}$$

Of particular interest are the Gaussian point processes for $d=2$, 3, and 5, whose PDFs correspond to the Wigner-GOE, -GUE, and -GSE (where -GSE refers to the Gaussian symplectic ensemble), respectively. The classical Wigner surmises for the NNSDs of RMT are usually defined only for $\beta=1$, 2, and 4, however, as noted in Ref. [2], $P_W(S;3)$ is identical to the NNSD of complex eigenvalues from Ginibre's ensemble of 2×2 general complex non-Hermitian random matrices [3]. We can therefore also conclude that the PDF $F(R;4)$ is formally equal to the Wigner surmise of the Ginibre ensemble (Wigner-Ginibre). As already stated, the above identity is only valid for $\beta\in\mathbb{N}$, given that $F(R;\beta+1)$ is meaningless for noninteger β. $P_W(S;\beta)$ is, however, valid for real $\beta>0$ (the corresponding Gaussian ensembles of random matrices are the so-called β-Hermite ensembles [16]).

For completeness, a comparison of Eq. (4) to the kth-NNSD for $\mathbf{P}_2$, which is given by [2,17]

$$D(S;k,2) = \frac{2[\mathcal{A}(k)]^k}{\Gamma(k)} S^{2k-1} \exp(-\mathcal{A}(k) S^2), \tag{7a}$$

where

$$\mathcal{A}(k) = \frac{\left[\Gamma\left(k+\frac{1}{2}\right)\right]^2}{[\Gamma(k)]^2}, \tag{7b}$$

can also be made. It is then obvious that, formally,

$$F(R;\beta+1) = D(S;(\beta+1)/2,2) = P_W(S;\beta), \tag{8}$$

where $\beta=1,3,5,\ldots$.

Consider now the following three-dimensional Gaussian distribution:

$$f(x_1,x_2,x_3;\sigma_3,3) = \frac{1}{\sqrt{(2\pi)^3}\sigma_3}\exp\left[-\frac{1}{2}\left(x_1^2+x_2^2+\frac{x_3^2}{\sigma_3^2}\right)\right]. \tag{9}$$

Note that we have set $\sigma_1^2=\sigma_2^2=1$ and that σ_3^2 will be restricted to take values in the range of 0 and 1. In terms of spherical polar coordinates we can write the probability of a random point being at a distance r from the origin, in a shell of thickness dr, as

$$F(r;\sigma_3,3)dr = \left[\frac{1}{\sqrt{(2\pi)^3}\sigma_3}\int_0^{2\pi}\int_0^{\pi}\exp\left(\frac{-r^2}{2\sigma_3^2}[\sigma_3^2\sin^2\phi\cos^2\theta+\sigma_3^2\sin^2\phi\sin^2\theta+\cos^2\phi]\right)r^2\sin\phi\, d\phi\, d\theta\right]dr. \tag{10}$$

It can then be shown that

$$F(r;\sigma_3,3) = \frac{r}{(1-\sigma_3^2)^{1/2}}\exp\left(\frac{-r^2}{2}\right)\mathrm{erf}\left[\left(\frac{1-\sigma_3^2}{2\sigma_3^2}\right)^{1/2} r\right], \tag{11a}$$

where

$$\bar{r} = \sqrt{\frac{2}{\pi}}[(1-\sigma_3^2)^{-1/2}\arctan\{(1-\sigma_3^2)^{1/2}/\sigma_3\}+\sigma_3]. \tag{11b}$$

The significance of this result will become clear momentarily, but first we must introduce a two-matrix ensemble that is used in RMT to describe a GOE-to-GUE transition.

Following Ref. [15] we will consider a time-reversal invariant Hamiltonian H_0 to which a time-reversal breaking part H_{break} is added such that

$$H = H_0 + H_{\mathrm{break}}. \tag{12a}$$

In order to study H from the perspective of RMT its components are written as a random matrix ensemble:

$$H_0 = S(v^2) \quad \text{and } H_{\mathrm{break}} = i\alpha A(v^2). \tag{12b}$$

S is a real symmetric matrix, A is a real antisymmetric matrix, and $0\le\alpha\le 1$. The statistically independent matrix elements of S are normally distributed with a mean of zero and a variance of v^2 for off-diagonal elements and a variance of $2v^2$ for diagonal elements. The independent matrix elements of A are also normally distributed with a mean of zero and a variance of v^2 for off-diagonal elements—the diagonal elements are however taken to be zero. As α is varied from 0 to 1 a transition from the GOE to the GUE occurs continuously. The properties of this ensemble (see Ref. [15] for a summary) have been worked out in detail by Pandey and Mehta [10–12] and for this reason the random matrix model of H is sometimes [18] referred to as the Pandey-Mehta two-matrix model. We prefer to call it the French-Kota–Pandey-Mehta two-matrix model given that French and Kota [9] discussed it first.

In the present study, we are interested in the Wigner-GOE-to-Wigner-GUE transition that occurs for the 2×2 form of Eq. (12), and set $v^2=1$ for convenience. It is known that the NNSD of eigenvalues from an ensemble of such matrices is [9,15,19,20]

$$P_H(s;\alpha)=\frac{s}{4(1-\alpha^2)^{1/2}}\exp\left(\frac{-s^2}{8}\right)\mathrm{erf}\left[\left(\frac{1-\alpha^2}{8\alpha^2}\right)^{1/2}s\right], \tag{13a}$$

where

$$\bar{s}=\sqrt{\frac{8}{\pi}}[(1-\alpha^2)^{-1/2}\arctan\{(1-\alpha^2)^{1/2}/\alpha\}+\alpha]. \tag{13b}$$

We can now compare Eqs. (11) and (13) to deduce that, formally, $\bar{s}=2\bar{r}$ and that α is equivalent to σ_3. [Performing the above calculations with $\sigma_1^2=\sigma_2^2=\sigma^2$, $0\leqslant\sigma_3^2\leqslant\sigma^2$, and v^2, we can deduce that in general $\bar{s}=(2v/\sigma)\bar{r}$ and that α is equivalent to σ_3/σ.]

This connection becomes obvious if we explicitly consider the structure of the 2×2 matrix of the two-matrix French-Kota–Pandey-Mehta model, and how one would go about studying its eigenvalues. H is first represented as

$$H=\begin{pmatrix} a & c \\ c & b\end{pmatrix}+i\alpha\begin{pmatrix} 0 & d \\ -d & 0\end{pmatrix}=\begin{pmatrix} a & e \\ e^* & b\end{pmatrix}, \tag{14}$$

where a, b, c, and d are all real numbers and $e=c+i\alpha d$. The eigenvalues of this matrix are

$$E_\pm=\frac{1}{2}[(a+b)\pm\sqrt{(a-b)^2+4|e|^2}], \tag{15}$$

from which we get the spacing of

$$s=E_+-E_-=\sqrt{(a-b)^2+(2c)^2+(2\alpha d)^2}. \tag{16}$$

Now, recalling that a and b are chosen from a Gaussian distribution having variance of 2 and c and d are chosen from a Gaussian distribution having a variance of 1, we can recast the above expression as

$$s=\sqrt{f^2+(2c)^2+(2\alpha d)^2}=2\sqrt{g^2+c^2+(\alpha d)^2}, \tag{17}$$

where f and g are real numbers that come from Gaussian distributions with zero means and variances of 4 and 1, respectively. [It should be pointed out that Eq. (13) is in fact derived using Eqs. (16) and (17); see Refs. [9,20].] Let us now compare this to the distance r of a random point chosen from the distribution $f(x_1,x_2,x_3;\sigma_3,3)$:

$$r=\sqrt{x_1^2+x_2^2+(\sigma_3 w)^2}. \tag{18}$$

Note that $x_3=\sigma_3 w$ and that w is a random variable chosen from a Gaussian having a variance of 1 and a mean of zero. It is clear that Eqs. (17) and (18) are statistically equivalent, with the exception of the important factor of 2 that has already been noted above [see Eqs. (11b) and (13b)]. Therefore, the eigenvalue spacing statistics as calculated from an ensemble of 2×2 matrices of the French-Kota–Pandey-Mehta model must inevitably be related to the statistics of the Gaussian point process obtained by randomly sampling the three-dimensional space using the PDF given by Eq. (9). Similar arguments to the one just given can also be used to show that $F(R;5)$ must be related to the Wigner-GSE and that the statistics of $\mathbf{P}_2$ must also be related to 2×2 RMT.

In summary, we have shown that the statistics of $\mathbf{G}_d$, like the spacing statistics of $\mathbf{P}_2$, are connected to the spacing statistics of eigenvalues of 2×2 random matrices. This answers the question that was asked at the beginning of this report. Recall now that the Wigner surmises are labeled by the "level repulsion" parameter β. The larger the value of β the more neighboring levels close to one another tend to "repel" one another. Analogous to this is the concept of "point repulsion" for the statistics of point processes. The notion of "point repulsion" was discussed in Ref. [21] where the repulsion was understood to be between nearest-neighbor (and kth nearest-neighbor) random points on a regular fractal. For $\mathbf{G}_d$ "point repulsion" from the origin is observed, and the parameter d can be interpreted as a "repulsion" parameter; this is apparent upon examining Eq. (6). The Wigner-GOE-to-Wigner-GUE transition that was examined for $\mathbf{G}_3$ in the latter half of this report can of course be extended by going to higher dimensions. For example, one can study a complete Wigner-GOE-to-Wigner-GUE-to-Wigner-Ginibre-to-Wigner-GSE transition by also solving the four- and five-dimensional versions of Eq. (10). However, we are not aware of an RMT model that describes such a transition (recall that eigenvalues of the Ginibre ensemble are complex) and so a comparison to results from RMT cannot be made at this point in time.

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