

Sublattice synchronization of chaotic networks with delayed couplings

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(Received 23 January 2007; published 28 September 2007)

Synchronization of chaotic units coupled by their time-delayed variables is investigated analytically. A type of cooperative behavior is found: Sublattice synchronization. Although the units of one sublattice are not directly coupled to each other, they completely synchronize without time delay. The chaotic trajectories of different sublattices are only weakly correlated but not related by generalized synchronization. Nevertheless, the trajectory of one sublattice is predictable from the complete trajectory of the other one. The spectra of Lyapunov exponents are calculated analytically in the limit of infinite delay times, and phase diagrams are derived for different topologies.

DOI: 10.1103/PhysRevE.76.035202

PACS number(s): 05.45.Xt, 05.45.Ra

Two identical chaotic systems which are coupled to each other can synchronize to a common chaotic trajectory. Both of the systems are chaotic, but after synchronization the two chaotic trajectories are locked to each other [1,2]. This phenomenon has attracted a lot of research, partly because chaos synchronization has the potential to be applied for novel secure communication systems [3,4]. In fact, synchronization and bit exchange with chaotic semiconductor lasers has recently been demonstrated over a distance of 120 km in a public fiber-optic network [5].

Typically, the coupling between chaotic units has a time delay due to transmission of the exchanged signal. Nevertheless, two chaotic systems can synchronize without delay, isochronically, although the delay time may be very long. This counterintuitive phenomenon has recently been demonstrated with chaotic semiconductor lasers [6–9], and it is discussed in the context of corresponding observations on correlated neuronal activity [10–12].

In this paper we give an analytic solution of small networks of chaotic units coupled by their time-delayed variables. Motivated by chaotic lasers, we consider the competition between delayed self-feedback and coupling. For regular networks which can be decomposed into two identical sublattices—e.g., a ring with $N=4$ units in Fig. 1(b)—a new kind of synchronization is observed: The units A and C as well as the units B and D are completely synchronized. Even for arbitrarily long delay times, the synchronization is isochronal; i.e., the trajectories of units A and C are identical without any time shift. Although synchronization of A and C is caused by the trajectory of B and D and vice versa, the trajectories of A and B are not synchronized. Actually, we do not even find generalized synchronization between A and B . The cross correlations are symmetric; there is neither a leader nor a laggard for the two sublattices. Each unit is passive; i.e., it has negative Lyapunov exponents when it is isolated and driven by an external signal. Nevertheless, the mutual interaction of passive elements leads to high-dimensional chaos with synchronized units in each sublattice.

We have found sublattice synchronization in several chaotic networks, in particular for rate equations describing chaotic semiconductor lasers [13]. In this paper we analyze sublattice synchronization using a simpler system—namely,

chaotic Bernoulli maps with delay, which allows analytical as well as comprehensive numerical investigations [14,15]. In the limit of long delay times, the spectra of Lyapunov exponents, which are calculated analytically, determine the phase boundaries between sublattice synchronization, complete synchronization, and complete chaos.

Two mechanisms are included in our model: (i) delayed feedback and (ii) delayed mutual coupling. The first mechanism can generate high-dimensional chaos for each single unit whereas the second one can synchronize the network. The phenomena caused by these two mechanisms are captured by a network of iterated maps, given by the following equations, $j, k=1, \dots, N$:

$$x_t^j = (1 - \varepsilon)f(x_{t-1}^j) + \varepsilon \left[\kappa f(x_{t-\tau}^j) + (1 - \kappa) \sum_k F_{j,k} f(x_{t-\tau}^k) \right]. \tag{1}$$

Each node j of the network contains a variable $x_t^j \in [0, 1]$ which is iterated in time t . $F_{j,k}$ is the weighted adjacency matrix which has a component $1/z_j$ if node j is

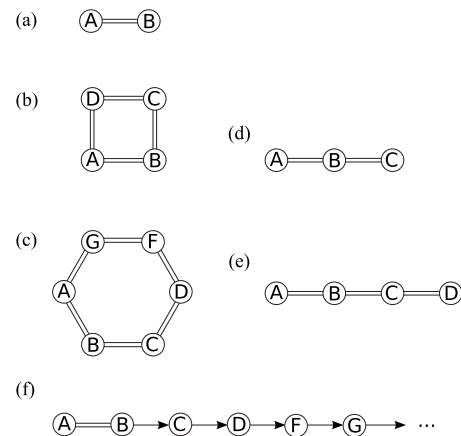


FIG. 1. Small networks of chaotic units. Double lines signify bidirectional couplings whereas arrows show unidirectional couplings. It turns out that each ring with an even number N of units has Lyapunov exponents which are identical to a chain with $N/2+1$ units.

driven by node k and 0 otherwise, and z_j is the number of connections to node j .

Analytic calculations are possible for the Bernoulli shift map $f(x)=(ax)\bmod 1$ which is chaotic for $a > 1$ [16]. τ is the delay time, which, for simplicity, is identical for the feedback as well as for the coupling. [Synchronization can also be found for nonidentical delay times. But in that case, the synchronization region in the parameter space is smaller (if it exists) and extensive analytical calculations are not possible anymore.] The parameter ε determines the strength of the delay terms while κ determines the relative strength of the self-feedback compared to the mutual coupling terms.

Now let us discuss stationary solutions of Eq. (1). Complete synchronization, $x_t^j = x_t^k$ for all (j, k) , is a solution of Eq. (1). In addition, several kinds of incomplete synchronization are solutions, as well. For example, sublattice synchronization, $x_t^j = x_t^k$ for all nodes (j, k) in each sublattice, is a solution, too, if the network can be decomposed into two interconnecting sublattices, as for the chains and rings in Fig. 1. But are these solutions stable? Their stability has to be determined from Eq. (1). Let δx_t^j be a small deviation from any given trajectory x_t^j . The linearized Eq. (1) is solved with the ansatz $\delta x_t^j = c^t v_j$, where the value $\lambda = \ln|c|$ is the corresponding Lyapunov exponents. We find that the vectors $(v_j)_{j=1, \dots, N}$ are eigenvectors of the adjacency matrix F and their corresponding eigenvalues $\mu_q, q=0, \dots, N-1$ determine the Lyapunov exponents by

$$\mu_q = -\frac{(1-\varepsilon)ac^{\tau-1} + \varepsilon\kappa a - c^\tau}{\varepsilon(1-\kappa)a}. \quad (2)$$

For each eigenvector, indexed by $q=0, \dots, N-1$, this polynomial equation (2) for c has τ complex solutions $c_{q,m}, m=1, \dots, \tau$, giving a total of $N\tau$ Lyapunov exponents, which are plotted in Fig. 2. The eigenvectors of F determine the N directions of perturbation $v_{q,j}$ which shrink or explode with the corresponding values of c .

Here we are interested in long delay times $\tau \rightarrow \infty$. It turns out that for each eigenvalue $\mu_q, q=0, \dots, N-1$, there is one Lyapunov exponent of order 1 and $\tau-1$ exponents of order $1/\tau$ (see Fig. 2). The large exponent is given by $\lambda_q^1 = \ln[(1-\varepsilon)a]$ and merges with the band of the other $\tau-1$ exponents at the critical point

$$\varepsilon_c = \frac{a-1}{a}. \quad (3)$$

Consequently, for $\varepsilon < \varepsilon_c$, stable synchronization does not exist.

The eigenvalues of the adjacency matrix F determine the stability of synchronized trajectories according to Eq. (2), and they depend on the topology of the lattice. Let us consider the rings and chains depicted in Fig. 1. We find that the phase diagram of all six topologies is given by Fig. 3. To see this, let us consider a ring of N mutually coupled chaotic units with self-feedback, described by Eq. (1). Figure 1 shows $N=2, 4$, and 6. By symmetry, the eigenvectors $v_{q,j}$ and eigenvalues μ_q of the adjacency matrix F are the Fourier modes ($q=0, \dots, N-1$):

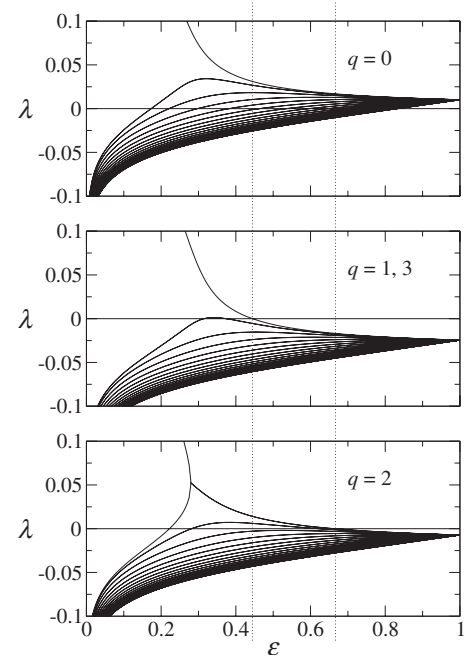


FIG. 2. Lyapunov spectra for the different modes q for $N=4$, $a=1.5$, $\tau=40$, and $\kappa=1/4$. At $\varepsilon=4/9$ the ($q=1$) mode becomes stable (\rightarrow sublattice synchronization). At $\varepsilon=2/3$ the ($q=2$) mode $(+1, -1, +1, -1)$ becomes stable, too (\rightarrow complete synchronization). See also Fig. 3.

$$v_{q,j} = \exp(2\pi i j q / N), \quad \mu_q = \cos\left(\frac{2\pi q}{N}\right). \quad (4)$$

These eigenvalues determine the Lyapunov spectrum by Eq. (2). Using the methods of Ref. [16], for large values of τ the maximal Lyapunov exponent is calculated as

$$\lambda_q^{\max} = \frac{1}{\tau} \ln \left| \frac{a\varepsilon \left[(1-\kappa)\cos\left(\frac{2\pi q}{N}\right) + \kappa \right]}{1-a(1-\varepsilon)} \right| \quad (\varepsilon > \varepsilon_c). \quad (5)$$

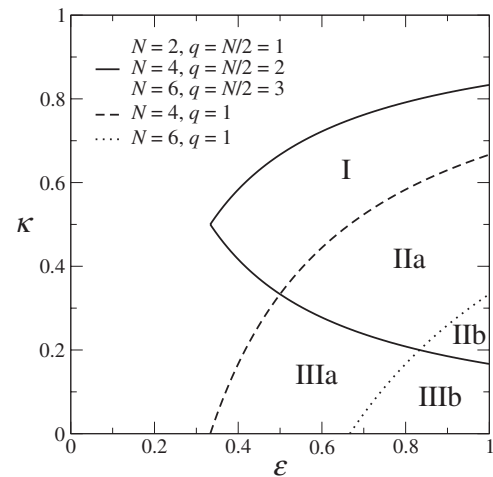


FIG. 3. Phase diagram for $a=1.5$. See text for assignment of the different regions to complete and sublattice synchronization.

The mode ($q=0$) with the corresponding eigenvector $(1, 1, 1, \dots)$ is always unstable, $\lambda_{q=0}^{\max} > 0$; hence, the system is chaotic for all parameters ε and κ . In fact, the system is in a state of high-dimensional chaos (hyperchaos), since the Kaplan-Yorke dimension increases linearly with delay time τ .

The superpositions of the modes ($q=1, q=N-1$), which have degenerate eigenvalues, are perturbations which destroy any structure of synchronization. They are unstable for

$$\kappa > \kappa_{q=1} = \frac{1-a}{a\varepsilon \left[1 - \cos\left(\frac{2\pi}{N}\right) \right]} + 1. \quad (6)$$

An analysis of Eq. (5) shows that for even values of N , the only perturbation which is relevant in the region $\kappa < \kappa_{q=1}$ is the perturbation of the mode ($q=N/2$). The network is unstable against this mode for

$$\kappa < \kappa_{q=N/2} = \frac{a-1}{2a\varepsilon}. \quad (7)$$

Since this mode corresponds to the eigenvector $(+1, -1, +1, -1, \dots, -1)$, it can destroy only complete but not sublattice synchronization.

Equations (6) and (7) immediately determine the phase diagram of the rings with N units in the (κ, ε) plane, Fig. 3. The ring with $N=4$ units, Fig. 1(b), is completely synchronized in region II (=IIa plus IIb). But complete synchronization cannot exist in region III, because there the $q=N/2$ mode is unstable. In this region the system is synchronized in a sublattice configuration: the trajectory of A is completely identical, without delay, with the one of C , and correspondingly, B is identical to D .

When increasing the number N of units the ($q=1$) line of Fig. 3 shrinks to the lower right corner. For $N=6$ we find complete synchronization in the region IIb, while sublattice synchronization occurs in region IIIb. In the latter region the trajectories of A , C , and F as well as B , D , and G are identical, but the two different trajectories do not synchronize.

Finally, for large system size N synchronization disappears completely. Complete synchronization exists up to $N < 2\pi / \arccos(\frac{3-a}{1+a})$, while sublattice synchronization remains for larger sizes up to $N < 2\pi / \arccos(\frac{1}{a})$. Consequently, even for arbitrary large rings one obtains regions of complete and sublattice synchronization if the chaos is sufficiently weak. Asymptotically, sublattice synchronization is stable if a is smaller than $a \sim 1 + 2(\pi/N)^2$.

In our model the rings with an even number N of chaotic units are equivalent to chains with $N/2+1$ units; i.e., the corresponding rings have identical, but degenerate, Lyapunov exponents. Hence the phase diagram of Fig. 3 describes synchronization of the chains Figs. 1(a), 1(d), and 1(e), as well. Regions II and III also describe the infinite chain with directed bonds of Fig. 1(f). If the left unit A is just a mirror, corresponding to a self-feedback of B with $\kappa=1$, the chain is in state of complete synchronization in these regions. If, however, A is a chaotic unit, the infinite chain

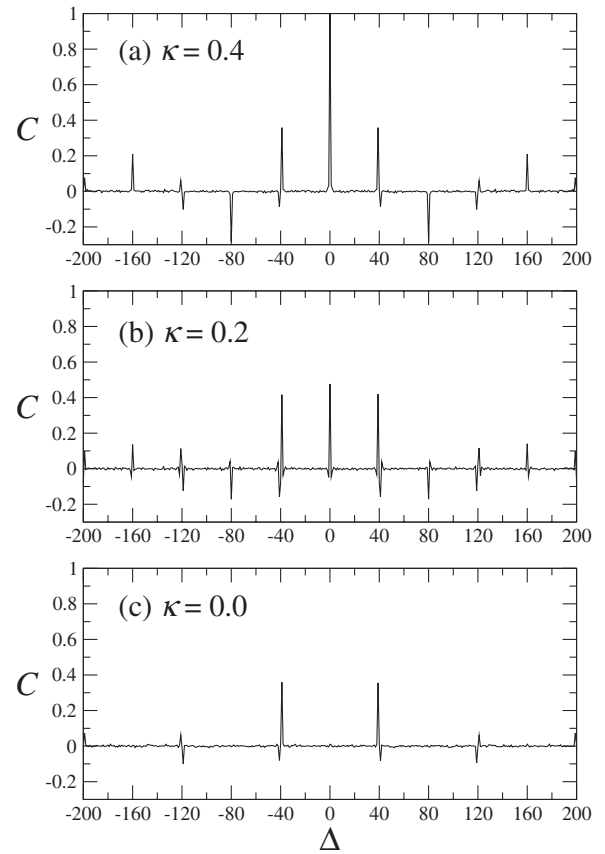


FIG. 4. Cross correlation C between A_t and $B_{t+\Delta}$ for $a=1.5$, $\tau=40$, and $\varepsilon=0.7$. (a) Complete synchronization. (b), (c) Sublattice synchronization.

switches to a state of complete synchronization in region II and sublattice synchronization in region III.

These findings are not specific to rings and chains, since we found sublattice synchronization in square lattices with periodic boundaries (with even side lengths) as well as for free boundaries (with even or odd side length). In addition, we found sublattice synchronization for a small triangular lattice with three sublattices. Each sublattice synchronizes completely, without time shifts, whereas the three trajectories have only weak correlations.

Synchronization of the units of one sublattice is relayed by the chaotic trajectory of the other one. Hence one would expect some relation between these two chaotic trajectories (or three for the triangular lattice).

The cross correlations between the two sublattices are shown in Fig. 4. For sublattice synchronization the central peak, a remainder of complete synchronization, still exists, in addition to correlations shifted by multiples of the delay time τ . Only when self-feedback is switched off, $\kappa=0$, do the isochronal correlations disappear. The correlations are completely symmetric in time shift $\pm\Delta$. In addition, when crossing the phase boundary between sublattice synchronization, complete synchronization, and complete chaos, the central peak of the cross correlation changes discontinuously, similar to a subcritical bifurcation. For large, but finite observation times, our simulations even show hysteresis effects.

These findings are in agreement with corresponding ex-

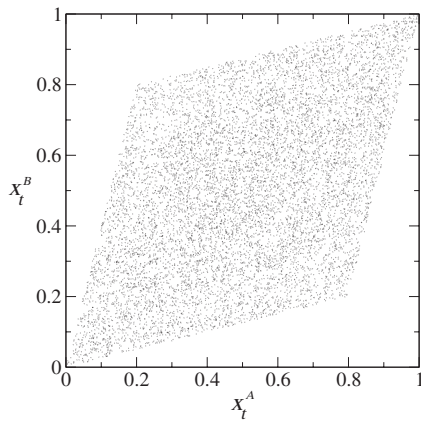


FIG. 5. Two trajectories of the sublattice synchronization phase for the networks of Fig. 1 with $\tau=1$, $\varepsilon=0.8$, $\kappa=0$, and $a=3.0$. Obviously, there is no functional relation between the two trajectories.

periments and numerical simulations of two bidirectionally coupled semiconductor lasers [17,18]. Without self-feedback, $\kappa=0$, the lasers do not show isochronal synchronization but they have strong correlations shifted by the transmission time. Such a state has been named *achronal synchronization*. For short time scales, however, the lasers show spontaneous symmetry breaking into a leader-laggard state which we do not observe for our model.

Apart from these weak correlations, is there any functional relation between the two different trajectories? In particular, are the trajectories in a state of *generalized synchronization* [19]? Several definitions of generalized synchronization are reported in the literature [20], and several methods have been developed to detect such a state, in

particular for driven systems. Generalized synchronization means that there exists a functional relation between the states of the two systems. For our model, it means that there exists a function between the sequence $(x_t^A, \dots, x_{t-\tau}^A)$ of A and $(x_t^B, \dots, x_{t-\tau}^B)$ of B . This function may be fractal [1] or may have several branches, which usually makes an analysis difficult. But at least for $\tau=1$ and $\tau=2$ the relation between these two vectors can be easily visualized. Figure 5 shows that the trajectories are not related by generalized synchronization in the phase of sublattice synchronization.

However, there is a related question: Can the trajectory of sublattice B be predicted from knowledge of the complete trajectory of A [21]? This question has a positive answer for the configurations of Fig. 1. In regions II and III all units are passive. This means, if we omit the drive, the last term of Eq. (1), each unit has negative Lyapunov exponents. Hence, if this unit is driven, it will relax to a unique trajectory. Since we find sublattice synchronization only for passive units, we can predict the trajectory of B from knowledge of the *complete* trajectory of A after some transient time. Note that for two units we find complete synchronization for active units, as well, but in this case, region I, the prediction is trivial.

Sublattice synchronization has not yet been found in experiments. However, a chain of three lasers, Fig. 1(d), has been investigated with semiconductor lasers without feedback, $\kappa=0$ [7,22]. Synchronization of the outer lasers has been observed, being relayed by a different chaotic trajectory of the internal laser, in agreement with our model. It would be interesting to realize a network of mutually coupled chaotic semiconductor lasers and measure their patterns of synchrony.

We would like to thank Jörg Reichardt, Andreas Ruttor, and Evi Kopelowitz for useful discussions.

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