Fractal dimension of chaotic light scattering in regular polyhedral mirror ball structures

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We experimentally observe fractal patterns of chaotic light scattering in regular polyhedral mirror ball structures that consist of spherical reflectors located at the vertices of polyhedra as optical scattering devices. We measure the fractal dimension of the basin boundaries of the light scattering patterns in the regular polyhedral mirror ball structures.

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I. INTRODUCTION

Photonic crystals have been intensively investigated for many years as optical devices to control the localization and propagation characteristics of photons. Recently, photonic fractals have been invented [1], which are three-dimensional fractal cavities based on the Menger sponge structure fabricated from epoxy resin by stereolithography. Localization of electromagnetic field at a resonant frequency can be achieved in the photonic fractals. The use of fractal characteristics may have a potential to control and localize photons in a different way from the conventional optical devices, and may lead to new engineering applications of photon localization devices.

As an example of fractal optical devices, a three-dimensional optical billiard has been proposed [2,3], in which the centers of four spherical reflectors are located at the vertices of a regular tetrahedron. Light was injected into the optical billiard and fractal patterns of light scattering with the Wada basin property were found. A similar configuration has been used as a chaos mirror [4], which provides free-space optical beam links for optical wireless networks. The chaos mirror has a feature that a one-dimensional spread of input optical rays results in a two-dimensional spread of the rays reflected from the chaos mirror. The characteristics of fractal optical devices need to be well understood to investigate the behavior of photons in fractal optical devices. The analysis of the fractal dimension [5] is one of the basic characteristics of fractal structures.

In this paper we experimentally observe fractal patterns of chaotic light scattering in regular polyhedral mirror ball structures. We analyze the fractal dimension of the basin boundaries of the light scattering patterns by using the models obtained from our experimental observation.

II. EXPERIMENTAL OBSERVATION

We extend the idea of the optical billiard and the chaos mirror with a tetrahedral structure [2–4] to regular polyhedral mirror ball structures. Figure 1 shows five examples of the regular polyhedral mirror ball structures. The five regular polyhedra correspond to a regular tetrahedron (4), hexahedron (6), octahedron (8), dodecahedron (12), and icosahedron (20), respectively, whose number corresponds with the number of the faces. The centers of spherical reflectors are located at the vertices of a polyhedron whose edge length is equal to the diameter of the spherical reflectors. Each of the spheres is in contact with the nearest-neighboring spheres. We used mirrored balls with 85-mm diameter as spherical reflectors in our experiment.

To observe chaotic light scattering in the regular polyhedral mirror ball structures, light emitting diodes (LEDs) with different colors are located at the center of the faces of the polyhedra. We used $N-1$ LEDs with different colors on the faces of the polyhedron, where N is the number of the faces. The light beams from the LEDs are injected and scattered in

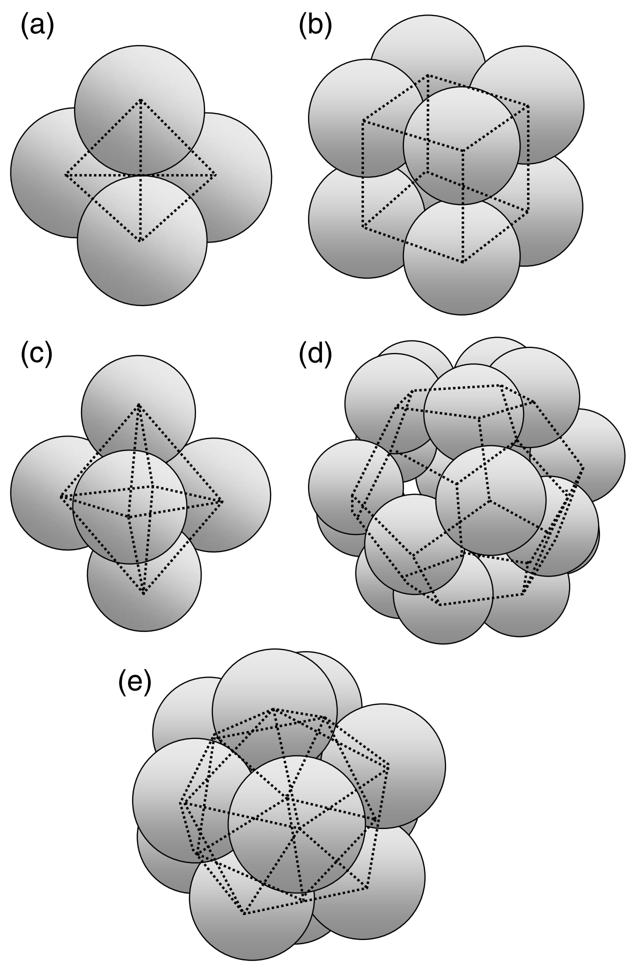

FIG. 1. Regular polyhedral mirror ball structures. (a) Regular tetrahedron, (b) regular hexahedron, (c) regular octahedron, (d) regular dodecahedron, and (e) regular icosahedron.

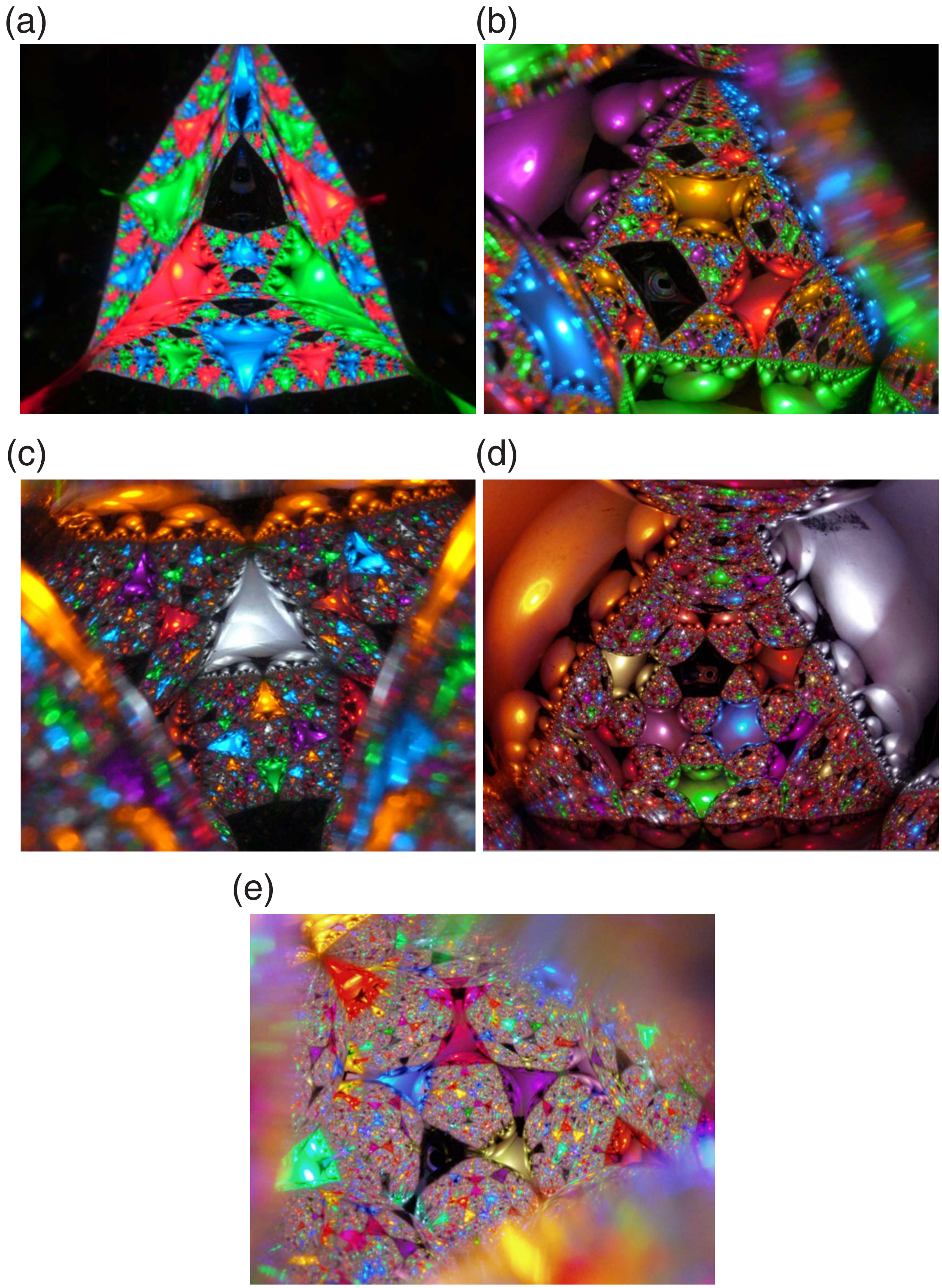

FIG. 2. (Color online) Pictures of fractal patterns of chaotic light scattering in the five regular polyhedral mirror ball structures. (a) Regular tetrahedron, (b) regular hexahedron, (c) regular octahedron, (d) regular dodecahedron, and (e) regular icosahedron.

the mirror ball structure. We observed a light scattering pattern in the structures from one face without LEDs by using a digital camera.

Figure 2 shows the pictures of the light scattering patterns in the five regular polyhedral mirror ball structures. Self-similarity of the light scattering patterns is observed as fractal. For the tetrahedral structure shown in Fig. 2(a), many triangle patterns with different colors and sizes are shown in the triangle region with some contraction ratios. These triangle patterns indicate the light source coming from one of the faces of the tetrahedron, which can be considered as a set of initial conditions of the light scattering. We thus call the

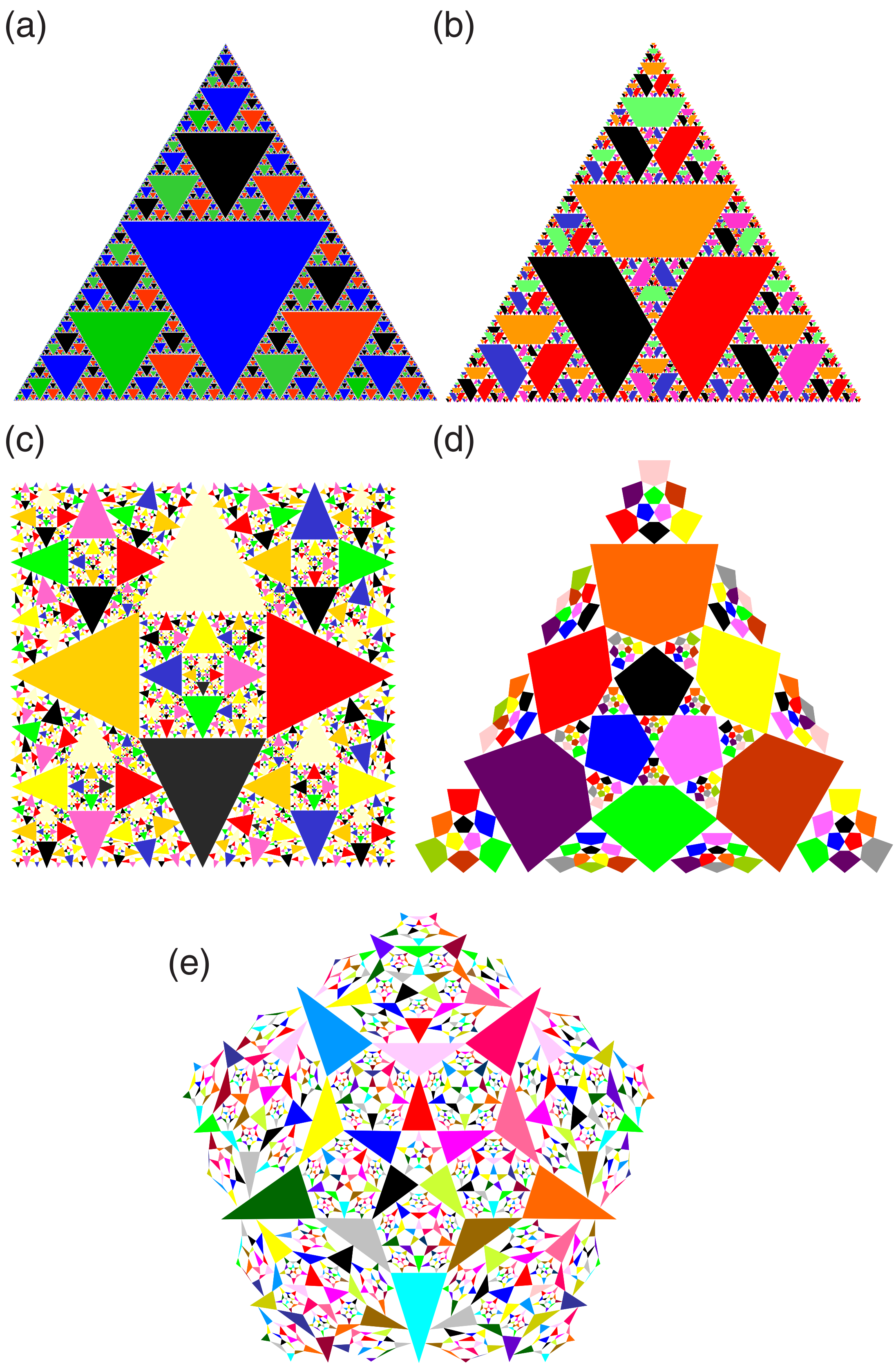

FIG. 3. (Color online) Models of fractal patterns of chaotic light scattering in the five regular polyhedral mirror ball structures. (a) Regular tetrahedron, (b) regular hexahedron, (c) regular octahedron, (d) regular dodecahedron, and (e) regular icosahedron.

TABLE I. Summary of the structures of the fractal models shown in Fig. 3.

Polyhedron	Each pattern	Entire region
Tetrahedron	Triangle	Triangle
Hexahedron	Square	Triangle
Octahedron	Triangle	Square
Dodecahedron	Pentagon	Triangle
Icosahedron	Triangle	Pentagon

color patterns *basins* [3]. The boundaries of these basins are very complex and it is considered as fractal basin boundaries [3,5]. Different fractal patterns are observed in the five regular polyhedral mirror ball structures, and the basin boundaries look more complex as the number of the faces is increased.

III. MODEL

Based on the experimental observation of Fig. 2, we created models for the observed fractal patterns in the five regular polyhedral mirror ball structures, as shown in Fig. 3. The color patterns indicate the basins of light scattering, corresponding to the faces of the polyhedron. We modified the distorted patterns at the edge of the entire region, so that we can extract the essence of the fractal patterns from Fig. 2. The structures of these models are summarized in Table I. Note that the shape of each color pattern is determined by the shape of the faces of the polyhedra, whereas the shape of the entire region is determined by the number of the nearest-neighboring vertices. The fractal patterns of the hexahedron and octahedron are complementary (triangle and square), as well as those of dodecahedron and icosahedron (pentagon and triangle), as shown in Table I. The basin boundaries are very complicated for all the five structures, as shown in Fig. 3.

IV. FRACTAL DIMENSION

A. Self-similar dimension

We calculated the fractal dimension of the basin boundaries of the light scattering patterns in the five regular polyhedral mirror ball structures. We refer to the dimension of the basin boundaries as measured in a two-dimensional slice as shown in Figs. 2 and 3 [3]. The fractal dimension is thus between 1 and 2. We used two methods to measure the fractal dimension: the self-similar and box-counting dimensions. We first calculated the self-similar dimension. The self-similar dimension D_s can be analytically calculated by [5],

$$\sum_{i=1}^{M} b_i(a_i)^{D_s} = 1, \tag{1}$$

where a is the contraction ratio of the size of self-similar patterns and b is the expansion ratio of the number of increasing self-similar patterns. The size and number of the self-similar patterns are changed with the ratios of a^n and b^n at the n-stage of iteration, respectively, where n indicates the number of reflection of the light on the spheres. M is the total number of the combination of different a_i and b_i. a and b may be different at different regions of the models. For example, the regular hexahedral mirror ball structure and the corresponding fractal pattern model are shown in Fig. 4. There are three different combinations of the distances of the neighboring spheres from one sphere, as indicated by A, B, and C in Fig. 4(a). The light reflection from these neighboring spheres results in the fractal patterns with different sizes, as indicated by A, B, and C in Fig. 4(b), and there exists different ratios of a and b in the model of Fig. 4(b). In fact, b corresponds to the number of neighboring spheres at a certain distance, i.e., three neighboring balls are located at a distance of A ($b_1=3$), other three neighboring balls are located at a distance of B ($b_2=3$), and one ball is located at a distance of C ($b_3=1$). On the other hand, a_i is estimated from the contraction ratio of the size of the corresponding part in the model [$a_1=0.400$ for A, $a_2=0.200$ for B, and $a_3=0.200$ for C in Fig. 4(b)].

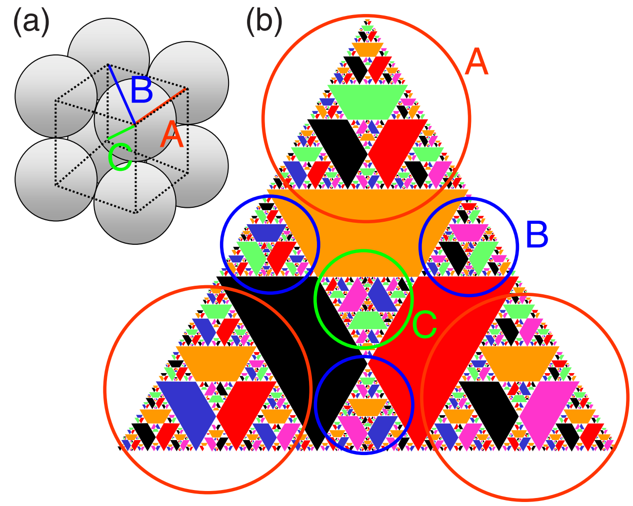

FIG. 4. (Color online) (a) Regular hexahedral mirror ball structure and (b) the corresponding fractal pattern model. (a) Three different distances of the neighboring spheres from one sphere are indicated by A, B, and C. (b) Three different patterns with different sizes are indicated by A, B, and C.

The values of a_i and b_i are obtained from the models in Fig. 3 and the self-similar dimension is calculated. The result of M, a_i, b_i, and D_s are summarized in Table II. D_s of the tetrahedron is identical to that of the Sierpinski gasket, since the models are exactly the same as shown in Fig. 3(a). D_s of the tetrahedron and hexahedron are very close to each other. D_s of the icosahedron is the maximum value of the fractal dimension in the five regular polyhedral mirror ball structures.

The analytical estimation of the self-similar dimension is effective. However, there is some inaccuracy of the contraction ratios of a_i for the octahedron, dodecahedron, and icosahedron, since some distortion of the fractal patterns occurs near the edges of the entire region due to the reflection from a spherical mirror.

TABLE II. Summary of the self-similar dimension of the fractal models for the five regular polyhedral mirror ball structures.

Polyhedron	M	a_i	b_i	D_s
Tetrahedron	1	0.500	3	1.585…
Hexahedron	3	0.400	3	1.597…
		0.200	3	
		0.200	1	
Octahedron	2	0.427	4	1.773…
		0.293	1	
Dodecahedron	5	0.258	3	1.560…
		0.172	6	
		0.102	6	
		0.0848	3	
		0.0781	1	
Icosahedron	3	0.357	5	1.854…
		0.194	5	
		0.123	1	

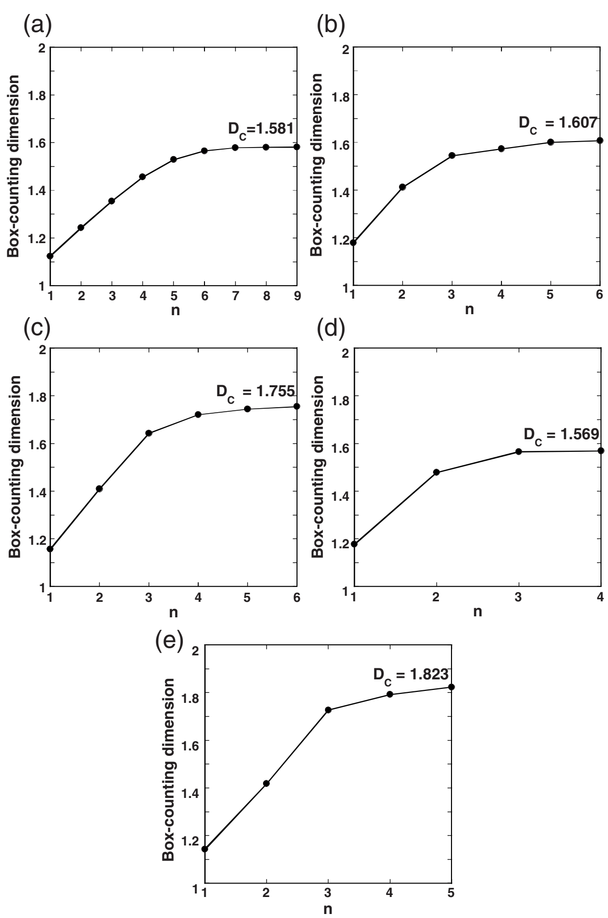

FIG. 5. Box-counting dimensions of the fractal models for the polyhedral mirror ball structures as a function of the number of iterations n. (a) Regular tetrahedron, (b) regular hexahedron, (c) regular octahedron, (d) regular dodecahedron, and (e) regular icosahedron.

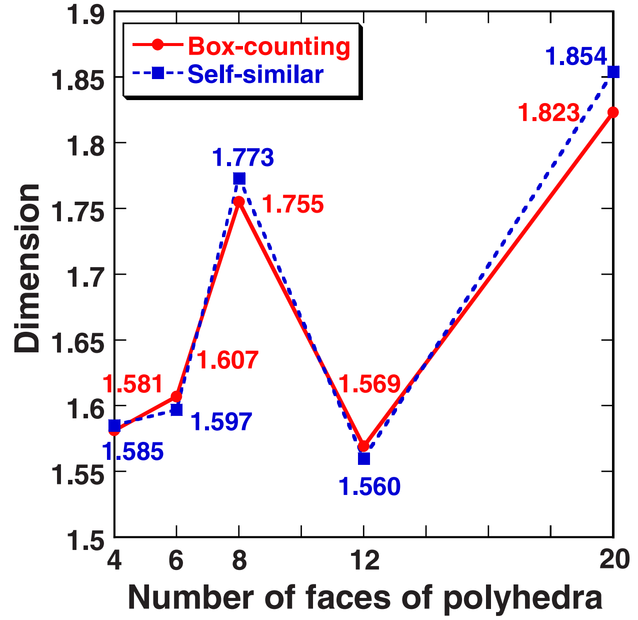

FIG. 6. (Color online) Summary of the self-similar and box-counting dimensions (D_s and D_c) of the fractal models for the five regular polyhedral mirror ball structures.

B. Box-counting dimension

We next used the box-counting dimension of the basin boundary based on the models shown in Fig. 3. The box-counting dimension D_c can be calculated by [5],

$$D_c = \lim_{\varepsilon \to 0} \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)}, \tag{2}$$

where ε is the a box size (length) and $N(\varepsilon)$ is the number of boxes needed to cover the basin boundary. We used a commercially available software (Benoit, TruSoft International Inc.) for the calculation of the box-counting dimension of the basin boundaries for the models shown in Fig. 3. We extracted the edges of all the fractal patterns and deleted the colors in the models. We covered the edges with ε-size boxes and counted the number of the boxes to calculate the box-counting dimension of the basin boundaries.

We calculated the box-counting dimension of the basin boundaries for the five regular polyhedral mirror ball structures. Figure 5 shows the result of box-counting dimension of the fractal models at the different stages of the iteration n (i.e., the number of light reflection from the spherical mirrors). As the number of the iteration n is increased, the box-counting dimension increases and saturates at a certain value, which corresponds to a reliable value of the box-counting dimension, as indicated in Fig. 5.

Figure 6 shows the summary of the self-similar and box-counting dimensions of the basin boundaries of the fractal models for the five regular polyhedral mirror ball structures. The self-similar and box-counting dimensions match each other with the maximum error of 1.7%. It is found that the fractal dimension increases as the number of the faces of the polyhedral structures is increased, except the dodecahedral

(12-face) structure. For the dodecahedral structure, the pentagonal color patterns share the large region and the basin boundaries are not as long as the other polyhedral structures.

The fractal dimension obtained in our study may be an important measure to characterize optical scattering devices with fractal structures. The fractal dimension may be related to the Q value as optical confinement devices. The investigation of the relationship between the fractal dimension and the functionality of fractal optical devices is our future work.

From physics point of view, we only take into account ray optics to analyze the fractal patterns since the size of the spherical structures is much larger than the wavelength of visible light. The effects of wave optics and quantum optics may become dominant when the size of these structures is decreased in the order of the optical wavelength. We expect to find more interesting behavior of photons in such nanoscale fractal optical devices.

V. CONCLUSION

We have experimentally observed fractal patterns of chaotic light scattering in the five regular polyhedral mirror ball structures that consist of spherical reflectors located at the vertices of polyhedra as optical scattering devices. We have obtained the pictures of the fractal patterns in the regular polyhedral mirror ball structures and constructed the models for the five fractal patterns. We have measured the fractal dimension of basin boundaries of the light scattering patterns in the regular polyhedral mirror ball structures by using the self-similar and box-counting dimensions. The two dimensions agree well with each other. We have found that the fractal dimension increases as the number of the faces of the polyhedral structures is increased, except the dodecahedral structure. The characteristics of the fractal patterns of chaotic light scattering may be useful for optical devices using fractal, such as photon localization devices.

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