Stochastic electromagnetic radiation of complex sources

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The emission of electromagnetic radiation by localized complex electric charge and current distributions is studied. A statistical formalism in terms of general dynamical multipole fields is developed. The appearing coefficients are treated as stochastic variables. Hereby as much as possible *a priori* physical knowledge is exploited. First results of simulated statistical electromagnetic fields as a function of position are presented. Sampling this field at one point approximates its resulting probability density.

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I. INTRODUCTION

Understanding the radiation of electromagnetic sources is becoming increasingly more important. Exact analytical calculation of the electromagnetic fields is only feasible in a few cases. Alternatively, one needs to make various approximations and/or perform extensive numerical calculations. For complex sources, however, such computations may become extremely cumbersome especially if there are no simplifications due to, for example, frequency band limits and distances. One may even doubt the usefulness of such a deterministic approach in many cases, because small changes in a complex source may drastically change the electromagnetic field pattern. These considerations actually stem from the study of electromagnetic fields in complex cavities—see [1–10].

Here we propose a statistical approach to address the electromagnetic radiation of complex sources. Since the present physical problem, however, essentially differs from the one in closed cavities, no methods can be taken over directly. As always in probability theory and stochastics the choice of random variables and the "establishment" of their *a priori* probability distributions is crucial. These depend on the actual system; i.e., the underlying physics and cannot be replaced by philosophical principles [11].

Only assuming that the sources are localized enables an expansion of the radiating fields outside the source region by means of generalized multipoles [12,13]. In this way one obtains exact general solutions of Maxwell's equations in terms of infinitely many unknown coefficients. These complex numbers are in principle exactly determined by the sources. In practice, of course, a finite number, essentially depending on the extension of the source, will yield the desired accuracy. Computation of the multipole coefficients for complex sources is usually illusory. Therefore we propose to treat them as stochastic variables, compressing all the intricate but hopefully irrelevant details of the source under consideration. Apart from respecting Maxwell's theory, it is tried to put in more available physical knowledge like an extension of the source, typical values of appearing currents, and (approximate) symmetries. Beyond that, various assumptions concerning the statistics need to be made. Only experiments can eventually justify our approach.

The general concept of random fields in radiation theory has of course been established for some time and can be found in textbooks like [11]. An important early contribution was given by Wolf [14]. An energy conservation law for randomly fluctuating electromagnetic fields is derived in [15]. Recent developments are a study of coherence for stochastic scalar electromagnetic sources and fields [16] and the investigation of stochastic interactions between electromagnetic fields and systems [17,18].

The outline of this paper is as follows. First, the general multipole expansion of radiation fields is reviewed. The main part introduces the statistics of the multipole coefficients. We have chosen an approach, which includes physical constraints due to, for instance, symmetries. It is discussed which quantities can be evaluated with the assumptions so far. Next, the choice of a probability density and the actual implementation are addressed. The following section presents field simulations exploiting pseudorandom generators for Gaussian and uniform probability distributions. Finally, we summarize the formalism and results and propose further studies for the near future.

II. MULTIPOLE EXPANSION

Electromagnetic radiation is generated by charge and current densities. Assuming that these source distributions are localized, a general, systematic expansion of the electromagnetic fields by means of multipoles is developed. It is valid everywhere outside the source region. For given harmonic systems, exact expressions for the multipole coefficients in terms of charge and current densities are derived. These results are valid for arbitrary source dimension, wavelength, and distance—with the caveat that the observation point be outside a spherical surface completely enclosing all sources. Examples of well-defined current distributions are explicitly worked out in order to compare several field expressions. Given the multipole coefficients the formalism enables the calculation of the radiation field at arbitrary locations (outside the above-mentioned sphere containing the electromagnetic sources).

We start with the theory of electrodynamics, in particular Maxwell's equations, Poynting's theorem, and the general multipole fields. Two examples of localized current distributions are treated next: linear current and circular loop current. The theoretical work in this section essentially follows the

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elaborate framework of Jackson ([12], Chap. 9); we include it to make this paper self-contained. SI units are used.

A. Maxwell equations and Poynting theorem

Maxwell's equations, describing the electric field $\vec{E}$, the magnetic field $\vec{H}$, and their interaction with charge and current density ρ and $\vec{j}$, respectively, can be written for harmonically varying fields and sources of frequency ω as

$$\epsilon_0 \vec{\nabla}\cdot\vec{E}=\rho, \quad \vec{\nabla}\times\vec{E}=ikZ_0\vec{H},$$

$$\vec{\nabla}\cdot\vec{H}=0, \quad \vec{\nabla}\times\vec{H}=\vec{J}-ikZ_0^{-1}\vec{E}, \tag{1}$$

where $Z_0=(\mu_0/\epsilon_0)^{1/2}$ is the impedance of free space, wave number $k=\omega/c$, and velocity of light $c=(\epsilon_0\mu_0)^{-1/2}$. The corresponding wavelength is given by $\lambda=2\pi c/\omega=2\pi/k$. Charge and current density are not independent; combining the source equations gives the continuity equation

$$i\omega\rho=\vec{\nabla}\cdot\vec{J}. \tag{2}$$

With the assumed time $e^{-i\omega t}$ dependence of field and sources we have as the physical electric field

$$\vec{E}(\vec{r},t)=\mathrm{Re}[\vec{E}(\vec{r})e^{-i\omega t}]=\tfrac{1}{2}[\vec{E}(\vec{r})e^{-i\omega t}+\vec{E}^*(\vec{r})e^{i\omega t}]. \tag{3}$$

Other quantities follow analogously. For time averages of products we take the real part of the product of one complex quantity with the complex conjugate of the other.

The complex Poynting vector is defined as

$$\vec{S}=\tfrac{1}{2}(\vec{E}\times\vec{H}^*). \tag{4}$$

The corresponding harmonic energy densities read

$$w_e=\tfrac{1}{4}\epsilon_0(\vec{E}\cdot\vec{E}^*), \quad w_m=\tfrac{1}{4}\mu_0(\vec{H}\cdot\vec{H}^*). \tag{5}$$

The total time-averaged energy density is given by their sum. Herewith the Poynting theorem for harmonic fields can be formulated:

$$\frac{1}{2}\int_V \vec{J}^*\cdot\vec{E}d^3r+2i\omega\int_V(w_e-w_m)d^3r+\oint_S\vec{S}\cdot\vec{n}da=0, \tag{6}$$

where $\vec{n}$ is the outward normal to the closed surface S, the boundary of the volume V.

With the aid of the time-averaged Poynting vector we get the time-averaged power per unit solid angle:

$$\frac{dP}{d\Omega}=\frac{1}{2}\mathrm{Re}[r^2\vec{e}_r\cdot(\vec{E}\times\vec{H}^*)], \tag{7}$$

where $\vec{e}_r$ is the unit vector in the $\vec{r}$ direction. The total power reads

$$P=\int\left(\frac{dP}{d\Omega}\right)d\Omega=\frac{1}{2}\int\mathrm{Re}[r^2\vec{e}_r\cdot(\vec{E}\times\vec{H}^*)]d\Omega. \tag{8}$$

B. Multipole expansion

In this section we present the general multipole expansion of the electromagnetic fields $\vec{E}$ and $\vec{H}$ in a source-free region of space. In other words, generic solutions of Maxwell's equations (1) with $\rho=0$ and $\vec{J}=0$ are given. For details of the derivation we refer to [12,13]; we restrict ourselves to the essential results. These read in spherical coordinates (r,θ,ϕ)

$$\vec{H}(\vec{r})=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[a_E(l,m)f_l(kr)\vec{X}_{lm}-\frac{i}{k}a_M(l,m)\vec{\nabla}\times g_l(kr)\vec{X}_{lm}\right],$$

$$\vec{E}(\vec{r})=Z_0\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[\frac{i}{k}a_E(l,m)\vec{\nabla}\times f_l(kr)\vec{X}_{lm}+a_M(l,m)g_l(kr)\vec{X}_{lm}\right], \tag{9}$$

with normalized vector spherical harmonics

$$\vec{X}_{lm}(\theta,\phi)=\frac{1}{\sqrt{l(l+1)}}\vec{L}Y_{lm}(\theta,\phi) \quad (l\geqslant 1) \tag{10}$$

and $\vec{X}_{lm}=0$ for $l=0$ [12]. We can therefore restrict ourselves to $l\geqslant 1$. The functions Y_{lm} are the well-known spherical harmonics, and the differential operator

$$\vec{L}=-i(\vec{r}\times\vec{\nabla}) \tag{11}$$

is familiar as $\hbar^{-1}$ times the orbital angular momentum operator from quantum mechanics. The radial functions f and g are of the form

$$f_l(kr)=C_l^{(1)}h_l^{(1)}(kr)+C_l^{(2)}h_l^{(2)}(kr),$$

$$g_l(kr)=A_l^{(1)}h_l^{(1)}(kr)+A_l^{(2)}h_l^{(2)}(kr), \tag{12}$$

with the spherical Hankel functions h. The latter are combinations of the spherical Bessel functions:

$$h_l^{(1)}(x)=j_l(x)+in_l(x),$$

$$h_l^{(2)}(x)=j_l(x)-in_l(x). \tag{13}$$

The coefficients $a_E(l,m)$ and $a_M(l,m)$ specify the amounts of electric and magnetic multipole fields. They are determined by the scalars $\vec{r}\cdot\vec{E}$ and $\vec{r}\cdot\vec{H}$ via

$$Z_0a_E(l,m)f_l(kr)=-\frac{k}{\sqrt{l(l+1)}}\int Y_{lm}^*\vec{r}\cdot\vec{E}d\Omega,$$

$$a_M(l,m)g_l(kr)=\frac{k}{\sqrt{l(l+1)}}\int Y_{lm}^*\vec{r}\cdot\vec{H}d\Omega. \tag{14}$$

These expressions also determine the relative proportions in Eqs. (12): knowledge of $\vec{r}\cdot\vec{E}$ and $\vec{r}\cdot\vec{H}$ at two different radii r_1 and r_2 completely specifies the fields. In the case of *only* outgoing waves at infinity one has $A_l^{(2)}=C_l^{(2)}=0$ in Eqs. (12). Thus we can choose $f_l(kr)=g_l(kr)=h_l^{(1)}(kr)$ in the expansion of the fields everywhere outside the source region. Another consequence of this boundary condition is that the scalars $\vec{r}\cdot\vec{E}$ and $\vec{r}\cdot\vec{H}$ at one radius r determine the electromagnetic field.

C. Sources of radiation

The multipole fields are generated by sources—i.e., charge and current distributions. We assume these to be harmonic and to be localized. Furthermore, the point $\vec{r}$ in which the fields are considered is supposed to be outside a spherical surface completely enclosing the sources. In this case it is possible to uniquely relate the multipole coefficients $a_E(l,m)$ and $a_M(l,m)$ to the charge and current distributions. In source-free regions the electric field is divergenceless. Since this is technically advantageous, the field

$$\vec{E}' = \vec{E} + \frac{i}{\omega\epsilon_0}\vec{J} \tag{15}$$

is accordingly introduced. Herewith, inhomogeneous wave equations for the scalars $\vec{r}\cdot\vec{H}$ and $\vec{r}\cdot\vec{E}'$ are obtained.

Their solutions read in terms of the angular momentum operator (11)

$$\vec{r}\cdot\vec{H} = \frac{i}{4\pi}\int \frac{e^{ik|\vec{r}-\vec{x}|}}{|\vec{r}-\vec{x}|}\vec{L}\cdot\vec{J}(\vec{x})d^3x,$$

$$\vec{r}\cdot\vec{E} = -\frac{Z_0}{4\pi k}\int \frac{e^{ik|\vec{r}-\vec{x}|}}{|\vec{r}-\vec{x}|}\vec{L}\cdot[\vec{\nabla}\times\vec{J}(\vec{x})]d^3x, \tag{16}$$

where it has been required that there be only outgoing waves at infinity. By means of the spherical-wave expansion of the Green function, one can derive [12] for the electric multipole coefficient

$$a_E(l,m) = \frac{-ik^2}{\sqrt{l(l+1)}}\int Y^*_{lm}\{c\rho\partial_r[rj_l(kr)] + ik(\vec{r}\cdot\vec{J})j_l(kr)\}d^3r. \tag{17}$$

Analogously, one eventually obtains as magnetic multipole coefficient

$$a_M(l,m) = \frac{-ik^2}{\sqrt{l(l+1)}}\int Y^*_{lm}\{\vec{\nabla}\cdot(\vec{r}\times\vec{J})j_l(kr)\}d^3r. \tag{18}$$

These expressions can be extended to include intrinsic magnetizations [12]. It should be emphasized that these results are exact, valid for arbitrary frequency and source size. We end this section by discussing two physically interesting limiting cases.

1. Small sources

If the source dimensions are very small compared to the wavelength, $kd \ll 1$, the multipole coefficients simplify considerably and can be related to the familiar multipole moments

$$a_E(l,m) \approx \frac{ck^{l+2}}{i(2l+1)!!}\sqrt{\frac{l+1}{l}}q_{lm}, \tag{19}$$

$$a_M(l,m) \approx \frac{ik^{l+2}}{(2l+1)!!}\sqrt{\frac{l+1}{l}}M_{lm}, \tag{20}$$

with electrostatic and magnetic multipole moments

$$q_{lm} = \int r^l Y^*_{lm}\rho d^3r, \tag{21}$$

$$M_{lm} = -\frac{1}{l+1}\int r^l Y^*_{lm}\vec{\nabla}\cdot(\vec{r}\times\vec{J})d^3r. \tag{22}$$

In this limit the charge density determines the electric multipole fields, whereas the magnetic multipole fields are related to the current density.

2. Radiation zone

The far (radiation) zone is defined as distances $r \gg \lambda$, implying $kr \gg 1$. For a general localized source distribution the radiation fields in the far-zone approach

$$\vec{H} \to \frac{\exp ikr}{kr}\sum_{l=1}^{\infty}\sum_{m=-l}^{l}(-i)^{l+1}[a_E(l,m)\vec{X}_{lm} + a_M(l,m)\vec{e}_r\times\vec{X}_{lm}],$$

$$\vec{E} \to Z_0\vec{H}\times\vec{e}_r. \tag{23}$$

The time averaged power radiated per unit solid angle [cf. Eq. (7)] is

$$\frac{dP}{d\Omega} = \frac{Z_0}{2k^2}\left|\sum_{l=1}^{\infty}\sum_{m=-l}^{l}(-i)^{l+1}[a_E(l,m)\vec{X}_{lm}\times\vec{e}_r + a_M(l,m)\vec{X}_{lm}]\right|^2. \tag{24}$$

The total radiated power (8) becomes an incoherent sum of different multipole coefficients:

$$P = \frac{Z_0}{2k^2}\sum_{l=1}^{\infty}\sum_{m=-l}^{l}[|a_E(l,m)|^2 + |a_M(l,m)|^2]. \tag{25}$$

We refer to [12] for further elucidations. Here we recall that Eq. (25) is derived for the radiation zone, where this power is supposed to be measured. In the remainder of this paper we tacitly imply this.

D. Examples

1. Linear antenna

As a first example of the multipole expansion of the radiation from a localized source, we consider a linear, infinitely thin, center-fed antenna [12]. It lies along the z axis from $-d/2 \leq z \leq d/2$, and its current is given by

$$I(z,t) = I(|z|)e^{-i\omega t}, \tag{26}$$

with $I(d/2)=0$. For $r < d/2$ the current density reads in spherical coordinates

$$\vec{J}(\vec{r}) = \frac{I(r)}{2\pi r^3}[\delta(\cos\theta - 1) - \delta(\cos\theta + 1)]\vec{r}. \tag{27}$$

The corresponding charge density follows from Eq. (2):

$$\rho(\vec{r}) = \frac{-i}{\omega}\partial_r I(r)\left[\frac{\delta(\cos\theta-1)-\delta(\cos\theta+1)}{2\pi r^2}\right]. \tag{28}$$

The physical, real charge and current densities are proportional to sin ωt and cos ωt, respectively.

The magnetic multipole coefficients $a_M(l,m)$ vanish because $\vec{r}\times\vec{J}=0$. The electric multipoles are nonzero for odd l and $m=0$:

$$a_E(l,0) = \frac{k}{2\pi}\sqrt{\frac{4\pi(2l+1)}{l(l+1)}} \times \int_0^{d/2}\{-\partial_r[rj_l(kr)\partial_r I] + rj_l(kr)(\partial_r^2 I + k^2 I)\}dr. \tag{29}$$

Since we want to compare multipole expansions for given current distributions, we once more exploit the assumption of an infinitely thin antenna. Consequently, modifications of the current due to radiation can be neglected. Here we prescribe the current as

$$I(z) = I\sin(kd/2 - k|z|). \tag{30}$$

The last part of the integrand vanishes in this case, and we readily obtain

$$a_E(l,0) = \frac{I}{\pi d}\sqrt{\frac{4\pi(2l+1)}{l(l+1)}}[(kd/2)^2 j_l(kd/2)], \quad l \text{ odd}. \tag{31}$$

Note that no assumptions concerning the source dimension relative to the wavelength have been made; actually, these results are studied for a half-wave ($kd=\pi$) and a full-wave antenna ($kd=2\pi$) in [12].

2. Circular loop

As the next example we consider an infinitely thin circular loop carrying a constant current I_0. The derivation of these multipole coefficients has, to the best of our knowledge, not been presented in the literature. Therefore we provide some details of this calculation. No assumptions concerning wavelength, radius, and observation distance are made—with the exception of the caveat $r>a$, where a is the radius of the loop.

The current density is in cylindrical coordinates given by

$$\vec{J}(\vec{r})e^{-i\omega t} = I_0\delta(z)\delta(\varrho - a)e^{-i\omega t}\hat{e}_\phi = j_\phi(\varrho,z)e^{-i\omega t}\hat{e}_\phi. \tag{32}$$

Since $\vec{\nabla}\cdot\vec{J}=0$, Eq. (2) yields a vanishing charge density ρ ($\omega\neq 0$). Moreover, we see that $\vec{r}\cdot\vec{J}=0$. As a consequence, the electric multipole coefficients vanish: $a_E(l,m)=0$. Computing $\vec{r}\times\vec{J}$ gives

$$\vec{r}\times\vec{J} = j_\phi(\varrho,z)(-z\hat{e}_\varrho + \varrho\hat{e}_z). \tag{33}$$

At this point, it is convenient to transform to spherical coordinates:

$$\vec{r}\times\vec{J} = -\frac{I_0}{r\sin\theta}\delta(r-a)\delta(\cos\theta)\hat{e}_\theta. \tag{34}$$

Then it is easily verified that

$$\vec{\nabla}\cdot(\vec{r}\times\vec{J}) = -\frac{I_0}{r\sin\theta}\delta(r-a)\partial_\theta\delta(\cos\theta). \tag{35}$$

Inserting this into Eq. (18) and doing the trivial r integration yields

$$a_M(l,m) = aj_l(ka)\frac{iI_0k^2}{\sqrt{l(l+1)}} \times \int_0^{2\pi}d\phi\int_0^{\pi}d\theta Y_{lm}^*(\theta,\phi))\partial_\theta\delta(\cos\theta). \tag{36}$$

We insert the explicit expression for the spherical harmonics and obtain, after integrating over the angle ϕ, only $m=0$ contributions:

$$\begin{aligned} a_M(l,0) &= aj_l(ka)\frac{2\pi iI_0k^2}{\sqrt{l(l+1)}}\sqrt{\frac{2l+1}{4\pi}} \times \int_0^{\pi}d\theta P_l(\cos\theta)\partial_\theta\delta(\cos\theta) \\ &= -aj_l(ka)\frac{2\pi iI_0k^2}{\sqrt{l(l+1)}}\sqrt{\frac{2l+1}{4\pi}}P_l^1(0). \end{aligned} \tag{37}$$

P_l and P_l^m are the Legendre polynomials and the associated Legendre functions, respectively. Only odd multipoles are nonvanishing because $P_l^1(0)\sim\sin\pi l/2$.

III. STATISTICS OF COEFFICIENTS

The exact calculation of multipole coefficients is only feasible for relatively simple source distributions like the examples presented above. For more complex charge and current distributions the integrations get cumbersome, even using a numerical approach. Moreover, the precise form of the source densities may even be unknown; this will be more the rule than the exception. Consider, for instance, electronic subsystems or various electronic components present on a printed circuit board (PCB). Lacking detailed source distributions, the challenge is to estimate the high-frequency radiation in the near and intermediate fields. The idea of a statistical approach for complex systems is to generate the relevant multipole coefficients by means of a probability density. Just as in statistical electromagnetics for complex, enclosed systems we try to put in "as much physics as possible" [3]; this is disputably known as "objective Bayesian statistics" [1,19]. In cavities, however, the crucial assumption is the very large number of modes [2]. Here we stick to harmonic analysis and the statistics relies on the complexity of the extended sources.

A. General formalism

In this section we introduce statistical electromagnetics for complex radiators, including the boundary condition of outgoing waves at infinity. Further physical input is discussed below. A joint probability distribution is assumed for the multipole coefficients $a_A(l,m)$, where the index A denotes E or M. These are complex quantities, with independent real and imaginary parts $a_A(l,m)=\alpha^A_{lm}+i\beta^A_{lm}$. Furthermore, we assume independence for different l,m values, implying that the joint distribution is a product:

$$P(\{\text{all } lm|\alpha^E_{lm},\beta^E_{lm},\alpha^M_{lm},\beta^M_{lm}\}) = \prod_{l=1}^{\infty}\prod_{m=-l}^{l} P^E_{lm}(\alpha^E_{lm})\tilde{P}^E_{lm}(\beta^E_{lm})P^M_{lm}(\alpha^M_{lm})\tilde{P}^M_{lm}(\beta^M_{lm}). \quad (38)$$

Single-variable distributions are normalized:

$$\int d\varphi P^A_{lm}(\varphi) = \int d\varphi \tilde{P}^A_{lm}(\varphi) = 1 \text{ for all } lm. \quad (39)$$

Means, variances, and covariances are defined in the usual way [11]. We presuppose that all means are zero:

$$\langle\alpha^A_{lm}\rangle = \langle\beta^A_{lm}\rangle = 0 \Leftrightarrow \langle a_A(l,m)\rangle = \langle\alpha^A_{lm}\rangle + i\langle\beta^A_{lm}\rangle = 0. \quad (40)$$

Consequently, we get for the covariances

$$\begin{aligned}\langle\langle a^A_{lm}a^{A'*}_{l'm'}\rangle\rangle &= \langle a^A_{lm}a^{A'*}_{l'm'}\rangle - \langle a^A_{lm}\rangle\langle a^{A'*}_{l'm'}\rangle \\ &= \langle\langle\alpha^A_{lm}\alpha^{A'}_{l'm'}\rangle\rangle + \langle\langle\beta^A_{lm}\beta^{A'}_{l'm'}\rangle\rangle \\ &= \delta_{ll'}\delta_{mm'}\delta_{AA'}\{(\sigma^{\alpha A}_{lm})^2 + (\sigma^{\beta A}_{lm})^2\} \\ &= \delta_{ll'}\delta_{mm'}\delta_{AA'}|\sigma^A_{lm}|^2, \end{aligned} \quad (41)$$

in terms of standard deviations. Note that we also have used the statistical independence of the various multipole coefficients. Instead of describing the complete electromagnetic field as stochastic with mean zero, one can actually add a deterministic field. The latter "mean field" may be expressed by the general multipole expansion as well—i.e., by their known coefficients.

The radiated power is expressed in multipole coefficients by Eq. (25) and can be written as

$$P = \sum_{l=1}^{\infty} P_l(k), \quad (42)$$

where the "angular power spectrum" [20] is given by

$$P_l(k) = \frac{Z_0}{2k^2}\sum_{m=-l}^{l}[|a_E(l,m)|^2 + |a_M(l,m)|^2]. \quad (43)$$

Its mean is obtained by replacing coefficients by expectation values—i.e.,

$$|a_E|^2 \to \langle a_E a_E^*\rangle = \langle\langle a_E a_E^*\rangle\rangle,$$

$$|a_M|^2 \to \langle a_M a_M^*\rangle = \langle\langle a_M a_M^*\rangle\rangle. \quad (44)$$

Hence it is explicitly given by

$$P_l(k) = \frac{Z_0}{2k^2}\sum_{m=-l}^{l}\{|\sigma^E_{lm}|^2 + |\sigma^M_{lm}|^2\}. \quad (45)$$

In this way a connection between the variances of the probability densities and the average radiated power is established. The latter is supposed to be measured in the far field.

B. Physical considerations

In studies of the cosmic background radiation [20], the question is whether the sky is statistically isotropic. The latter implies that the analog multipole coefficients for the temperature are realizations of Gaussian random variables with zero mean and standard deviations independent of m. In the problem under consideration, the radiation stems from a complex source distribution—in principle, yielding the multipole coefficients. Explicitly, using statistics means we need to presuppose probability densities $P^A_{lm}(X)$. The random variable is here denoted by X; it should be noted that its realization is frequency dependent. In other words, the densities may depend on k as well. Recall that pure mathematics is in general not able to decide which *a priori* density is valid for a certain situation [11]. Here we therefore continue by means of physical and geometrical arguments.

1. Dimensional analysis

It is useful to note that the multipole coefficients have the dimension of a magnetic field—unit ampere per meter. Let us assume a typical (average) current I and a typical extension d; the remaining independent dimensionful parameter in the problem is the wave number k. The last two combine into the dimensionless variable $u=kd$. Using expressions (14), (17), and (18) and dimensional arguments yields for a coefficient $a_A(l,m)$

$$\sqrt{4\pi}a_A(l,m) = C_l\frac{I}{d}\phi_A(l,m) = C_l u^2\frac{I}{d}\psi_A(l,m) = C_l u\frac{I}{d}\xi_A(l,m), \quad (46)$$

with $C_l=\sqrt{\frac{1}{l(l+1)}}$. Here we have introduced dimensionless functions, depending on u, resulting from the integrations

$$\phi_A(l,m) \propto \int v^2 j_l(v)\cdots dv, \quad \psi_A(l,m) \propto \int j_l(v)\cdots dv,$$

$$\xi_A(l,m) \propto \int v j_l(v)\cdots dv. \quad (47)$$

An infinite number of choices is of course possible. Our choices correspond to inclusion of the Jacobian in ϕ, its omission in ψ—corresponding to the results for linear and circular current: cf. Eqs. (31) and (38)—and an intermediate choice ξ. For the power, now considered as a function of u, we respectively obtain

$$\begin{aligned}4\pi P(u) &= \frac{Z_0 I^2}{2u^2}\sum_{l=1}^{\infty}\sum_{m=-l}^{l} C_l^2[|\phi^E_{lm}(u)|^2 + |\phi^M_{lm}(u)|^2] \\ &= \frac{1}{2}Z_0 I^2 u^2\sum_{l=1}^{\infty}\sum_{m=-l}^{l} C_l^2[|\psi^E_{lm}(u)|^2 + |\psi^M_{lm}(u)|^2] \\ &= \frac{1}{2}Z_0 I^2\sum_{l=1}^{\infty}\sum_{m=-l}^{l} C_l^2[|\xi^E_{lm}(u)|^2 + |\xi^M_{lm}(u)|^2]. \end{aligned} \quad (48)$$

It is clear that all u dependence is contained in ξ, whereas some powers of u are factored out in ϕ and ψ. In a statistical treatment this should be reflected in the respective variances (45).

2. Power, source extension, and maximal l

Given the relation between the total power and the variances of the multipole coefficients, two approaches are in principle possible. If one has some idea about the total radiated power, or has measured it, then it serves as a constraint on the variances. Their sum, given by Eq. (45), must be equal to the known power. Alternatively, one may specify the probability densities of the multipole coefficients. Then the power density and its mean, etc., follow. In that case no information on the total power is put in. Randomly drawing the coefficients, with a chosen density, gives a sample of the power distribution in both cases. Recall that we have introduced some typical value of the current I. It appears natural to consider the dimensionless "power" [12]

$$\mathcal{P} = \frac{4\pi P}{Z_0 I^2} = \frac{1}{2}\sum_{l=1}^{\infty} C_l^2 \sum_{m=-l}^{l} \{|\xi_E(l,m)|^2 + |\xi_M(l,m)^2|\}, \quad (49)$$

which trivially follows from Eq. (48).

The extension of the complex source is important. Of course, the length scale is governed by the wavelength as is expressed by the dimensionless variable $u=kd$. It is the argument of the appearing Bessel functions $j_l(u)$. Depending on the choice of dimensionless function ϕ, ψ, or ξ, the Bessel function is multiplied by u^n, $n=0,1,2$. Other intricate details of the sources are hidden in the defining integrals (47) and are supposed to be covered statistically. Note that in an actual (sub)problem the quantity u is fixed. As a rule of thumb one may state "the larger u, the more multipoles l are necessary" in the expansion of the electromagnetic field. Let us try to refine this somewhat.

The spherical Bessel functions $j_l(x)$ are oscillating functions which have their first maximum somewhere around $x_M \approx l+1$. This is good enough in the range of our interest and for our purposes; for large l values this number needs to be corrected. For $l=50$, the maximum is around $x_M=53$. Starting from the first maximum the envelope of the $j_l(x)$ is approximately given by $1/x$. This means that for $x \geqslant x_M$, $xj_l(x)=O(1)$. Next, it is observed that contributions can be neglected for $x \ll x_M$. These features are illustrated in Fig. 1. Alternatively, one considers $xj_l(x)$ for given x and increasing l; see Figs. 2 and 3. As a consequence, for a given u it is sufficient to take $\max(\text{int}(u),1)$ l values into account; note that we are on the safe side since there is an additional l-dependent geometrical suppression factor in Eq. (48), even taking into account that there are $2l+1$ terms for a given l. In practice, it may turn out that even fewer multipoles might be sufficient.

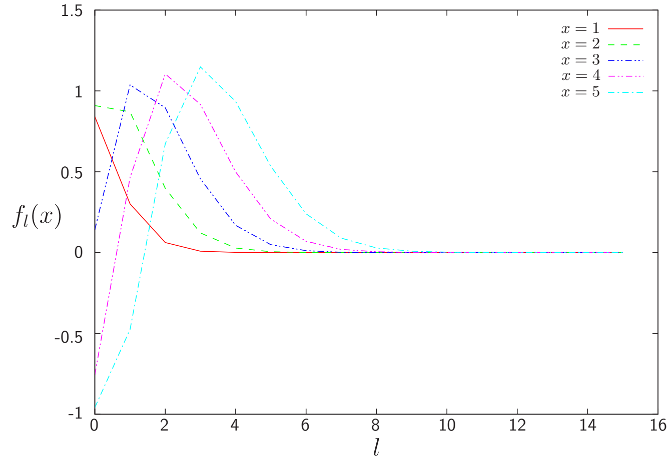

FIG. 2. (Color online) Varying l: $xj_l(x)$, $x=1,2,\ldots,5$.

The functions ξ_{lm} are obtained by integrating $vj_l(v)\ldots$, which is oscillating and order 1. The integral, therefore, is also order 1. Just as integrating, say, $\exp iv$, no additional length factor appears. This is confirmed by the examples we have explicitly calculated, i.e.—the linear antenna and the circular loop. Therefore we choose the $\xi_{lm}(u)$ as independent random variables with mean 0 and variance 1—i.e., $\langle\xi\rangle=0$ and $\langle\langle|\xi|^2\rangle\rangle=\langle\langle\xi\xi^*\rangle\rangle=\langle\xi\xi*\rangle=1$. Recall their relation to the original multipole coefficients [cf. Eq. (47)],

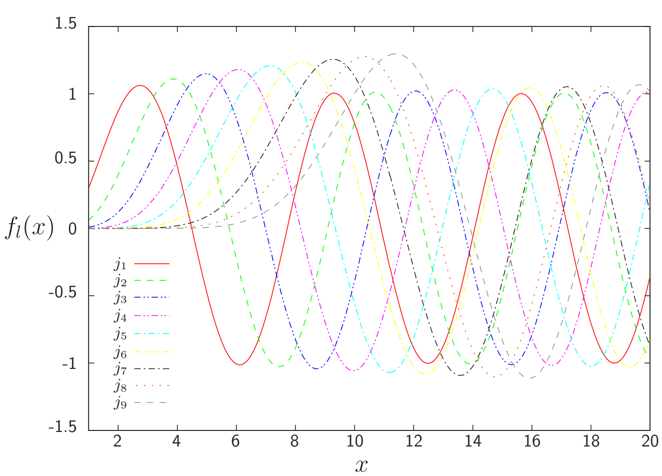

FIG. 1. (Color online) $f_l(x)=xj_l(x)$ for $l=1,9$.

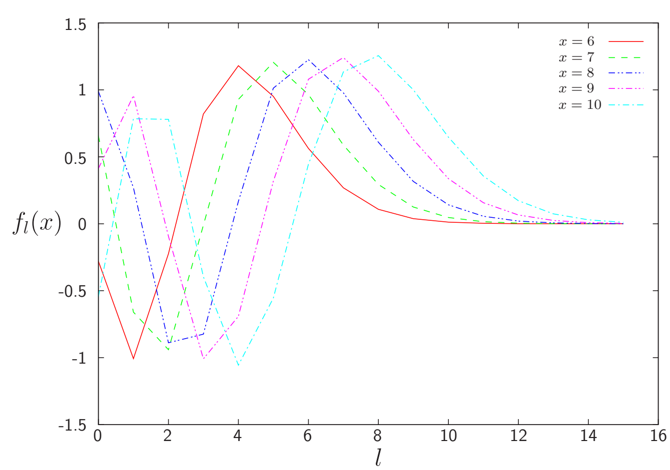

FIG. 3. (Color online) Varying l: $xj_l(x)$, $x=6,7,\ldots,10$.

$$\sqrt{4\pi}a_E(l,m) = C_l k I \xi_E(l,m),$$

$$\sqrt{4\pi}a_M(l,m) = C_l k I \xi_M(l,m). \tag{50}$$

As stochastic variables, the $\xi(l,m)$ become independent of u. For different u, of course, different variates generally will appear. Note that the linearly related random multipole coefficients $a(l,m)$ depend on u, or eventually on the wave number k. Obviously, the variances increase for increasing frequency and current. The typical extension of the source does not show up explicitly in the $\xi(l,m)$ and, consequently, in the electromagnetic fields and power. It implicitly, however, determines the number of multipoles. In other words, the variances of $\xi(l,m)$ are set to zero beyond the maximal l.

Alternatively, one still may include some additional dependence on u in the variances of $\xi_A(l,m)$. This requires, however, more *a priori* knowledge of the complex source. If this is not available, we propose to stick to the framework outline above. This is also in line with the principle of a "minimum description length" [21].

3. Symmetries

Invariance under transformations of equations describing physical phenomena or systems are called symmetries. Invariance up to a factor of -1 appears as well; sometimes such an equation or system is called antisymmetric. Maxwell's equations, governing electromagnetics, are invariant under rotations, spatial reflections, and time reversal [12]. Symmetry in a particular problem—i.e., a symmetric electromagnetic source distribution—reduces the number of independent, nonzero multipoles. It may become clear that some multipoles are necessarily zero or it can yield relations between coefficients. This reduction in of the number of degrees of freedom is a well-known property in classical physics as well as in quantum mechanics. Symmetries are also related to conservation laws—for example, in classical mechanics translation invariance implies momentum conservation.

The transformation properties of electromagnetic fields and sources are studied in [12]. The active view is adopted; i.e., the physical system transforms—for example, rotates—whereas the coordinate axes are fixed. A table in [12] contains the transformation properties of mechanical and electromagnetic physical quantities. With respect to space inversion it is important to realize that the magnetic field is a pseudovector. The electric field, current density, and Poynting vector are polar vectors. The charge density is a scalar. Relations (14) are useful to investigate the consequences of symmetry for the multipole coefficients in a particular problem. Note that $\vec{r}\cdot\vec{E}$ is a scalar while $\vec{r}\cdot\vec{H}$ is a pseudoscalar. Alternatively, one may use the relations (17) and (18) to study symmetries.

From these equations it immediately follows that (a) radial current, $a_M(l,m)=0$; (b) longitudinal (curl-free) current, $a_E(l,m)=0$; and (c) angular current (zero radial component) *and* vanishing charge density, $a_E(l,m)=0$. In the case of azimuthal symmetry—i.e., the system is invariant under rotation around the z axis—all multipole coefficients with $m \neq 0$ vanish; $a_E(l,0)$ and $a_M(l,0)$ are possibly nonzero.

Spatial reflections are relevant for the applications we are interested in. Invariance under such a reflection corresponds to a symmetry. Of course, a particular system may reflect more symmetries. In the case of an antisymmetry an additional factor (-1) needs to be inserted. It can be shown that (a) $z\to -z$ symmetry, $a_E(l,m)=0$ for $l+m$ odd and $a_M(l,m)=0$ for $l+m$ even; (b) $x\to -x$ symmetry, $a_E(l,m)=a_E(l,-m)$ and $a_M(l,m)=-a_M(l,-m)$; and (c) $y\to -y$ symmetry, $a_E(l,m)=(-1)^m a_E(l,-m)$ and $a_M(l,m)=(-1)^{m+1}a_M(l,-m)$. A planar configuration of sources is approximately invariant under such a reflection.

It may be elucidating to review the presented examples of a linear antenna and circular loop. The linear antenna has a $z\to -z$ antisymmetry, $x\to -x$ and $y\to -y$ symmetries, and azimuthal symmetry. Combining the consequences yields vanishing multipoles except for $a_E(l,0)$ with odd l. The spatial symmetries of the circular loop are reversed and it also reflects azimuthal symmetry. As a consequence, only $a_M(l,0)$ with odd l are nonzero. These *a priori* implications of invariances are consistent with the results of the explicit calculations.

Complex electromagnetic current sources still may reflect a symmetry, perhaps in an approximate way. In our physics-based statistical approach, we aim at taking into account such an invariance. This means that either a subset of multipole coefficients is set to zero from the onset or particular coefficients are related to each other. The number of independent random multipole variables is reduced in this way. It renders the model simpler, again in accordance with the philosophy of a minimum description length [21], and puts in "as much physics as possible" once more [3]. The explicit consequences for the statistical approach are presented below.

4. Quest for the probability distribution

By means of the framework developed so far, various averaged quantities can already be expressed in the wave number k, typical current I, and extension d. The statistics of the coefficients has been presented in Eqs. (39)–(44). The random variables $a_A(l,m)$ and $\xi_A(l,m)$ are related via Eqs. (50). For the latter zero mean and unit variance have been assumed. All averages of quantities that are linear and/or quadratic in the multipole coefficients follow. In other words, means depending on first- and second-order moments can be determined without further specification of the probability distribution of the stochastic variables $\xi(l,m)$.

In order to calculate higher-order moments—and quantities involving these—or to determine statistical fields one explicitly needs probability distributions in order to generate samples of the random variables $\xi(l,m)$. The principal difficulties are outlined in [11]. Even if we are convinced that we have selected the appropriate random variables for our problem, we can at best make a definite probability distribution plausible. In the end only experiments can decide.

At this point we propose the Gaussian probability density for the coefficients $\xi(l,m)$—that is, for their real and imaginary parts, also assumed to be independent. It can be merely seen as a hypothesis. Alternatively, one may argue that such a coefficient receives many contributions from various current elements present in the extended complex source. Assuming independence, or at least a sufficient weak dependence, and implicitly some other mathematical conditions [11] in order to validate the use of the central limit theorem, would indeed, in the limit of "many $\to\infty$," guarantee a Gaussian probability density. Alternatively, one can apply the principle of maximum entropy [22,23] to the stochastic variables $\xi(l,m)$. Given the constraints of prescribed mean and variance, this principle also yields the Gaussian density. Although maximum entropy is an appealing concept, it only can be applied with the necessary prudence [11]. Noting these caveats, we will select the Gaussian distribution for further research, in particular for doing simulations. For comparison, a uniform probability density will be used as well.

IV. IMPLEMENTATION

In order to do simulations of stochastic electromagnetic fields, we first introduce the maximum l value—that is, $L=\max(\text{int}(u),1)$, $1\leqslant l\leqslant L$. Next, the actual real random variables for which variates will be generated are defined. Implications of possible symmetries are derived. Finally, results of the simulations are presented.

A. Real random variables

The complex random variables are written as

$$\xi_E(l,m)=\rho^E_{lm}+i\zeta^E_{lm},\quad \xi_M(l,m)=\rho^M_{lm}+i\zeta^M_{lm}, \tag{51}$$

where the various ρ and ζ are real independent random variables. Because $\langle\xi\rangle=0$ and $\langle|\xi^2|\rangle=1$, they have mean zero and variance $\sigma^2=\langle\rho^2\rangle=\langle\zeta^2\rangle=\frac{1}{2}$. As mentioned above, we choose their probability distribution to be Gaussian. For the dimensionless power we now get

$$\mathcal{P}=\frac{1}{2}\sum_{l=1}^{L}C_l^2\sum_{m=-l}^{l}[(\rho^E_{lm})^2+(\zeta^E_{lm})^2+(\rho^M_{lm})^2+(\zeta^M_{lm})^2]. \tag{52}$$

The stochastic fields are given by

$$\vec{H}(\vec{r})=\frac{I}{\sqrt{4\pi}}\sum_{l=1}^{L}\sum_{m=-l}^{l}C_l[(\rho^E_{lm}+i\zeta^E_{lm})kf_l(kr)\vec{X}_{lm} - i(\rho^M_{lm}+i\zeta^M_{lm})\vec{\nabla}\times g_l(kr)\vec{X}_{lm}],$$

$$\vec{E}(\vec{r})=\frac{Z_0I}{\sqrt{4\pi}}\sum_{l=1}^{L}\sum_{m=-l}^{l}C_l[i(\rho^E_{lm}+i\zeta^E_{lm})\vec{\nabla}\times f_l(kr)\vec{X}_{lm} + (\rho^M_{lm}+i\zeta^M_{lm})kg_l(kr)\vec{X}_{lm}]. \tag{53}$$

The probability densities for the dimensionless power and electromagnetic field in principle follow by transformation of variables and is given by an integral [11]. Because of the different factors C_l^2 in Eq. (52), we cannot further simplify the integral expression for the power (for constant $C_l^2=C$ the result would be a γ distribution [11]). Since the real and imaginary parts of the field components are weighted sums of Gaussians, their density is also Gaussian with mean zero and a variance given by the sum of the weight factors. This is an extension of a corollary in [11]. It can be proven as follows. Let the x_n, $n=1$, N, be independent Gaussian random variables with mean zero and variance σ^2 and let $y=\sum_{n=1}^{N}f_nx_n$. Its probability distribution is given by

$$P(y)=\int\cdots\int\delta\left(y-\sum_{n=1}^{N}f_nx_n\right)P_G(x_1)\cdots P_G(x_N)dx_1\cdots dx_N, \tag{54}$$

where $P_G(z)=(2\pi\sigma^2)^{-1/2}\exp(-z^2/2\sigma^2)$. For the characteristic function one obtains

$$\begin{aligned}\Phi(k)&=\int e^{iky}P(y)dy=\int\cdots\int\exp\left(ik\sum_{n=1}^{N}f_nx_n\right)P_G(x_1)P_G(x_2)\cdots P_G(x_N)dx_1\cdots dx_N\\&=\int\exp(ikf_1x_1)P_G(x_1)dx_1\int\exp(ikf_2x_2)P_G(x_2)dx_2\cdots\int\exp(ikf_Nx_N)P_G(x_N)dx_N=\Phi_G(kf_1)\Phi_G(kf_2)\cdots\Phi_G(kf_N),\end{aligned} \tag{55}$$

where $\Phi_G(k)=\exp\left(-\frac{1}{2}\sigma^2k^2\right)$ is the Gaussian characteristic function (mean zero, variance σ^2). Therefore, we get

$$\Phi(k)=\exp\left(-\tfrac{1}{2}\sigma^2Fk^2\right),\quad \text{with } F=f_1^2+f_2^2+\cdots+f_N^2. \tag{56}$$

Transforming back finally yields a density

$$P(y)=\frac{1}{2\pi}\int\Phi(k)e^{-iky}dk=\frac{1}{\sqrt{2\pi F\sigma^2}}\exp\left(-\frac{y^2}{2F\sigma^2}\right), \tag{57}$$

which completes the proof.

For the average dimensionless power and the power density one easily gets

$$\langle \mathcal{P} \rangle = \sum_{l=1}^{L} (2l+1) C_l^2 = \frac{4\pi}{Z_0 I^2} \langle P \rangle. \tag{58}$$

For a fixed frequency, $\langle \mathcal{P} \rangle$ is determined by L and thus by the typical extension d. Consequently, power and current are not independent but related by Eq. (58). On the other hand, having values or estimates of power and current at fixed frequency determines L and therefore the typical source extension d.

B. Consequences of symmetry

It is useful to distinguish two cases. Coefficients may identically vanish, or some coefficients are related to other ones. That respectively means that some random variables are not present or that random variables are dependent of others—in fact, they are identical up to a phase factor. In general, combinations are possible as well.

1. Vanishing coefficients, fewer random variables

In the case of vanishing multipole coefficients due to symmetry, there are no concomitant random variables and their number therefore decreases. Since the angular power spectrum (43) gets contributions of fewer terms—say, n_l instead of $2l+1 (1 \leqslant n_l \leqslant 2l+1)$—we define $\tilde{C}_l = \sqrt{\frac{2l+1}{n_l}} C_l$ for the electric contribution and analogously $\bar{C}_l$ for the magnetic one. This yields for the dimensionless power

$$\begin{aligned}\mathcal{P} &= \frac{1}{2}\left\{ \sum_{l=1}^{L}{}' \sum_{m=-l}^{l}{}' \tilde{C}_l^2 |\xi_E(l,m)|^2 + \sum_{l=1}^{L}{}'' \sum_{m=-l}^{l}{}'' \bar{C}_l^2 |\xi_M(l,m)^2| \right\} \\ &= \frac{1}{2}\left\{ \sum_{l=1}^{L}{}' \tilde{C}_l^2 \sum_{m=-l}^{l}{}' (\rho_{lm}^E)^2 + (\zeta_{lm}^E)^2 \right. \\ &\quad \left. + \sum_{l=1}^{L}{}'' \bar{C}_l^2 \sum_{m=-l}^{l}{}'' (\rho_{lm}^M)^2 + (\zeta_{lm}^M)^2 \right\},\end{aligned} \tag{59}$$

where the prime and double prime denote that certain lm are to be excluded from the summation. For the remaining random variables we adopt the same statistics as above: ρ and ζ are Gaussian with mean zero and $\sigma^2 = 1/2$. The average of $\mathcal{P}$ then becomes

$$\langle \mathcal{P} \rangle = \frac{1}{2}\left\{ \sum_{l=1}^{L}{}' (2l+1) C_l^2 + \sum_{l=1}^{L}{}'' (2l+1) C_l^2 \right\}. \tag{60}$$

The expression for the stochastic multipole fields follows analogously as

$$\begin{aligned}\vec{H}(\vec{r}) &= \frac{I}{\sqrt{4\pi}} \left\{ \sum_{l=1}^{L}{}' \sum_{m=-l}^{l}{}' \tilde{C}_l [(\rho_{lm}^E + i\zeta_{lm}^E) k f_l(kr) \vec{X}_{lm}] \right. \\ &\quad \left. - i \sum_{l=1}^{L}{}'' \sum_{m=-l}^{l}{}'' \bar{C}_l [(\rho_{lm}^M + i\zeta_{lm}^M) \vec{\nabla} \times g_l(kr) \vec{X}_{lm}] \right\},\end{aligned}$$

$$\begin{aligned}\vec{E}(\vec{r}) &= \frac{Z_0 I}{\sqrt{4\pi}} \left\{ \sum_{l=1}^{L}{}' \sum_{m=-l}^{l}{}' \tilde{C}_l [i(\rho_{lm}^E + i\zeta_{lm}^E) \vec{\nabla} \times f_l(kr) \vec{X}_{lm}] \right. \\ &\quad \left. + \sum_{l=1}^{L}{}'' \sum_{m=-l}^{l}{}'' \bar{C}_l [(\rho_{lm}^M + i\zeta_{lm}^M) k g_l(kr) \vec{X}_{lm}] \right\}.\end{aligned} \tag{61}$$

Their mean is obviously zero. The covariance matrices $\langle\langle H_j^* E_n \rangle\rangle$ $\langle\langle H_j^* H_n \rangle\rangle$ and $\langle\!\langle E_j^* E_n \rangle\!\rangle$ do not vanish. We omit the explicit, rather complicated expressions since they are not of immediate use.

2. Related coefficients, dependent random variables

Apart from vanishing, coefficients can be related to each other. We restrict ourselves to the following relations:

$$\xi_E(l,-m) = e^{i\phi_E(m)} \xi_E(l,m), \quad \xi_M(l,-m) = e^{i\phi_M(m)} \xi_M(l,m), \tag{62}$$

a slight generalization from those discussed in Sec. III B 3.

Herewith we obtain for the dimensionless power

$$\begin{aligned}\mathcal{P} &= \frac{1}{2}\left\{ \sum_{l=1}^{L}{}' \tilde{C}_l^2 \left[|\xi_E(l,0)|^2 + 2\sum_{m=1}^{l}{}' |\xi_E(l,m)|^2 \right] \right. \\ &\quad \left. + \sum_{l=1}^{L}{}'' \bar{C}_l^2 \left[|\xi_M(l,0)|^2 + 2\sum_{m=1}^{l}{}'' |\xi_M(l,m)^2| \right] \right\} \\ &= \frac{1}{2}\left\{ \sum_{l=1}^{L}{}' \tilde{C}_l^2 \left[(\rho_{l0}^E)^2 + (\zeta_{l0}^E)^2 + 2\sum_{m=1}^{l}{}' (\rho_{lm}^E)^2 + (\zeta_{lm}^E)^2 \right] \right\} \\ &\quad + \frac{1}{2}\left\{ \sum_{l=1}^{L}{}'' \bar{C}_l^2 \left[(\rho_{l0}^M)^2 + (\zeta_{l0}^M)^2 + 2\sum_{m=1}^{l}{}'' (\rho_{lm}^M)^2 + (\zeta_{lm}^M)^2 \right] \right\}.\end{aligned} \tag{63}$$

It can readily be checked that its average is once again given by Eq. (60). Note that we still allow for vanishing coefficients.

Finally, we get for the random electromagnetic fields

$$\begin{aligned}\vec{H}(\vec{r}) &= \frac{I}{\sqrt{4\pi}} \left\{ \sum_{l=1}^{L}{}' \sum_{m=1}^{l}{}' \tilde{C}_l [(\rho_{lm}^E + i\zeta_{lm}^E) k f_l(kr) \vec{X}_{lm}] \right. \\ &\quad - i \sum_{l=1}^{L}{}'' \sum_{m=1}^{l}{}'' \bar{C}_l [(\rho_{lm}^M + i\zeta_{lm}^M) \vec{\nabla} \times g_l(kr) \vec{X}_{lm}] \\ &\quad + \sum_{l=1}^{L}{}' \sum_{m=-l}^{-1}{}' \tilde{C}_l [(\rho_{l\mu}^E + i\zeta_{l\mu}^E) e^{i\phi_E(\mu)} k f_l(kr) \vec{X}_{lm}] \\ &\quad - i \sum_{l=1}^{L}{}'' \sum_{m=-l}^{-1}{}'' \bar{C}_l [(\rho_{l\mu}^M + i\zeta_{l\mu}^M) e^{i\phi_M(\mu)} \vec{\nabla} \times g_l(kr) \vec{X}_{lm}] \\ &\quad + \sum_{l=1}^{L}{}' \tilde{C}_l [(\rho_{l0}^E + i\zeta_{l0}^E) k f_l(kr) \vec{X}_{l0}] \\ &\quad \left. - i \sum_{l=1}^{L}{}'' \bar{C}_l [(\rho_{l0}^M + i\zeta_{l0}^M) \vec{\nabla} \times g_l(kr) \vec{X}_{l0}] \right\}\end{aligned}$$

$$\vec{E}(\vec{r}) = \frac{Z_0 I}{\sqrt{4\pi}} \Bigg\{ \sum_{l=1}^{L}{}' \sum_{m=1}^{l}{}' \tilde{C}_l [i(\rho_{lm}^E + i\zeta_{lm}^E)\vec{\nabla} \times f_l(kr)\vec{X}_{lm}] + \sum_{l=1}^{L}{}'' \sum_{m=1}^{l}{}'' \bar{C}_l [(\rho_{lm}^M + i\zeta_{lm}^M) k g_l(kr)\vec{X}_{lm}] + \sum_{l=1}^{L}{}' \sum_{m=-l}^{-1}{}' \tilde{C}_l [i(\rho_{l\mu}^E + i\zeta_{l\mu}^E) e^{i\phi_E(\mu)} \vec{\nabla} \times f_l(kr)\vec{X}_{lm}] + \sum_{l=1}^{L}{}'' \sum_{m=-l}^{-1}{}'' \bar{C}_l [(\rho_{l\mu}^M + i\zeta_{l\mu}^M) e^{i\phi_M(\mu)} k g_l(kr)\vec{X}_{lm}] + \sum_{l=1}^{L}{}' \tilde{C}_l [i(\rho_{l0}^E + i\zeta_{l0}^E)\vec{\nabla} \times f_l(kr)\vec{X}_{l0}] + \sum_{l=1}^{L}{}'' \bar{C}_l [(\rho_{l0}^M + i\zeta_{l0}^M) k g_l(kr)\vec{X}_{l0}] \Bigg\}, \quad (64)$$

with $\mu=-m$. Obviously, to explicitly derive the covariances is straightforward but rather tedious.

V. SIMULATIONS

With the formalism developed we can simulate statistical radiation fields of complex electromagnetic sources. As explained above, several starting points can be chosen, all for a fixed frequency. First, one uses information on typical source extension and currents; this determines the average radiated power. Second, using *a priori* given extension and power fixes the typical current. Third, one obtains the typical source dimension by specifying the current and power. In all cases one can implement consequences of (approximate) symmetries. Initially we restrict ourselves to considering the first case; that is, we assume certain typical values of source extension and current in our simulations. Implementation of the other cases is straightforward. If there are no symmetries, all multipole coefficients $a_E(l,m), a_M(l,m)$ contribute for $1 \leqslant l \leqslant L$. They are determined by the complex Gaussian random variables $\xi_E(l,m)$ and $\xi_M(l,m)$, the latter having real and imaginary parts with mean 0 and variance 1/2. The simulations are done with two random generators for Gaussian variates, the first one stemming from [24], the second one having been developed by us. Since no differences are found, we only show results obtained with the first random generator. In all simulations the frequency was fixed to 960 MHz. We have chosen to depict the real part of the third component of the electric field E_z.

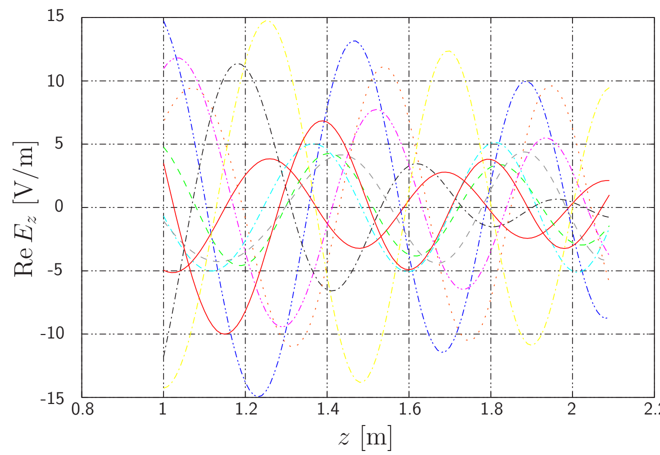

FIG. 5. (Color online) Ten simulations, $I=0.5$ A: $L=5$, $P=28.0$ W.

First, we show Re E_z as a function of z, $1.0\text{ m} \leqslant z \leqslant 2.1$ m, for fixed $x=1.0$ m and $y=1.0$ m in Fig. 4. The results of ten simulations are shown for $L=2$. Recall that L is directly related to the extension of the source. Analogous results for $L=5,9$ are presented in Figs. 5 and 6. Note that the power increases for increasing L.

Second, we have fixed the point of observation to $\vec{r}=(1.0,1.0,1.49)$ m. In repeated simulations we computed 10^5 samples of the electric field value, thereby approximating its probability distribution. We repeated this for the values $1 \leqslant L \leqslant 8$. The obtained results are given in Fig. 7.

Indeed, the densities resemble Gaussian ones with L-dependent standard deviations. This has been predicted in Sec. IV A. In order to further establish this apparent agree-

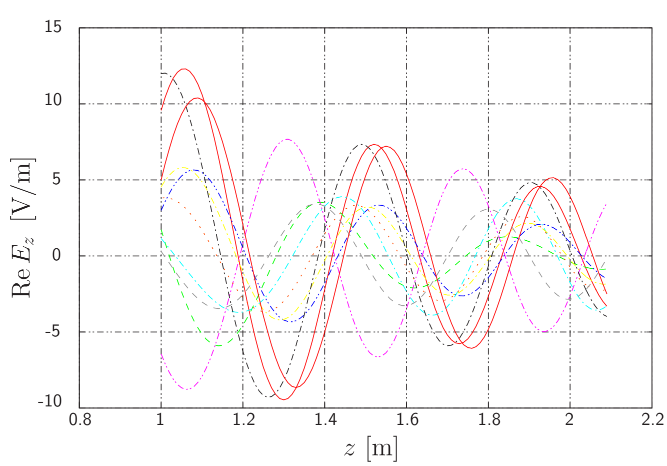

FIG. 4. (Color online) Ten simulations, $I=0.5$ A: $L=2$, $P=17.5$ W.

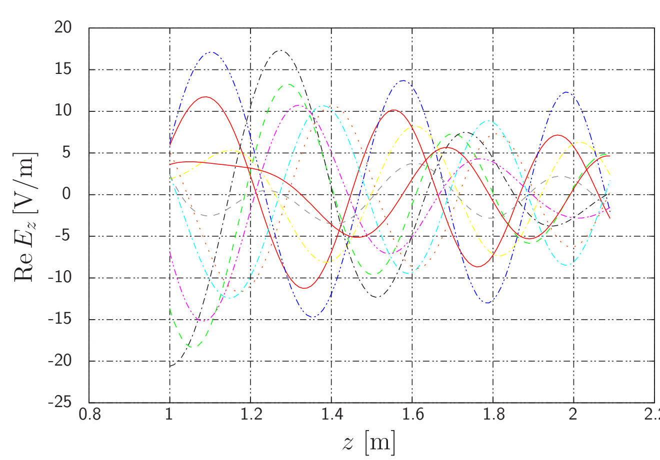

FIG. 6. (Color online) Ten simulations, $I=0.5$ A: $L=9$, $P=35.7$ W.

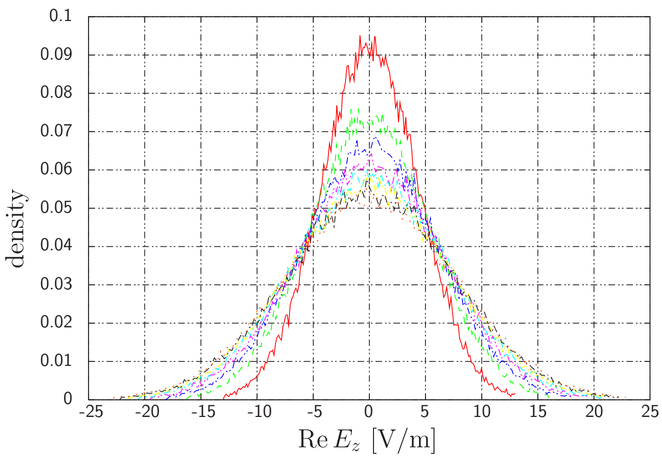

FIG. 7. (Color online) Simulated field probability densities at a fixed point for $L=1,2,\ldots,8$: Gaussian distribution.

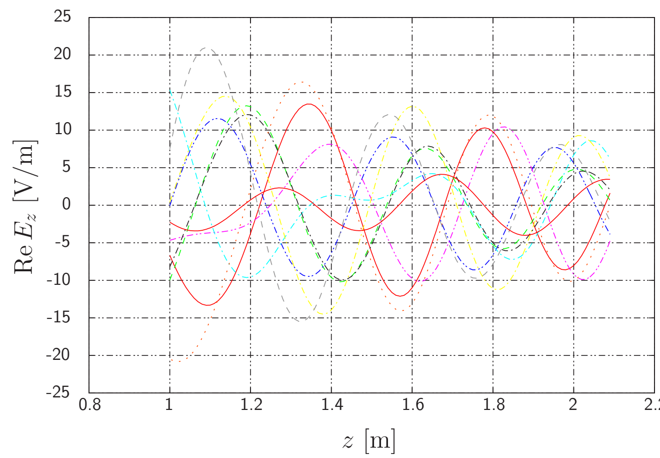

FIG. 9. (Color online) Ten simulations, uniform distribution, $I=0.5$ A: $L=9$, $P=35.7$ W.

ment, Kolmogorov-Smirnov tests [24] have been performed. In all cases it has been confirmed that the obtained field variates are consistent with the Gaussian probability density.

Simulations starting with a uniform distribution for the multipole coefficients have been performed as well. As can be seen in Fig. 8 the obtained densities actually also resemble the Gaussian. For small $L \leqslant 3$, however, this is not confirmed by the Kolmogorov-Smirnov test. For $L \geqslant 7$ this test indicates that the distribution becomes Gaussian. For the intermediate L the test does not yield conclusive results. We interpret these results as the onset of the region where the central limit theorem is valid: the number of independent random variables becomes sufficiently large. For completeness we also show the results for the field component Re E_z obtained with the uniform probability density in Fig. 9. Next, we have fixed the total power and have performed simulations for increasing extension and, consequently, L. The typical current I decreases in this case. The field component Re E_z is shown in Figs. 10 and 11 for $L=3$ and $L=8$, respectively. Finally, we have once more approximated the resulting probability density for Re E_z at a fixed location by repeated sampling. Since the current I decreases for increasing L, the widths are much less varying than in the previous example with fixed current. If one starts with Gaussian distributions for the coefficients, it is confirmed that the resulting distribution is also Gaussian. For the uniform density, Gaussianity is achieved for larger extension, $L>3$.

VI. SUMMARY AND OUTLOOK

In order to describe the electromagnetic radiation by complex localized sources, a statistical formalism has been developed. It is based on the general multipole expansion of electromagnetic fields. In this way exact solutions of the Maxwell equations valid outside the source region are constructed. No further restriction on distances, source extension, and/or frequencies need to be imposed. Electric and magnetic fields are expressed in the so-called multipole coefficients. These are in principle determined by the electromagnetic charge and current distributions. In practice, the latter may be unknown for complex sources or, even if they are known, the coefficients possibly cannot be calculated in a

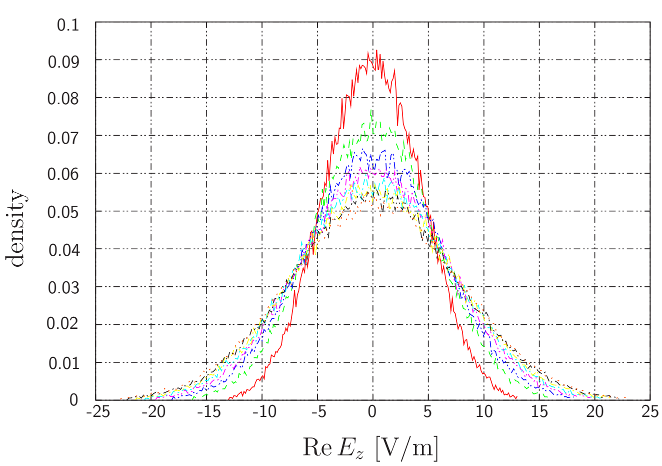

FIG. 8. (Color online) Simulated field probability densities at a fixed point for $L=1,2,\ldots,8$: uniform distribution.

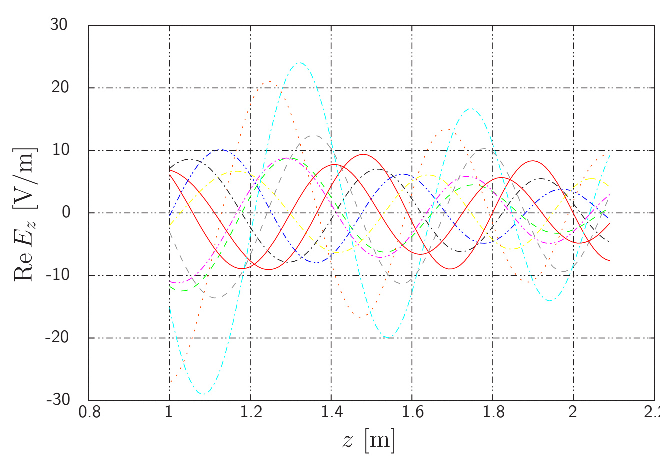

FIG. 10. (Color online) Ten simulations, $P=42.7$ W: $L=3$, $I=0.70$ A.

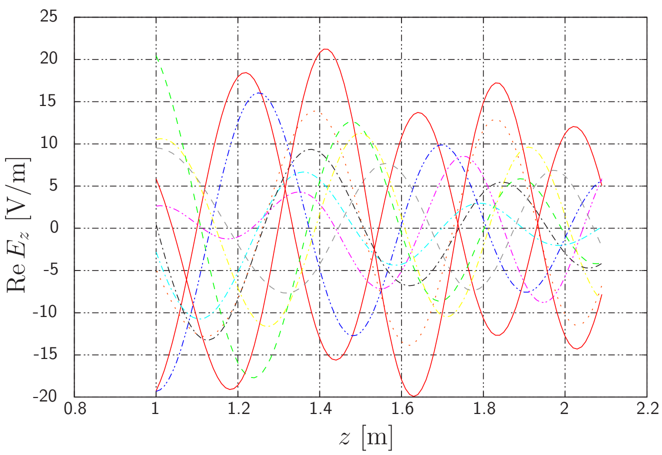

FIG. 11. (Color online) Ten simulations, $P=42.7$ W: $L=8$, $I=0.56$ A.

reliable and useful way. Therefore, it is proposed to consider the multipole coefficients as random variables.

In formulating this stochastic problem, we try to incorporate physics and geometry as much as possible. Symmetries imply that certain coefficients vanish or that the number of independent coefficients diminishes. Using dimensional arguments, the multipole coefficients are linearly related to dimensionless stochastic variables for which mean zero and unit variance are plausible assumptions. The total radiated power can be expressed in the latter by only assuming a typical value of the current. The harmonic electromagnetic fields follow immediately as a function of their wave number. The typical extension of the localized source is related to the number of necessary multipole coefficients. Herewith, various averages of fields and related quantities only depending on the first and second moments of the random coefficients can be calculated. It should be noted that field correlations are automatically included in this framework.

In order to proceed, a definite probability distribution has to be chosen. We have presented some arguments favoring the Gaussian density—for instance, the principle of maximum entropy. Strictly speaking, however, it is a hypothesis to be eventually tested with experimental data. Exploiting this density, the necessary variates can be generated in order to simulate statistical radiation fields of complex sources.

Simulations using this formalism have been performed. For a given frequency and extension of the source, results for the fields are obtained for fixed typical current and for fixed radiated power. The variates of the multipole coefficients are obtained by pseudorandom generators for the Gaussian probability density; for comparison, the uniform distribution has also been used. The field variables at a fixed point in space become stochastic as well. For Gaussian coefficients, it is derived and confirmed that the resulting field probability distribution is once more Gaussian. In the case of uniformly distributed coefficients, simulations only yield a Gaussian field density for a sufficiently large number of contributing multipoles.

In the near future, it is planned to extend such simulations. The implications of symmetries are interesting to include. We eventually aim at a comparison with experiments in order to judge the validity of this statistical formalism. The framework developed may be applied to various complex radiating sources. In particular, we aim at applications in the field of complex antennas and integrated circuits. Electromagnetic band-gap structure antennas, for example, show considerable variations in their measured performance which appear to be impossible to grasp by means of deterministic computations [25]. Other examples are dual-band antennas to be integrated in airplanes and the aforementioned integrated circuits. Statistical theory is also planned to be used in analyzing the electromagnetic radiation of larger sources, usually a structure enclosing internal sources, like electronic apparatus, ships, buildings, etc. Finally, we note that it may be valuable in assigning manufacturing tolerances, allowing variations in electromagnetic performance of electronic components and electric equipment.

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