Discrete surface growth process as a synchronization mechanism for scale-free complex networks

A. L. Pastore y Piontti,[1] P. A. Macri,[1] and L. A. Braunstein[1,2]

[1]*Departamento de Física, Facultad de Ciencias Exactas y Naturales, Universidad Nacional de Mar del Plata, Funes 3350, 7600 Mar del Plata, Argentina*

[2]*Center for Polymer Studies, Boston University, Boston, Massachusetts 02215, USA*

We consider the discrete surface growth process with relaxation to the minimum [F. Family, J. Phys. A **19**, L441 (1986)] as a possible synchronization mechanism on scale-free networks, characterized by a degree distribution $P(k) \sim k^{-\lambda}$, where k is the degree of a node and λ its broadness, and compare it with the usually applied Edward-Wilkinson process (EW) [S. F. Edwards and D. R. Wilkinson, Proc. R. Soc. London, Ser. A **381**, 17 (1982)]. In spite of both processes belonging to the same universality class for Euclidean lattices, in this work we demonstrate that for scale-free networks with exponents $\lambda < 3$ the scaling behavior of the roughness in the saturation cannot be explained by the EW process. Moreover, we show that for these ubiquitous cases the Edward-Wilkinson process enhances spontaneously the synchronization when the system size is increased. This nonphysical result is mainly due to finite size effects due to the underlying network. Contrarily, the discrete surface growth process does not present this flaw and is applicable for every λ.

The study of the dynamics on complex networks became a subject of great interest in the last few years since it was realized that they are useful tools to understand biological, social, and communications systems [1,2]. Networks are constituted by nodes associated to individuals, organizations or computers and by links representing their interactions. The classical model for random networks is the Erdős-Rényi (ER) model [3–5] characterized by a Poisson degree distribution $P(k) = \exp(-\langle k \rangle) \langle k \rangle^k / k!$ where k is the degree or number of links that a node has and $\langle k \rangle$ is the average degree. However, it was found [1] that many real networks are characterized by a scale-free (SF) degree distribution given by

$$P(k) = (\lambda - 1) k_{\min}^{\lambda-1} \frac{1}{1 - (k_{\min}/k_{\max})^{\lambda-1}} k^{-\lambda}, \qquad (1)$$

for $k_{\min} < k < k_{\max}$ where $k_{\min}$ is the smaller degree that a node can have, $k_{\max}$ is a cutoff that diverges when the system size $N \to \infty$ and λ represent the broadness of the distribution. Most of the observed networks such as Internet, the World Wide Web, and metabolic networks have $2 < \lambda < 3$ [1,2].

It was shown that the topology of the network is very relevant to determine their statics and dynamics properties, such as robustness and percolation thresholds [6,7], the average shortest path length [8] and transport [9]. An important quantity characterizing networks is its diameter (maximal hopping) d. In a network of a total of N nodes, d scales as $\ln N$ [5], which leads to the concept of "small worlds" or "six degrees of separation" [10]. For scale-free (SF) networks with $\lambda < 3$ [11] d scales as $\ln \ln N$, which leads to the concept of ultrasmall worlds [2,12].

Very recently, the research focus is changing from the study of the network topology to the study of dynamical processes on the underlying network. Of particular interest are the studies on the dynamics and fluctuations of task completion landscapes of queuing networks. If for each node on the network there is a scalar h which specifies the time it takes to finish a job or the amount of work that has been assigned to it, the fluctuations on h indicates how synchro-

nized or balanced is the system. Jobs synchronization and load balance are required in many applications such as packet routing on the Internet [13] or in parallel computing [14,15].

These synchronization processes are usually mapped into a nonequilibrium surface growth via an Edwards-Wilkinson (EW) equation [16] on complex networks [17–19]. The EW equation for the evolution of the growing interface in complex networks is given by

$$\frac{\partial h_i}{\partial t} = \nu \sum_{j=1}^{N} A_{ij}(h_j - h_i) + \eta_i, \qquad (2)$$

where $h_i \equiv h_i(t)$ is the height of the interface of node i, A_{ij} is the element ij of the adjacency matrix that take the value 1 if i and j are connected and zero otherwise, N is the system size, ν is a coefficient that represent the "surface tension" and $\eta_i \equiv \eta_i(t)$ is a random Gaussian uncorrelated noise with $\{\eta_i\} = 0$; $\{\eta_i \eta_j\} = 2D\delta_{ij}\delta(t - t')$, where D is the diffusion coefficient and $\{\cdots\}$ represent averages over configurations. The interface is characterized by its roughness $W(t)$ at time t,

$$W(t) = \left(\frac{1}{N} \sum_{i=1}^{N} (h_i - \langle h \rangle)^2 \right)^{1/2}, \qquad (3)$$

that represents the fluctuations of the height of the interface around the mean value $\langle h \rangle$.

There are several technical advantages of using the continuous EW equation to model queue synchronization or load balance processes [17–19] mainly because it is a linear continuous equation. However, some real implementations of this process are intrinsically discrete. For this reason, in this work we use the discrete growth model of surface relaxation to the minimum (SRM), which is very well known on Euclidean lattices [20,21], on SF networks. It is also a well-known fact that on Euclidean lattices this discrete model belongs to the same universality class of the EW equation. This might be one of the motivations of using EW to model these discrete processes. In the SRM model at each time step

a node i is chosen with probability $1/N$. If we denote by v_i the nodes nearest neighbors of i, then

$$\text{if} \begin{cases} h_i \leq h_j \quad \forall\, j \in v_i & \Rightarrow h_i = h_i + 1, \text{ else,} \\ \text{node } j \text{ has the minimum height } \in v_i & \Rightarrow h_j = h_j + 1. \end{cases} \tag{4}$$

These rules mimic a process where the higher loaded node distributes the excess of load to one of the neighbors which is less charged. To generate SF graphs of size N, we employ the Molloy-Reed algorithm (MR) [22]: initially the degree of each node is chosen according to a SF distribution, where each node is given a number of open links or "stubs" according to its degree. Then, stubs from all nodes of the network are interconnected randomly to each other with the two constraints that there are no multiple edges between two nodes and that there are no looped edges with identical ends.

We use for the simulation $k_{\min}=2$ because when $k_{\min}>1$ there is a high probability that the network is fully connected [12] which is required in this work to analyze the interface.

At $t=0$ we initialize all the values of h_i with random numbers taken from a uniform distribution in [0, 1]. At each time step we select a node with probability $1/N$ and use the rules given by Eq. (4), then the time is increased by $1/N$. We compute $W(t)$ for SF networks with $\lambda>2$ and different values of N.

In Figs. 1(a) and 1(b) we plot $W(t)$ for the SRM as function of t for $\lambda=2.5$ and $\lambda=3.5$, respectively. In both figures we can see a very short growing regime for $W(t)$ after which the system saturates with a width W_s. This fast regime before the saturation can be explained in terms of finite size effects. For almost all growth processes the correlation length grows with time until it reaches the characteristic length of the system [20], which for complex random networks is the diameter d. As explained above, the diameter is very small, and the system reaches the saturation time very fast. We focus the attention on the steady state of W_s because only at the steady state does it matter to analyze the fluctuations in the load balance of multiprocessors in parallel computing or synchronization of queues.

For $\lambda=2.5$ we found by a linear fitting of $W(t)$ in the steady state that W_s behaves with N as $W_s \sim \ln N$ [see the inset of Fig. 1(a)]. The same scaling behavior was obtained for all other values of $\lambda<3$. We also run all the simulations for an initial flat interface and found no differences in W_s [23]. For $\lambda=3.5$, W_s does depend weakly on the system size for big enough networks [see the inset of Fig. 1(b)]. Korniss reported this lack of finite size effect for the growing network model of Barabási-Albert [1] that has $\lambda=3$ [24].

As mentioned above, it is well known that this model in Euclidean lattices belongs to the EW universality class represented by Eq. (2), so it is expected that Eq. (2) will show the same scaling behavior as the SRM model. In this work we demonstrate that surprisingly this is not generally true. In Figs. 2(a) and 2(b) we show $W(t)$ as function of t for different values of N from the numerical integration of Eq. (2) with $\nu=1$ and $D=1$ for SF networks with $k_{\min}=2$ for $\lambda=2.5$ and $\lambda=3.5$.

Counterintuitive, for $\lambda=2.5$ W_s decreases with the system size, which is a nonexpected result for any growth model. If this were the case, increasing the system size will be a simple strategy to minimize the roughness and thus improving synchronization of queues or balance in the load of multiprocessors in parallel computing.

We next show that the decreasing of the width for $\lambda<3$ is mainly due to finite size effects introduced by the MR construction. It was shown in [24] that for the EW process in
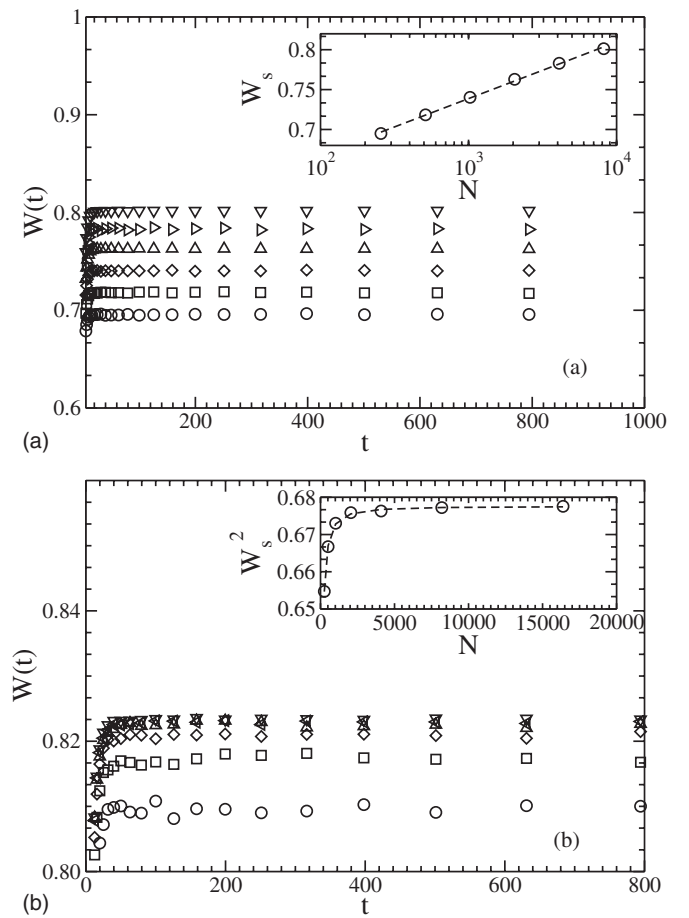

FIG. 1. Plots of $W(t)$ for SRM and different system size N, $N=256$ ($\bigcirc$), $N=512$ ($\square$), $N=1024$ ($\diamond$), $N=2048$ ($\triangle$), $N=4096$ ($\triangleleft$), and $N=8192$ ($\triangledown$) for the following: (a) $\lambda=2.5$, we can see that W_s increases with the system size N. In the inset figure we show W_s as function of N in linear log scale ($\bigcirc$). The dashed lines represent the logarithmic fitting supporting that $W_s \sim \ln N$. (b) $\lambda=3.5$, we can see that $W(t)$ depend weakly on N. In the inset figure we show in symbols W_s^2 as a function of N. The dashed lines represent the fitting of W_s^2 with Eq. (7) ($A \approx 10$ and $B \approx 0.25$). In all the inset of data's figures we do not display the errors bars because they are of the size of the symbols.

unweighted networks the absolute lower bound of the W_s^2 is

$$W_{\min}^2 = (1 - 1/N)^2 \frac{1}{\langle k \rangle}. \qquad (5)$$

The decreasing on the width observed in our numerical results is because $\langle k \rangle$ increases with N. As a consequence of the MR construction which introduces the natural cutoff $k_{\max} = k_{\min} N^{1/(\lambda-1)}$ [25], $\langle k(N) \rangle$ is given by

$$\langle k(N) \rangle = k_\infty \frac{1 - 1/N^{(\lambda-2)/(\lambda-1)}}{1 - 1/N}, \qquad (6)$$

where $k_\infty \equiv k(N \to \infty)$. Taking into account the results presented in Eq. (5) where we replace $\langle k \rangle$ by $\langle k(N) \rangle$ we propose that

$$W_s^2 \sim W_s^2(\infty) \left(1 - \frac{A}{N} + \frac{B}{N^{(\lambda-2)/(\lambda-1)}} \right), \qquad (7)$$

where $W_s(\infty) \equiv W_s(N \to \infty)$.

The fitting of W_s^2 with Eq. (7) shows an excellent agreement [see the inset of Fig. 2(a)] with the simulations supporting that the decrease in the width for $\lambda < 3$ is mainly due to the MR construction and for large N, $W_s^2 \sim$ const.

Thus, the scaling behavior of W_s for the SRM model with $\lambda < 3$ is not well represented by the EW equation with constant coefficients ν and D, despite the fact that it is often used in synchronization problems.

Next we analyze finite size effects for $\lambda > 3$. For the SRM model W_s was well fitted by Eq. (7) [see the inset in Fig. 1(b)]. Thus, in this regime, the finite size effects can be attributed to the MR construction. For the EW equation we find the best fitting $W_s^2 \sim W_s^2(\infty)(1 - A/N)$ with $B \approx 0$ in Eq. (7). This behavior cannot be explained as finite size effects due the MR construction because for $\lambda > 3$, N diverges faster than $N^{(\lambda-2)/(\lambda-1)}$ and maybe is due to the EW process. The separation between the finite size effect of the EW process and the MR construction is still an open question that goes beyond the aim of this paper and could be the subject of future researches.

In summary, we simulate the SRM model in SF networks and compare the results with the EW process. We show that a discrete model and a continuous model which share the same scaling properties on Euclidean lattices does not exhibit this equivalence on complex networks. For the SRM model in SF networks W_s diverges with the system size as $\ln N$ for $\lambda < 3$. For $\lambda > 3$ for both, the model and the EW equation, when $N \to \infty$, $W_s \to$ const. In order to compare the results of the SRM model with a continuous equation further investigation including higher order of the Laplacian in the continuous equation are needed. Also the dynamics could introduce some weights on the links on the underlying unweighted network that even at a linear approximation could
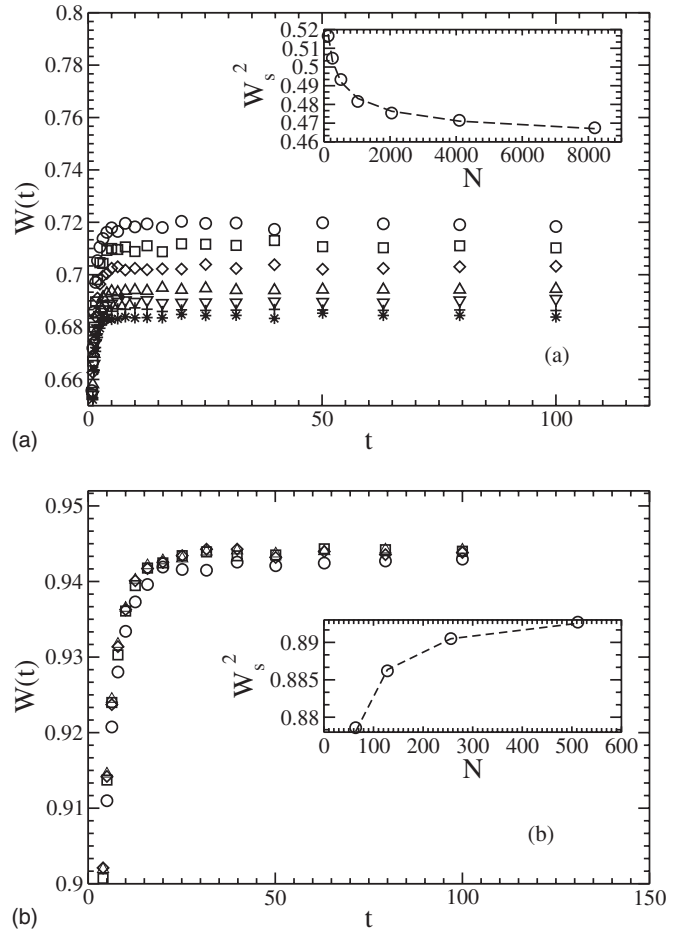

FIG. 2. Plots of $W(t)$ from the integration of the EW equation and different system size N. (a) For $\lambda = 2.5$, $N = 128$ ($\bigcirc$), $N = 256$ ($\square$), $N = 512$ ($\diamond$), $N = 1024$ ($\triangle$), $N = 2048$ ($\triangledown$), $N = 4096$ ($\times$), and $N = 8192$ ($*$). We can see that W_s decreases with N. In the inset figure we show in symbols W_s^2 as a function of N. The dashed line is the fitting with Eq. (7) ($A \approx 0.10$ and $B \approx 0.75$). (b) For $\lambda = 3.5$, $N = 128$ ($\bigcirc$), $N = 256$ ($\square$), $N = 512$ ($\diamond$), and $N = 1024$ ($\triangle$). The dashed line represents the fitting with Eq. (7) ($A \approx 1.15$ and $B \approx 0$).

affect the EW unweighted process. This is the aim of our future research. Finally, we can conclude that despite the fact that the SRM model and the EW equation belongs to the same universality class in Euclidean networks, in SF networks they do not have the same behavior.

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