Roles of mixing patterns in cooperation on a scale-free networked game

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We investigate how the degree-mixing pattern affects the emergence of cooperation in the networked prisoner's dilemma game. Our study shows that when a network becomes assortative mixing by degree, the large-degree vertices (hubs) tend to interconnect to each other closely, which destroys the sustainability among cooperators and promotes the invasion of defectors, whereas in disassortative networks, the isolation among hubs protects the cooperative hubs in holding onto their initial strategies to avoid extinction.

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Understanding the emergence of cooperation has drawn considerable attention in diverse fields including sociology, economics, and biology [1], and evolutionary game theory provides a powerful framework to study it [2]. As a general metaphor to investigate the conflict among altruists and selfish individuals, the prisoner's dilemma (PD) has been extensively explored [3]. In the PD game, two individuals decide whether to cooperate or defect to obtain some payoffs. They receive the reward R for mutual cooperation and the punishment P for mutual defection. If one chooses cooperation and the other chooses defection, the former's payoff is S and the latter gets T as the temptation to defect. The order of four payoffs is $T>R>P>S$ in the PD game. Since the pioneering work of Nowak and May [4], spatial structures, such as lattices [5], regular random graphs [6], small-world networks [7], and scale-free networks [8–11], have been widely investigated and well recognized as one of the key mechanisms for the cooperative dynamics in the networked PD game [12]. More relevant to the subject of this Brief Report, Santos *et al.* and Pacheco have shown that cooperation is the dominating trait in scale-free networks [8–10], and they found that the interconnections among the hubs favor the dominance of cooperation in the PD game [10]. More recently, Gomez-Gardenes *et al.* pointed out that in scale-free networks, there exists a single cluster composed of the hub cooperators (C-hubs), which provides further insight to understand the cooperative dominance in heterogeneous networks [11].

It has been witnessed that real networks display different mixing patterns of degree [13–15]. To measure the degree mixing of a network, Newman defined the assortativity coefficient in terms of the Pearson correlation coefficient: $r_k=(\langle ij\rangle-\langle i\rangle\langle j\rangle)/(\langle i^2\rangle-\langle i\rangle^2)$, where i and j are the remaining degrees at the two ends of an edge and $\langle\cdot\rangle$ means the average over all edges [13]. When a network is assortatively (disassortatively) mixed by degree, r_k is positive (negative) and the highly connected vertices tend to choose those vertices with similar (dissimilar) degrees as neighbors. An uncorrelated network exhibits the neutral degree-mixing pattern whose $r_k=0$. Usually, social networks are assortatively mixed, while many technological and biological networks generally display disassortatively mixing patterns.

It is well known that the assortative degree-mixing pattern of a network significantly influences the collective dynamical behaviors, such as epidemic spreading [13], node percolation [14], and synchronization [15], of the network. However, to the best of our knowledge, the role of the degree-mixing pattern in the cooperative behavior in the PD game has not been addressed to date. In this Brief Report, we focus on how the degree-mixing pattern influences the evolution of cooperation in the PD game. Our investigation shows that an uncorrelated network promotes the emergence of cooperation. However, when the network becomes assortative, highly connected vertices will stick together as a tight core group sharing many neighbors among them, which inhibits the sustainability among cooperators and promotes the invasion of defectors. In a network displaying the disassortative mixing pattern, the isolation among hubs will make the initial C-hubs insist on their cooperative strategies for a wide range of temptation.

Consider a network where each individual occupies a site. These individuals update their strategies according to the replicator dynamics adopted in [8]: At every the generation, each individual i plays the game with its neighbors and obtains its accumulated payoff P_i. When individual i updates its strategy, it randomly selects a neighbor j and adopts j's strategy with probability $(P_j-P_i)/[(T-P)k_{\max}]$, where $k_{\max}$ is the highest degree between i and j. During the evolution process, all individuals update their strategies synchronously. Following [4], we set $P=0$, $T=b>1$, $R=1$, and $S=0$, where b is the temptation to defect. To generate different degree-mixing networks, we employ the Xulvi-Brunet–Sokolov (XS) algorithm proposed in [14]: To generate an assortative network, at each step, two edges in a given network with four different vertices are randomly selected; then, one edge links the two vertices with smaller degrees and the other connects the two vertices with larger degrees. Multiple connections are forbidden in this process. By repeating this operation, an assortative network is generated without changing the vertex degrees of the original network; i.e., the number of edges is unchanged. Similarly, a disassortative network can be produced with the rewiring operation in the mirror method.

We first use the Barabási-Albert (BA) model [16] to initialize a neutrally degree-mixing scale-free network with 5000 vertices and the average degree $\langle k\rangle=4$. Since the degree-mixing patterns of most real-world networks fall in the region of [−0.3, 0.3] [13], we reproduce a group of networks with the XS algorithm to regulate their assortative coefficients into this range. An equal percentage of coopera-

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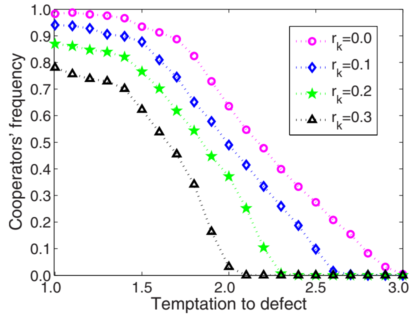

FIG. 1. (Color online) Frequency of cooperators, f_c, as a function of the temptation to defect, b, in the scale-free networks with different assortative coefficients r_k=0.0,0.1,0.2,0.3, respectively. The networks have 5000 vertices and 4 connections per vertex. Each datum is averaged over 10 different networks with 10 runs for each network.

tion or defection strategies is randomly initialized among the individuals. We characterize the cooperative behaviors with two crucial quantities: one is the extinction threshold of cooperators, b_c, above which the cooperators are extinct in the network, and the other is the frequency of cooperators, f_c, which is obtained by averaging over 10 000 generations after the transient time of 20 000 generations.

Figure 1 shows the results of the cooperators' frequency f_c versus the temptation to defect b of a group of scale-free networks, whose assortative coefficients vary from 0.0 to 0.3 [17]. We observe that, compared with the case of the uncorrelated network (r_k=0.0), the cooperation in assortative networks is significantly inhibited, because not only the cooperators' frequency of assortative networks is lower than that of uncorrelated networks, but also the cooperators are earlier to disappear in the former case than in the latter one.

Now, we focus our attention on the role of the assortative mixing to the extinction threshold of cooperators. It has been pointed out that in a BA scale-free network, the hubs play a prominent role in maintaining the cooperation [10], which uniformly select vertices with large degree or small degree as their neighbors. Therefore, on the one hand, hubs can directly communicate with each other. On the other hand, they share a few common neighbors and a defector is hard to invade a C-hub. For example, we set the most connected vertex of a BA network as defector and the other vertices as cooperators at the initial state. Figure 2 shows how the hub defector (D-hub) influences its cooperative neighbors in the network. In Fig. 2(a), since the D-hub obtains a higher payoff at the beginning, its small-degree neighbors have the tendency to learn its strategy; therefore, the number of cooperators around the D-hub decreases rapidly and the D-hub's payoff in return reduces quickly. However, the payoffs of those large-degree neighbors around the D-hub decrease slightly because they share a few neighbors. In the BA network of Fig. 2, the second largest-degree vertex is directly connected to the D-hub with 12 shared neighbors. Consequently, the cooperative neighbors around the C-hub only decrease 10% under the influence of the D-hub. Hence, the C-hub always holds on its initial strategy and its payoff is higher than those of the defectors, which results in the fact

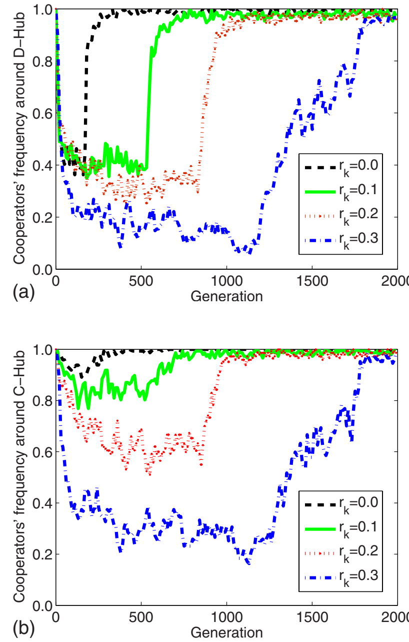

FIG. 2. (Color online) Cooperation frequency of hubs' neighbors as a function of generation when the assortative coefficients r_k of a group of scale-free networks vary from 0.0 to 0.3. The network have 5000 vertices and average degree of $\langle k \rangle$=4. Initially only the maximum degree vertex is the defector and others are cooperators. The temptation to defect b=1.5. Evolution of cooperators around (a) the largest-degree defector with 149 connections and (b) the second largest-degree cooperator with 147 neighbors are plotted. As r_k increases from 0.0 to 0.3, the two hubs share 12, 25, 43, 65 neighbors, respectively.

that the D-hub has the tendency to learn the strategy of its cooperator neighbor and become cooperator. Therefore, in uncorrelated scale-free networks, C-hubs are efficiently protected from the invasion of defectors, which promotes the emergence of cooperation.

However, when an uncorrelated scale-free network becomes more and more assortative mixing, the large-degree vertices tend to compose a core group sharing more neighbors than before, which promotes the diffusion of defectors and inhibits the emergence of cooperation in assortatively mixing networks. We illustrate this phenomenon with the subgraph shown in Fig. 3, where two hubs share m lowly connected neighbors. At the initial state, only one hub is defector and the other vertices are cooperators. The C-hub has the tendency to become defector because its payoff $\rho_c m$ is always less than the D-hub's payoff $\rho_c mb$, where ρ_c is the cooperator percentage of neighbors among the two hubs. Then, the rest of cooperators (lowly connected vertices) will vanish in the network with two D-hubs. Figure 2(b) shows

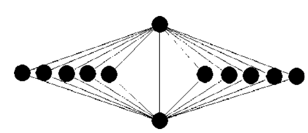

FIG. 3. Illustration of a subgraph, in which two connected central vertices have the same neighbors.

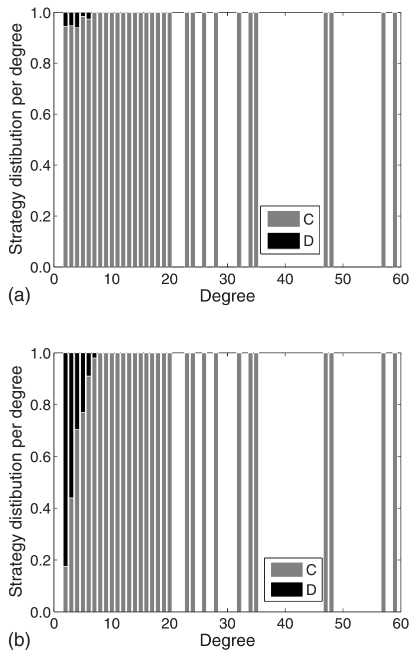

FIG. 4. Distribution of stationary strategies in a scale-free network with (a) r_k=0.0, (b) r_k=0.3, and b=1.5. Cooperators (C) and defectors (D) are denoted by gray bars and black bars, respectively. Each bar adds up to a total fraction of 1 per degree, and the gray and black fractions are directly proportional to the relative percentage of the respective strategy for each degree k. The data is obtained by averaging 10 runs on a network with 1000 vertices and 4 connections per vertex so that the figures can be seen clearly.

how the neighbors of the C-hub with the second highest degree are invaded by the D-hub with the highest degree in an assortative network. It can be observed that the more shared neighbors exist between the two hubs, the more vertices are willing to become defectors and the fewer cooperator neighbors surround the C-hub. Therefore, the ability to self-sustain the cooperation becomes weaker when the hubs closely stick together, which means a defector is easier to invade the hubs and to diffuse among the individuals, resulting in the cooperators vanishing earlier in assortative scale-free networks.

Moreover, when b is less than b_c, the cooperators' frequency in assortative networks is also lower than that of the uncorrelated ones. To explain it, we plot the relative strategy distribution per degree of the networks with r_k=0.0 and 0.3 at the steady state, as shown in Fig. 4. There exist more defectors with small degrees in an assortative network than those in an uncorrelated network, and the C-hubs sustain each other in both categories of networks. In an assortative network, vertices with similar degrees tend to be interconnected, and thus the lowly connected vertices are willing to select vertices with a small degree rather than hubs as their neighbors. Therefore, the influence of the hubs to small-degree vertices is weaker in assortative networks, and the defection action is easier to diffuse among the small-degree vertices. Hence, the assortatively mixing pattern inhibits the emergence of cooperation.

Similarly, we investigate how the emergence of cooperation is affected by the degree-mixing pattern in disassortative

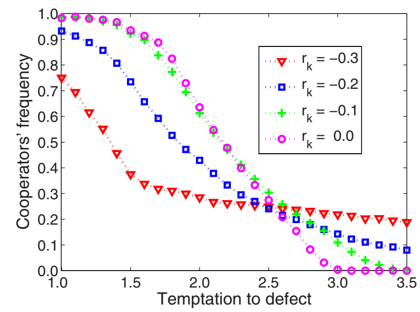

FIG. 5. (Color online) Frequency of cooperators, f_c, as a function of b in the scale-free networks with r_k=−0.3,−0.2,−0.1,0.0, respectively. Other network parameters are the same as in Fig. 3.

networks. As shown in Fig. 5, the frequency of cooperators in disassortative networks is lower than that in the uncorrelated case for a wide range of b. In a disassortative network, there are few interconnections among hubs, the sustainability of cooperation among hubs is destroyed, and a hub can only influence the actions of its local neighbors. In Fig. 6(a), we initialize half of the vertices as cooperators and the other half as defectors. Figure 6(b) shows that at steady state the hubs still hold onto their initial strategies and only those small-degree vertices change their strategies. Therefore, the frequency of cooperators of disassortative networks is lower than that of uncorrelated networks. Furthermore, as shown in Fig. 5, the extinction threshold of cooperators, b_c, becomes higher when r_k decreases, which implies that the cooperators can be sustained for a larger temptation in disassortative networks. That is due to the fact that a C-hub is only surrounded

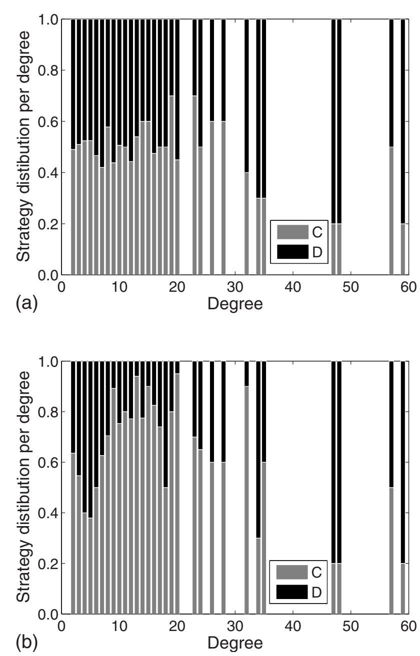

FIG. 6. Distribution of strategies in a disassortative network with r_k=−0.3. (a) Initial strategies distribution and (b) stationary strategies distribution for b=1.5, where the strategies distribution of hubs at the end time are unchanged comparing with the initial case. Other network parameters are same as in Fig. 4.

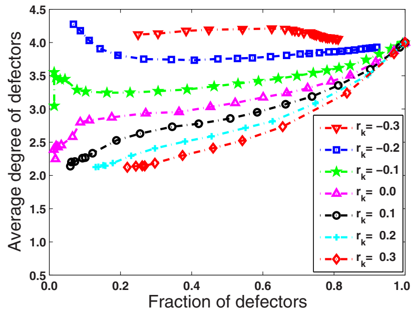

FIG. 7. (Color online) Average degree of defectors as a function of the fraction of defectors in the scale-free networks with different mixing patterns. Networks parameters are the same as those in Figs. 3 and 6.

by low-degree neighbors; it is easy to insist on its initial strategies and compose a cluster of cooperators with its neighbors to withstand the invasion of defectors. Therefore, the disassortatively mixing pattern destroys the communication among hubs, which, on the one hand, leads to the disappearance of cooperation sustainability among hubs and the decrease of cooperators' frequency and, on the other hand, protects the cooperation from extinction for a larger region of temptation.

Finally, we observe the average degree of defectors in the networks with different degree-mixing patterns in Fig. 7. It can be found that for the same fraction of defectors, the average degree of defectors in assortative networks is always lower than that in uncorrelated networks, which indicates that the defectors have fewer connections in the former case than those in the latter case. When the degree-mixing pattern becomes disassortative, there exist hubs which hold on their initial defection strategy, and thus the average degree of defectors increases.

In conclusion, in this Brief Report we addressed the roles of degree-mixing patterns in cooperative evolution. Our study supports the conjecture that uncorrelated networks are more willing to promote the emergence of cooperation than assortative or disassortative networks [18]. In particular, in the case of assortative networks, both the frequency and the extinction threshold of cooperators decrease, since the dense core group weakens not only the cooperation sustainability among hubs but also the communications from hubs to small-degree vertices. In contrast, when a network is mixed disassortatively, the isolation among hubs results in the fact that a hub tends to hold on its initial strategy and the cooperators' frequency of a disassortative network is therefore less than that of an uncorrelated network, while the cooperators are difficult to disappear in the former case. Last but not least, we should point out that our findings in this Brief Report mainly focus on the influence of degree mixing on the cooperation emergence of static networks. On the other hand, the coevolution of strategy and structure lead to self-organized dynamical networks exhibiting abundant degree-mixing patterns [19], where the roles of mixing patterns in cooperation deserve further more investigations in the near future.

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