Parametric frequency conversion, nonlinear Schrödinger equation, and multicomponent cnoidal waves

V. M. Petnikova and V. V. Shuvalov
M. V. Lomonosov Moscow State University, Vorob'evy Gory, Moscow 119992, Russia
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It is shown that, in a three-coupled-mode approximation, exact analytical solution of a system of equations, that describes a steady-state parametric frequency conversion, can be obtained as a solution of three closed nonlinear Schrödinger equations, coupled only through their boundary conditions. A reason for such a possibility lies in the description of the competition of two simultaneous second-order nonlinear processes in terms of an effective cascade cubic nonlinearity. Specific features and nontrivial asymptotes of complex self-consistent periodic solutions of the nonlinear Schrödinger equation are discussed.

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I. INTRODUCTION

Despite numerous presentations devoted to self-consistent multicomponent periodic solutions of nonlinear Schrödinger equation (NSE) and some other nonlinear equations [1–7], solutions of such type—multicomponent cnoidal waves (MCWs)—are usually considered as something exotic in laser physics. While these equations take into account the lowest (cubic) terms in nonlinear polarization expansion, it is generally accepted that MCWs are important in a rather limited range of problems. These are one-dimensional (1D) problems of stable propagation of pulse trains through optical fibers [3–6,8] and of parametric generation under synchronous pumping [9], as well as two- and three-dimensional (2D and 3D) problems of nondiffractive propagation of laser beams with special transverse structure through photorefractive crystals [7,10] and media with quadratic nonlinearity [11]. At the same time, MCWs become popular in other fields of physics. The concept of MCW is widely used in nonlinear hydrodynamics [1,12] and plasma physics [2,13], in description of coupled wave packets—quasiparticles (excitons, biexcitons, superconducting pairs, etc.) formed by electronic wave functions in physics of 1D chains (conjugated polymers) [14] and 2D planes (ferromagnetics and high-temperature superconductors) [15].

Below, we show that the MCW concept plays a key role in one of the classical problems of nonlinear optics—in parametric up- and down-frequency conversion, including second harmonic generation (SHG) and parametric amplification (PA) in media with quadratic nonlinearity. Using a nonstandard (for this physical problem, see [16]) technique, we demonstrate that all such solutions are defined as self-consistent periodic solutions of three steady-state NSEs for complex amplitudes of the interacting modes. We present our paper in the following way. First (Sec. II), we show that, in a three-coupled-mode approximation, solution of a steady-state parametric frequency conversion problem can be reduced to analytical solution of three closed NSEs, coupled only through their boundary conditions. Then (Sec. III), we present several examples of the analytical solutions and show that NSE must have a complex type of self-consistent periodic solutions with oscillating phases. Their specific features and nontrivial asymptotes are discussed in Secs. IV and V. Finally (Sec. VI), we show that the above complex periodic solutions of NSE can be useful in other fields of physics, particularly in problems of stable propagation of trains of laser pulses in optical fibers upon the phase modulation by information signal or chirp.

II. PARAMETRIC FREQUENCY CONVERSION AND NONLINEAR SCHRÖDINGER EQUATION

Consider the collinear nonlinear interaction of three plane monochromatic waves (modes) with the frequencies $\omega_{1,2}=\omega$ and $\omega_3=2\omega$, the amplitudes A_{1-3} and the wave vectors $\mathbf{k}_{1-3}$. We assume that the modes propagate from the plane $z=0$ along the z axis through a medium with quadratic nonlinearity. Neglecting anisotropy and absorption, we assume that the medium occupies the half-space $z\geq 0$ and the type II parametric process (so-called OEE interaction [16]) is realized. In this case, the problem under consideration is described using a well-known system of equations [16],

$$\partial A_1/\partial z = -i\beta A_2^* A_3 \exp(-i\Delta z), \tag{1a}$$

$$\partial A_2/\partial z = -i\beta A_1^* A_3 \exp(-i\Delta z), \tag{1b}$$

$$\partial A_3/\partial z = -i2\beta A_1 A_2 \exp(+i\Delta z). \tag{1c}$$

Here, β is the constant of nonlinear coupling and $\Delta=k_1+k_2-k_3$ is the wave mismatch. Systems (1a)–(1c) have two standard integrals of motion,

$$I_1(z)+I_2(z)+I_3(z)=I_{10}+I_{20}+I_{30}, \quad I_1(z)-I_2(z)=I_{10}-I_{20}, \tag{2}$$

where $I_i(z)=A_i(z)A_i^*(z)$ (below—the intensity) is proportional to the energy flux density of the ith ($i=1-3$) wave and $I_{i0}=I_i$ ($z=0$). The first integral describes conservation of the total energy flux, while the second one represents the so-called Manley-Rowe relations [16].

Using expressions (2), we can reduce systems (1a)–(1c) to three closed nonlinear equations describing self-consistent periodic solutions for $A_i(z)$. For this purpose we change variables

$$A_j(z)=\tilde{A}_j(z)\exp(-i\alpha_j z), \tag{3}$$

select such constants α_{1-3}, that

$$\alpha_1+\alpha_2-\alpha_3-\Delta=0, \tag{4}$$

and rewrite (1a)–(1c) in the form

$$\partial\tilde{A}_1/\partial z-i\alpha_1\tilde{A}_1=-i\beta\tilde{A}_2^*\tilde{A}_3, \tag{5a}$$

$$\partial\tilde{A}_2/\partial z-i\alpha_2\tilde{A}_2=-i\beta\tilde{A}_1^*\tilde{A}_3, \tag{5b}$$

$$\partial\tilde{A}_3/\partial z-i\alpha_3\tilde{A}_3=-i2\beta\tilde{A}_1\tilde{A}_2. \tag{5c}$$

Simple transformations with regard to integrals (2) yield an equation for the amplitude $\tilde{A}_1\tilde{A}_2$ of nonlinear polarization wave at the frequency ω_3 in the form

$$\partial(\tilde{A}_1\tilde{A}_2)/\partial z=i(\alpha_1+\alpha_2)\tilde{A}_1\tilde{A}_2-i\beta(I_{10}+I_{20}+I_{30}-\tilde{A}_3\tilde{A}_3^*)\tilde{A}_3. \tag{6}$$

By differentiating (5c) and substituting (6) into the result obtained, we derive the equation

$$\partial^2\tilde{A}_3/\partial z^2-i(\alpha_1+\alpha_2+\alpha_3)(\partial\tilde{A}_3/\partial z)+2\beta^2[I_{10}+I_{20}+I_{30}-(\alpha_1+\alpha_2)\alpha_3-\tilde{A}_3\tilde{A}_3^*]\tilde{A}_3=0. \tag{7}$$

It is seen that the second term in (7) can easily be eliminated. Indeed, taking into account that a specific choice of α_{1-3} values is not yet unique [see (4)], we can simply set

$$\alpha_1+\alpha_2=\Delta/2,\quad \alpha_3=-\Delta/2. \tag{8}$$

Then, we finally obtain a closed equation for $\tilde{A}_3$ in a form of steady-state NSE,

$$\partial^2\tilde{A}_3/\partial z^2+2\beta^2(I_{10}+I_{20}+I_{30}+\Delta^2/8\beta^2-\tilde{A}_3\tilde{A}_3^*)\tilde{A}_3=0. \tag{9}$$

Note that, since (9) is a second-order equation, we must be interested in its solutions satisfying the boundary condition

$$\partial\tilde{A}_3/\partial z|_{z=0}=-i\Delta\tilde{A}_{30}/2-i2\beta\tilde{A}_{10}\tilde{A}_{20}, \tag{10}$$

which follows from (5c). Here $\tilde{A}_{i0}=\tilde{A}_i(z=0)$.

Repeating the above successive transformations, we easily obtain

$$\partial(\tilde{A}_1^*\tilde{A}_3)/\partial z=-i(\alpha_1-\alpha_3)\tilde{A}_1^*\tilde{A}_3+i\beta(-2I_{10}+4I_{20}+I_{30}-4\tilde{A}_2\tilde{A}_2^*)\tilde{A}_2. \tag{11}$$

Then, by differentiating (5b), taking (11) into account, and choosing the values of α_{1-3} such that

$$\alpha_1-\alpha_3=\Delta/2,\quad \alpha_2=\Delta/2, \tag{12}$$

we find a similar closed NSE for $\tilde{A}_2$,

$$\partial^2\tilde{A}_2/\partial z^2-\beta^2(-2I_{10}+4I_{20}+I_{30}-\Delta^2/4\beta^2-4\tilde{A}_2\tilde{A}_2^*)\tilde{A}_2=0. \tag{13}$$

As in the previous case, we must be interested in solutions of (13) satisfying the boundary condition

$$\partial\tilde{A}_2/\partial z|_{z=0}=i\Delta\tilde{A}_{20}/2-i\beta\tilde{A}_{10}^*\tilde{A}_{30}. \tag{14}$$

With allowance for the symmetry of the problem, a closed NSE for $\tilde{A}_1$ can be obtained by interchanging the subscripts $1\leftrightarrow 2$. Therefore,

$$\partial^2\tilde{A}_1/\partial z^2-\beta^2(4I_{10}-2I_{20}+I_{30}-\Delta^2/4\beta^2-4\tilde{A}_1\tilde{A}_1^*)\tilde{A}_1=0 \tag{15}$$

for

$$\alpha_2-\alpha_3=\Delta/2,\quad \alpha_1=\Delta/2, \tag{16}$$

and the boundary condition

$$\partial\tilde{A}_1/\partial z_{z=0}=i\Delta\tilde{A}_{10}/2-i\beta\tilde{A}_{20}^*\tilde{A}_{30}. \tag{17}$$

The reduction of the system (1a)–(1c) to three closed NSEs for $\tilde{A}_{1-3}$ seems to be surprising. Indeed, the possibility to represent equations in the NSE form is usually attributed to the existence of a cubic nonlinearity [3–6]. However, there is no paradox here, because transition to closed equations (9), (13), and (15) is simply equivalent to the description of the competition of two simultaneous second-order nonlinear processes (merging $\omega_1+\omega_2\rightarrow\omega_3$ and decomposition $\omega_3\rightarrow\omega_1+\omega_2$ of quanta) in terms of an effective cascade cubic nonlinearity [18].

Note that Eqs. (9), (13), and (15) do not form a system of equations due to different values of α_{1-3} [see (8), (12), and (16)] and relate to each other through boundary conditions (10), (14), and (17). However, it is much more important that $\tilde{A}_i$ can be complex here. Therefore, in contrast to many other nonlinear problems, desired dependences of $\tilde{A}_i$ magnitude and phase on z can be very complicated. Thus, well-known NSE analytical solutions [16] proportional to Jacobian elliptic functions $\mathrm{sn}(\gamma z)$, $\mathrm{cn}(\gamma z)$, and $\mathrm{dn}(\gamma z)$, where the constant γ is determined in Sec. III, do not exhaust all possible solutions of (1a)–(1c) but determine the branches of such solutions for which the phase of $\tilde{A}_i$ is fixed [the phase of A_i is proportional to z, see (8) and (12) or (16)].

Note also that solutions of (9), on the one hand, and of (13) and (15), on the other hand, must be fundamentally different. This can be explained in the framework of a very simple analogy with the problem of self-action involving the so-called Kerr nonlinearity. In this context, Eq. (9) corresponds to the case of defocusing nonlinearity, whereas Eqs. (13) and (15) correspond to the focusing one [10]. When we are not interested in the phases of all of the interacting waves, it suffices to solve only one of Eqs. (9) and (13) or (15). After that, the desired dependences $I_i(z)$ for two other waves can be found from relationship (2).

III. EXAMPLES OF ANALYTICAL SOLUTIONS

As an example of the above approach, let us analyze one of the simplest solutions of problems (1a)–(1c) which describes SHG in the case when $I_{30}=0$. With regard to the results from [10], we can search for $\tilde{A}_3$ in the form

$$\tilde{A}_3 = B_3 \,\mathrm{sn}(\gamma z). \tag{18}$$

Substituting (18) into (9), we obtain

$$\gamma^2 = \beta^2[2(I_{10}+I_{20}) + \Delta^2/4\beta^2 - B_3B_3^*], \tag{19}$$

$$k^2 = B_3B_3^*/[2(I_{10}+I_{20}) + \Delta^2/4\beta^2 - B_3B_3^*].$$

Here, $1 \geq k \geq 0$ is the modulus of Jacobian elliptic functions. Two limiting cases $k \to 0$ and $k \to 1$ correspond to harmonic $[\mathrm{sn}(\gamma z) \to \sin(\gamma z),\ \mathrm{cn}(\gamma z) \to \cos(\gamma z),\ \mathrm{dn}(\gamma z) \to 1]$ and aperiodic $[\mathrm{sn}(\gamma z) \to \tanh(\gamma z),\ \mathrm{cn}(\gamma z) \to 1/\cosh(\gamma z),\ \mathrm{dn}(\gamma z) \to 1/\cosh(\gamma z)]$ solutions [17]. Note that condition $1 \geq k^2 \geq 0$ is ever fulfilled owing to the obvious relation $I_{10}+I_{20} \geq B_3B_3^*$. Moreover, $A_3(z)$ oscillations cannot be harmonic $(k=0)$ because in the case when $B_3B_3^*=0$ with allowance for expression (2) we obtain

$$I_1(z) = I_{10}, \quad I_2(z) = I_{20}, \quad A_3(z) = A_{30} = 0. \tag{20}$$

It follows from (10) that $A_{10}=0$ or $A_{20}=0$, so that, at least one of the fields at the frequency ω in the plane $z=0$ vanishes. At the same time, solution (18) can be aperiodic $(k=1)$, but only in the case of the total energy transfer to the mode A_3 $(I_{10}=I_{20}$, $B_3B_3^*=2I_{10})$ and the synchronous SHG process $(\Delta=0)$.

The amplitude B_3 can now be found from (10), which immediately yields

$$(B_3B_3^*)^2 - 2(I_{10}+I_{20}+\Delta^2/8\beta^2)B_3B_3^* + 4I_{10}I_{20} = 0 \tag{21}$$

and leads to the final result

$$I_1(z) = I_{10} - \frac{1}{2}\left[I_{10}+I_{20}+\frac{\Delta^2}{8\beta^2} - \sqrt{(I_{10}-I_{20})^2 + (I_{10}+I_{20})\frac{\Delta^2}{4\beta^2} + \left(\frac{\Delta^2}{8\beta^2}\right)^2}\right]\mathrm{sn}^2(\gamma z), \tag{22a}$$

$$I_2(z) = I_{20} - \frac{1}{2}\left[I_{10}+I_{20}+\frac{\Delta^2}{8\beta^2} - \sqrt{(I_{10}-I_{20})^2 + (I_{10}+I_{20})\frac{\Delta^2}{4\beta^2} + \left(\frac{\Delta^2}{8\beta^2}\right)^2}\right]\mathrm{sn}^2(\gamma z), \tag{22b}$$

$$A_3(z) = \sqrt{I_{10}+I_{20}+\frac{\Delta^2}{8\beta^2} - \sqrt{(I_{10}-I_{20})^2 + (I_{10}+I_{20})\frac{\Delta^2}{4\beta^2} + \left(\frac{\Delta^2}{8\beta^2}\right)^2}}\,\mathrm{sn}(\gamma z)\exp\left(+i\frac{\Delta}{2}z\right). \tag{22c}$$

It is quite reasonable that solutions (22a)–(22c) coincide with a well-known analytical solution for the simplest SHG case [16,19]. However, we emphasize here that analytical solutions of problems (1a)–(1c) can now be obtained for any boundary conditions. Let us illustrate this statement using the second example that describes the SHG with complete depletion of the field A_2 due to the total energy transfer to the field A_3. With regard to this fact and results from [10], we can search for solution of (13) in the form

$$\tilde{A}_2 = B_2 \,\mathrm{cn}(\gamma z). \tag{23}$$

By substituting (23) into (13) and taking into account the boundary conditions, we obtain

$$I_1(z) = I_{10} - I_{20}\,\mathrm{sn}^2(\gamma z), \tag{24a}$$

$$A_2(z) = A_{20}\,\mathrm{cn}(\gamma z)\exp(-i\Delta z/2), \tag{24b}$$

$$I_3(z) = 2I_{20}[\Delta^2/8\beta^2 I_{10} + \mathrm{sn}^2(\gamma z)], \tag{24c}$$

$$\gamma^2 = 2\beta^2 I_{10}[1 + (\Delta^2/8\beta^2 I_{10})(1 - I_{20}/I_{10})],$$

$$k^2 = I_{20}/[I_{10} + (\Delta^2/8\beta^2)(1 - I_{20}/I_{10})]. \tag{24d}$$

It follows from (24d) that solutions of this type only exists for $I_{10} \geq I_{20}$ and the amplitude A_{30} can be arbitrarily small only in the case of synchronous interaction $(\Delta \to 0)$. When $I_{10}=I_{20}$, solutions (24a)–(24d) becomes aperiodic $(k=1)$ and

$$I_1(z) = I_2(z) = I_{10}/\cosh^2(\gamma z), \tag{25a}$$

$$I_3(z) = \Delta^2/4\beta^2 + 2I_{10}\tanh^2(\gamma z), \tag{25b}$$

$$\gamma^2 = 2\beta^2 I_{10}. \tag{25c}$$

To the best of our knowledge, solution (24a)–(24d) of problems (1a)–(1c) have not been reported.

To conclude this section, we consider one more exact solution of (1a)–(1c) that we have not found in the literature and which corresponds to the case of SHG with an incomplete depletion of the field A_2. With regard to this fact, we can search for solution of (13) in the form

$$\tilde{A}_2 = B_2 dn(\gamma z). \tag{26}$$

By substituting (26) into (13), we obtain that either $k^2=0$ and

$$I_1(z)=I_{10}, \tag{27a}$$

$$I_2(z)=I_{10}(1+8\beta^2 I_{10}/\Delta^2), \tag{27b}$$

$$I_3(z)=2I_{10}(1+\Delta^2/8\beta^2 I_{10}) \tag{27c}$$

(so-called parametric bleaching, when the rates of the processes $\omega_1+\omega_2\to\omega_3$ and $\omega_3\to\omega_1+\omega_2$ are equal), or

$$I_1(z)=I_{10}\,\mathrm{cn}^2(\gamma z)+(\Delta^2/8\beta^2 I_{10})(I_{20}-I_{10})\mathrm{sn}^2(\gamma z), \tag{28a}$$

$$A_2(z)=A_{20}\,\mathrm{dn}(\gamma z)\exp(-i\Delta z/2), \tag{28b}$$

$$I_3(z)=(\Delta^2/4\beta^2 I_{10})I_{20}\,\mathrm{cn}^2(\gamma z)+2I_{10}(1+\Delta^2/8\beta^2 I_{10})\mathrm{sn}^2(\gamma z), \tag{28c}$$

$$\gamma^2=2\beta^2 I_{20},\quad k^2=I_{10}/I_{20}-(\Delta^2/8\beta^2 I_{10})(I_{20}-I_{10})I_{20}. \tag{28d}$$

It can easily be demonstrated that solution (28a)–(28d) of problem (1a)–(1c) exists only when

$$I_{10}\le I_{20}\le I_{10}(1+8\beta^2 I_{10}/\Delta^2). \tag{29}$$

All of the above solutions can be obtained by any other approach. However, in our opinion, the technique used here is much more convenient, that it enables us to find some solutions and analyze their specific features. Because the considered problem is completely integrable, all of these solutions must be stable and the same set of solutions A_{1-3} can be obtained if we specify the boundary conditions A_{10-30} for all three interacting modes and start our consideration from any equation [Eq. (9) for A_3, Eq. (13) for A_2, or Eq. (15) for A_1].

IV. COMPLEX SOLUTIONS OF NONLINEAR SCHRÖDINGER EQUATION

Using representation $\tilde{A}_i\to\tilde{A}_i'+i\tilde{A}_i''$ with two real variables $\tilde{A}_i'$ and $\tilde{A}_i''$, we convert each of Eqs. (9), (13), and (15) into a system of two coupled NSEs, describing MCW composed of two noninterfering (quadrature) components $\tilde{A}_i'(z)$ and $\tilde{A}_i''(z)$. Since the equations in each of the three systems are identical, such a change of variables makes it possible to build only degenerate MCW of the so-called Manakov type, for which $\tilde{A}_i'(z)$ and $\tilde{A}_i''(z)$ are proportional to the same elliptic function and the phase $\varphi_i=\arctan[\tilde{A}_i''(z)/\tilde{A}_i'(z)]$ does not depend on z [10]. This means that NSE must have one more (with regard to the solutions from [10]) complex type of self-consistent periodic solution $\tilde{A}_i(z)$, which describes nondegenerate MCWs in parametric frequency conversion.

A. Focusing nonlinearity

First, we consider the case of focusing nonlinearity [Eqs. (13) and (15)] and represent NSE in a standard normalized form

$$\partial^2\tilde{A}/\partial z^2+2(\tilde{A}\tilde{A}^*-\delta)\tilde{A}=0, \tag{30}$$

where δ is constant. Assuming that $\tilde{A}(z)=X(z)\exp[i\varphi(z)]$, where $X(z)$ and $\varphi(z)$ are real analytic functions, we obtain the system of equations

$$\partial^2 X/\partial z^2-X(\partial\varphi/\partial z)^2+2(X^2-\delta)X=0, \tag{31a}$$

$$X\partial^2\varphi/\partial z^2+2(\partial X/\partial z)(\partial\varphi/\partial z)=0. \tag{31b}$$

After integration of (31b), we find that

$$X^2\,\partial\varphi/\partial z=\left.(X^2\,\partial\varphi/\partial z)\right|_{z=0}=\text{const}. \tag{32}$$

It follows from (31b) and (32) that, if there is a point z_0, where $X|_{z=z_0}=0$ and $\partial X/\partial z|_{z=z_0}\neq 0$, then for all other points z, where $X(z)\neq 0$, the equality $\partial\varphi/\partial z=0$ must be satisfied and therefore φ=const. After integration of (32), we easily obtain

$$\partial^2 X/\partial z^2-(\varphi_0')^2 I_0^2/X^3+2(X^2-\delta)X=0, \tag{33a}$$

$$\varphi(z)=\varphi_0+\varphi_0' I_0\int_0^z dz'/X^2(z'). \tag{33b}$$

Here $I(z)=X^2(z)$, $X|_{z=0}=X_0$; $X^2|_{z=0}=X_0^2=I_0$; $\varphi|_{z=0}=\varphi_0$, and $\partial\varphi/\partial z|_{z=0}=\varphi_0'$.

We search for an analytical solution of (33a) in the form

$$X(z)=\pm\sqrt{I_0+\Delta I\,\mathrm{sn}^2(\gamma z)}, \tag{34}$$

where ΔI specifies the difference between two $I(z)$ extremes located at the points $z=0$ and $z=2K/\gamma$, and $K(k)$ is the complete elliptic integral of the first kind [17]. The sign $\pm$ in (34) reflects an invariance of (4) with regard to the replacement $X(z)\to -X(z)$ and determines two branches of possible solutions, which we identify below as the plus and minus branches.

Differentiating (34), substituting the result in (33a), representing $\mathrm{cn}^2(\gamma z)$ and $\mathrm{dn}^2(\gamma z)$ in terms of $\mathrm{sn}^2(\gamma z)$, and assuming that coefficients of different (from 0 to 3) powers of $\mathrm{sn}^2(\gamma z)$, we obtain an algebraic system of equations for ΔI, γ, and k, which determines the solution of (30),

$$\tilde{A}(z)=\pm\sqrt{I_0+\Delta I\,\mathrm{sn}^2(\gamma z)}\exp\left[i\left(\varphi_0+\varphi_0' I_0\int_0^z dz'/[I_0+\Delta I\,\mathrm{sn}^2(\gamma z)]\right)\right], \tag{35a}$$

$$\Delta I=\delta-3I_0/2+\sqrt{(\delta-I_0/2)^2+(\varphi_0')^2 I_0}, \tag{35b}$$

$$\gamma^2=-\delta+3I_0/2+\sqrt{(\delta-I_0/2)^2+(\varphi_0')^2 I_0},\quad k^2=-\Delta I/\gamma^2, \tag{35c}$$

existing only when

$$I_0\ge\delta+(\varphi_0')^2/2. \tag{36}$$

The limit $I_0=\delta+(\varphi_0')^2/2$ corresponds to the parametric bleaching (see above) when $I(z)=I_0$ and

$$\tilde{A}(z) = \pm \sqrt{I_0}\exp[i(\varphi_0 + \varphi_0' z)]. \quad (37)$$

Asymptotes of solutions (35a)–(35c) can be characterized as nontrivial. When $(\varphi_0')^2 \neq 0$, the branches of (35a)–(35c) do not cross, since $X(z) \neq 0$. However, in the case when $(\varphi_0')^2 = 0$, the domain of region existence for this solution is divided into two intervals

$$2\delta \geq I_0 \geq \delta, \quad (38a)$$

$$I_0 \geq 2\delta. \quad (38b)$$

If condition (38a) is satisfied, then

$$\tilde{A}(z) = \pm \sqrt{I_0}\,\mathrm{dn}(\sqrt{I_0} z)\exp(i\varphi_0), \quad (39a)$$

$$k^2 = 2(I_0 - \delta)/I_0, \quad (39b)$$

and the signs of $X(z)$ on both branches are different but constant. Otherwise, we have

$$\tilde{A}(z) = \pm \sqrt{I_0}\,\mathrm{cn}[\sqrt{2(I_0 - \delta)}z]\exp(i\varphi_0), \quad (40a)$$

$$k^2 = I_0/[2(I_0 - \delta)]. \quad (40b)$$

The boundary $I_0 = 2\delta$ of the intervals given by expressions (38a) and (38b) corresponds to the localized (solitary) solution

$$\tilde{A}(z) = \pm \sqrt{2\delta}\cos\mathrm{h}^{-1}(\sqrt{2\delta}z)\exp(i\varphi_0). \quad (41)$$

Such a drastic metamorphosis follows from crossing of the plus and minus branches of (35a)–(35c). In the case when $I_0 \geq 2\delta$, a series of the points

$$z_n = (2n+1)K\{\sqrt{I_0/[2(I_0 - \delta)]}\}/\sqrt{2(I_0 - \delta)}, \quad (42)$$

$$n = 0, \pm 1, \pm 2, \ldots,$$

where $X(z_n) = 0$ for both branches, appears on the z axis. The analyticity requirement for $X(z)$ and $\varphi(z)$ successively joins the parts of solutions from two different branches, which forms two solutions (43) with alternating sign and constant phase.

B. Defocusing nonlinearity

In the case of defocusing nonlinearity [see (9)], Eqs. (31a), (31b), and (33a) turn into

$$\partial^2 \tilde{A}/\partial z^2 c - 2(\tilde{A}\tilde{A}^* - \delta)\tilde{A} = 0, \quad (43)$$

$$\partial^2 X/\partial z^2 - (\varphi_0')^2 I_0^2/X^3 - 2(X^2 - \delta)X = 0. \quad (44)$$

Solution to (44) can again be searched in form (34). Using similar procedures, one can easily obtain an algebraic system of desired equations for ΔI, γ, and k. In contrast to the case of focusing nonlinearity, there are three types of possible solutions of (44). The first one

$$\tilde{A}(z) = \pm \sqrt{\Delta I}\,\mathrm{sn}(\sqrt{2\delta - \Delta I}z)\exp(i\varphi_0), \quad (45a)$$

$$k^2 = \Delta I/(2\delta - \Delta I), \quad (45b)$$

with alternating sign of $X(z)$ corresponds to the case when

$$I_0 = 0, \quad \delta \geq \Delta I \geq 0, \quad (\varphi_0')^2 = 0, \quad (46)$$

and is related to a well-known class of so-called dark cnoidal waves [10] with asymptote ($\Delta I \to \delta$) in the form of a standard dark soliton [3–6]

$$\tilde{A}(z) = \pm \sqrt{\delta}\tanh(\sqrt{\delta}z)\exp(i\varphi_0). \quad (47)$$

As in the previous case, the sign of $X(z)$ in (45a), (45b), and (47) changes because the plus and minus branches intersect in the points

$$z_n = 2nK[\sqrt{\Delta I/(2\delta - \Delta I)}]/\sqrt{2\delta - \Delta I}, \quad n = 0, \pm 1, \pm 2, \ldots. \quad (48)$$

The second type of possible solutions of (44),

$$\tilde{A}(z) = \pm \sqrt{I_0}\exp[i(\varphi_0 + \varphi_0' z)] \quad (49)$$

with a constant sign of $X(z)$ is realized when

$$I_0 = \delta - (\varphi_0')^2/2, \quad (\varphi_0')^2 \leq 2\delta, \quad (50)$$

and corresponds to the parametric bleaching (see above). Finally, the third type of solutions

$$\tilde{A}(z) = \pm \sqrt{I_0 + \Delta I\,\mathrm{sn}^2(\gamma z)}\exp\left[i\left(\varphi_0 + \varphi_0' I_0\int_0^z dz'/[I_0 + \Delta I\,\mathrm{sn}^2(\gamma z)]\right)\right], \quad (51a)$$

$$\Delta I = \delta - 3I_0/2 - \sqrt{(\delta - I_0/2)^2 - (\varphi_0')^2 I_0}, \quad (51b)$$

$$\gamma^2 = \delta - 3I_0/2 + \sqrt{(\delta - I_0/2)^2 - (\varphi_0')^2 I_0}, \quad k^2 = \Delta I/\gamma^2, \quad (51c)$$

looking like a kind of spatial shock wave or kink with a constant sign of $X(z)$, is realized when

$$(\varphi_0')^2 \geq 2\delta/3. \quad (52)$$

Note two interesting facts. First, in contrast to the case of focusing nonlinearity, solutions (45a), (45b), and (47) cannot be obtained as asymptotes of (51a)–(51c). While in the limit

$$I_0 \to 2[\delta + (\varphi_0')^2 - |\varphi_0'|\sqrt{2\delta + (\varphi_0')^2}], \quad (53)$$

solutions (51a)–(51c) become aperiodic (solitary),

$$\tilde{A}(z) = \pm \sqrt{I_0 + \Delta I\tanh^2(\gamma z)}\exp\left[i\left(\varphi_0 + \varphi_0' I_0\int_0^z dz'/[I_0 + \Delta I sn^2(\gamma z)]\right)\right], \quad (54a)$$

$$\Delta I = \gamma^2 = -2\delta - 3(\varphi_0')^2 + 3|\varphi_0'|\sqrt{2\delta + (\varphi_0')^2}, \quad (54b)$$

it significantly differs from dark soliton (47). Because in this case the condition $(\varphi_0')^2 \geq 2\delta/3$ must also be satisfied, the sign of $X(z)$ in (54a) remains unchanged. Only in the region

$$2\delta \geq (\varphi_0')^2 \geq 2\delta/3, \tag{55}$$

when $I_0 \to \delta - (\varphi_0')^2/2$, solutions (49) and (51a)–(51c) coincide. Second, because $z_n = 0$ for $n = 0$ [see (48)], solutions (45a), (45b), and (47) are determined through the boundary condition

$$\partial X/\partial z|_{z=0} = \pm \sqrt{\Delta I(2\delta - \Delta I)}, \quad \delta \geq \Delta I \geq 0, \tag{56}$$

which can be recalculated to the boundary condition for X and I of the point shifted by $K[\sqrt{\Delta I/(2\delta - \Delta I)}]/\sqrt{2\delta - \Delta I}$ along the z axis.

V. FEATURES OF COMPLEX MCW SOLUTIONS

A specific character of the above class of NSE complex periodic solutions becomes evident if we introduce real and imaginary parts of $\tilde{A}(z)$,

$$\tilde{A}(z) = \tilde{A}'(z) + i\tilde{A}''(z). \tag{57}$$

Then, after substitution of (57) into (30) and (43), we obtain a classical system of two coupled NSEs with respect to the real variables $\tilde{A}'(z)$ and $\tilde{A}''(z)$ in the form

$$\partial^2 \tilde{A}'/\partial z^2 \pm 2[(\tilde{A}')^2 + (\tilde{A}'')^2 - \delta]\tilde{A}' = 0, \tag{58a}$$

$$\partial^2 \tilde{A}''/\partial z^2 \pm 2[(\tilde{A}')^2 + (\tilde{A}'')^2 - \delta]\tilde{A}'' = 0. \tag{58b}$$

If we try now to construct a two-component MCW using the scheme from [10], we find that $\tilde{A}'$ and $\tilde{A}''$ must be proportional to the same elliptic function $\theta(z)$,

$$\tilde{A}'(z) = \theta(z)\cos \varphi_0, \tag{59a}$$

$$\tilde{A}''(z) = \theta(z)\sin \varphi_0, \tag{59b}$$

because δ in (58a) and (58b) is the same.

One can formally assume that both components $\tilde{A}'$ and $\tilde{A}''$ can be obtained by projecting the one-component solution $|\tilde{A}(z)| = \theta(z)$ along the axes of a system of coordinates, rotated by a fixed (independent of z) angle $\alpha = \varphi_0 = \arctan(\tilde{A}''/\tilde{A}')$ [10]. By the same substitution (57), the complex solutions described in the preceding section can be rewritten as

$$\tilde{A}'(z) = \sqrt{I_0 + \Delta I \,\mathrm{sn}^2(\gamma z)}\cos\left(\varphi_0 + \varphi_0' I_0 \int_0^z dz'/[I_0 + \Delta I \,\mathrm{sn}^2(\gamma z)]\right), \tag{60a}$$

$$\tilde{A}''(z) = \sqrt{I_0 + \Delta I \,\mathrm{sn}^2(\gamma z)}\sin\left(\varphi_0 + \varphi_0' I_0 \int_0^z dz'/[I_0 + \Delta I \,\mathrm{sn}^2(\gamma z)]\right). \tag{60b}$$

It is seen that, while solutions (60a) and (60b) look like a Manakov-type solution with the magnitude $|\tilde{A}(z)| = \sqrt{I_0 + \Delta I \,\mathrm{sn}^2(\gamma z)}$, its projection angle $\alpha(z) = \varphi_0 + \varphi_0' I_0 \int_0^z dz'/[I_0 + \Delta I \,\mathrm{sn}^2(\gamma z)]$ exhibits rather complicated nonlinear oscillations, matched with oscillations of $|\tilde{A}(z)|$. The character of such nonlinear oscillations can be found, for example, in [19].

VI. CONCLUSIONS

Thus, we have shown above that, in a three-coupled-mode approximation, solution of a problem of a steady-state parametric frequency conversion, including SHG and PA in a medium with quadratic nonlinearity, can be reduced to analytical solution of three closed nonlinear Schrödinger equations. Each of them is related to two others only due to its boundary conditions and describes a complex type of non-Manakov MCWs containing two noninterfering (quadrature) components. The projection angle (the phase) of the corresponding one-component solution exhibits complicated nonlinear oscillations, matched with nonlinear oscillations of its magnitude. The above approach is based on the description of a competition between simultaneous processes of the merging $\omega_1 + \omega_2 \to \omega_3$ and decomposition $\omega_3 \to \omega_1 + \omega_2$ of quanta in terms of an effective cascade cubic nonlinearity. Moreover, it enables one to change the type of boundary conditions. This makes it possible to use the same solution for SHG and PA problems. For example, shifting the argument $\xi \to \xi + K$ of elliptic functions by one-quarter of their period K is described by the standard substitutions $\mathrm{sn}(\xi) \to \mathrm{cn}(\xi)/\mathrm{dn}(\xi)$, $\mathrm{cn}(\xi) \to -\sqrt{1-k^2}\,\mathrm{sn}(\xi)/\mathrm{dn}(\xi)$, and $\mathrm{dn}(\xi) \to \sqrt{1-k^2}/\mathrm{dn}(\xi)$ [17].

Note also that such solutions can be extrapolated to the half-space $z < 0$ (by filling this half-space with the same nonlinear medium) and, then, set into motion at constant velocity v along the z axis (by the replacement $z \to \eta = z - vt$, where η is the running coordinate and t is the time). This follows from an invariance of the wave equation with respect to the Lorentz transformation. Therefore, with regard to rather universal character of the nonlinear Schrödinger equation, one can expect that the above complex periodic solutions (39a), (39b), (51a)–(51c), (54a), and (54b) of NSE can be useful in many other physical problems, where a derivative of the solution phase can be not equal to zero at least in one fixed plane $\eta = \eta_0 = \mathrm{const}$. In our opinion, one of the most interesting candidates is a problem of solitonlike propagation of trains of laser pulses along optical fibers upon the phase modulation by an information signal or by the chirp.

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