Fokker-Planck-Kramers equations of a heavy ion in presence of external fields

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(Received 10 April 2007; published 3 August 2007)

In this work, we use the same strategy studied in our previous work [J. I. Jiménez-Aquino and M. Romero-Bastida, Phys. Rev. E **74**, 041117 (2006)] to solve exactly the Fokker-Planck (FP) and Fokker-Planck-Kramers (FPK) equations of a charged Brownian particle in a fluid (a heavy ion in a light gas) under the influence of external fields: a constant magnetic field and, in general, time-varying mechanical and electric fields. In our proposal, a time-dependent rotation matrix is introduced to transform the Langevin equation in the phase-space $(\mathbf{r},\mathbf{u})$ to a new space $(\mathbf{r}',\mathbf{u}')$. As a result, the transformed Langevin equations are very similar to those of ordinary Brownian motion in the presence of those time-varying external forces only, without the magnetic field; therefore, the associated FP and FPK equations can easily be solved in those transformed spaces. To solve these equations, we use the methods of solution developed by Chandrasekhar in the field-free case of ordinary Brownian motion. We also calculate a more general transition probability density in the velocity space by assuming an initial heavy-ion Maxwellian distribution at a temperature generally different from that corresponding to equilibrium, the same as that used by Ferrari [Physica A **163**, 596 (1990)].

DOI: 10.1103/PhysRevE.76.021106 PACS number(s): 05.40.−a, 02.50.−r

I. INTRODUCTION

In 2005, Simões and Lagos (SL) [1] proposed an exact solution method to the Fokker-Planck-Kramers (FPK) equation corresponding to a heavy ion immersed in a fluid in the presence of external fields, such as mechanical and electromagnetic fields, by combining both Czopnik and Garbaczewski's (CG) [2] "rotated" Stokes force and Ferrari's [3] gauge. As established in Ref. [1], the method of solution relies upon a transformation of the FPK equation into a similar field-free equation, in a similar manner to that advanced by Ferrari [3]. Accordingly, the solution of such a field-free equation is obtained by applying CG's strategy, with an assumed Gaussian distribution for the correlation functions of the appropriate variables. Another interesting problem for which the solution of the FPK equation is required is the anisotropic diffusion of charged particles in an external magnetic field, as studied by Holod *et al.* [4] and Zagorondy and Holod [5]. As an alternative method, in this work we extend the strategy of Ref. [6] to solve the FP and FPK equations to the situation in which the Brownian charged particle is under the action of mechanical and electromagnetic forces. Here, we assume the influence of a constant magnetic field $\mathbf{B}$ and in general time-dependent, but space-independent, mechanical $\mathbf{F}_{\text{mec}}(t)$ and electric $\mathbf{E}(t)$ external forces. Our proposal introduces a time-dependent rotation matrix to transform the Langevin equations in phase space $(\mathbf{r},\mathbf{u})$ to a new one $(\mathbf{r}',\mathbf{u}')$. As a consequence of this fact, the respective Langevin equations are very similar to those of ordinary Brownian motion in the presence of the transformed time-dependent external forces only, without the influence of the magnetic field. In the transformed space, the rotated noise term contains the magnetic field. It is shown that the statistical properties of the original noise and those of the rotated noise are the same, if the former satisfies the properties of Gaussian white noise. Under these circumstances, the FP and FPK equations associated, respectively, with the transformed velocity and phase spaces, are quite similar to those of the ordinary Brownian motion in the presence of an external field, and are therefore more easily solved than those formulated in the original space. To solve these equations in the transformed space, we use the same methods developed by Chandrasehar [7] in the field-free case, and extend them to the cases in which external fields are present. By returning to the original variables, we obtain the corresponding transition probability densities (TPDs), which are referred to as "fundamental" solutions of the FP and FPK equations, having as initial conditions the Dirac δ functions. However, for arbitrary initial probability distributions, it is possible, in principle, to calculate more general probability densities (PDs) than those of the fundamental TPD's. Such is the case of the velocity-space probability distribution, for which we calculate a more general PD by assuming an initial heavy-ion Maxwellian velocity distribution at a temperature T_0 different from that of equilibrium, T, like that studied by Ferrari [8] and Berman [9]. Our general PD will be compared with that obtained by Ferrari.

On the other hand, our phase-space fundamental solution is similar to, but not exactly the same as, that proposed by SL. The main reason is that our solution (which is a Gaussian distribution) is not obtained directly from the FPK equation in the original variables. Rather, it is calculated in a stricter way in the transformed phase space, whereas SL's solution is also a Gaussian distribution function, but imposed as an ansatz, as done by CG. To show the consistency of our phase-space fundamental solution, we first compare it with Ferrari's [3] fundamental solution, when only a time-varying electric field is present in the diffusion process. Next, we calculate the velocity-space solution through an integration process over the whole complete configuration space of the phase-space solution. The obtained velocity-space solution coin-

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cides with that calculated through the solution of the FP equation. In a similar way, the configuration-space solution is obtained through an integration over the whole velocity space of the phase-space solution. In addition, from the configuration-space fundamental solution, we can calculate the mean square displacements along the x, y, and z axes. A more general phase-space probability density is still under investigation. If we define P as the TPD for a single Brownian particle, then the probability density of a swarm of $n_{\rm tot}$ mutually independent large particles in a dense fluid of much smaller particles, defined as W, is given by $W \equiv n_{\rm tot}P$. Finally, to complement our work, we introduce Appendixes A and B, where we study the solutions of the transformed FP and FPK equations, respectively.

II. THE FP EQUATION OF A CHARGED PARTICLE IN EXTERNAL FIELDS

The Langevin equation describing the diffusion process of a charged particle embedded in a fluid in the presence of electromagnetic (via the Lorentz force) and mechanical $F_{\rm mec}$ forces can be written as

$$\dot{\mathbf{u}} = -\beta\mathbf{u} + \frac{q}{m}\mathbf{u}\times\mathbf{B} + \frac{q}{m}\mathbf{E} + \frac{\mathbf{F}_{\rm mec}}{m} + \mathbf{A}(t), \tag{1}$$

where q denotes the charge of the particle of mass m, and $\mathbf{A}(t)$ satisfies the properties of Gaussian white noise with zero mean value and correlation function

$$\langle A_i(t)A_j(t')\rangle = 2\lambda\,\delta_{ij}\delta(t-t'), \tag{2}$$

where $\lambda = \beta k_B T/m$ is the noise intensity and k_B is the Boltzmann constant. $\mathbf{F}_{\rm mec}$ and $\mathbf{E}$ are, in general, both space-independent and time-varying external forces. The magnetic field is assumed for simplicity to be constant and with direction along the z axis of a Cartesian reference frame, that is, $\mathbf{B}=(0,0,B)$ with B a constant. If we define the acceleration $\mathbf{a}(t) \equiv \mathbf{F}_{\rm mec}(t)/m + q\mathbf{E}(t)/m$, the foregoing equation can also be written as

$$\dot{\mathbf{u}} = -\beta\mathbf{u} + \mathbb{W}\mathbf{u} + \mathbf{a}(t) + \mathbf{A}(t), \tag{3}$$

where $\mathbb{W}$ is a real antisymmetric matrix given by

$$\mathbb{W} = \begin{pmatrix} 0 & \Omega & 0 \\ -\Omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4}$$

$\Omega = qB/mc$ being the Larmor frequency. To establish the Fokker-Planck equation associated with Eq. (3), this last equation must be written as

$$\dot{\mathbf{u}} = -\Lambda\mathbf{u} + \mathbf{a}(t) + \mathbf{A}(t), \tag{5}$$

where the matrix $\Lambda = \beta\mathbb{I} - \mathbb{W}$ now reads as

$$\Lambda = \begin{pmatrix} \beta & -\Omega & 0 \\ \Omega & \beta & 0 \\ 0 & 0 & \beta \end{pmatrix}, \tag{6}$$

$\mathbb{I}$ being the unit matrix. Equation (5) corresponds to a coupled system of equations in the (u_x, u_y) plane and is independent of the coordinate u_z, for which the corresponding evolution equation is the Langevin equation of the ordinary Brownian motion. The drift and diffusion coefficients are

$$D_i = -\Lambda_{ij}u_j + a_i,$$

$$D_{ij} = \lambda\,\delta_{ij}, \tag{7}$$

a_i being the components of vector $\mathbf{a}(t)$. Thus, the associated Fokker-Planck equation for the transition probability density of the velocity conditioned by initial data $\mathbf{u}(0) \equiv \mathbf{u}_0$ at time $t=0$, i.e., $P(\mathbf{u},t|\mathbf{u}_0)$, reads as

$$\frac{\partial P}{\partial t} + \mathbf{a}\cdot \mathrm{grad}_{\mathbf{u}}P = \mathrm{div}_{\mathbf{u}}(\Lambda\mathbf{u}P) + \lambda\nabla^2_{\mathbf{u}}P, \tag{8}$$

subject to the initial condition

$$P(\mathbf{u},0|\mathbf{u}_0) = C_0\delta(\mathbf{u}-\mathbf{u}_0), \tag{9}$$

where C_0 is a constant that will be determined later on. The strategy of solution of Eq. (8) following Chandrasekhar's [7] idea requires the solution of its associated first-order equation without the Laplacian, which involves the Lagrangian subsidiary system

$$\dot{\mathbf{u}} = -\Lambda\mathbf{u} + \mathbf{a}(t) \tag{10}$$

together with the three integrals

$$e^{\Lambda t}\mathbf{u} - \bar{\mathbf{a}}(t) = \mathbf{I}_1, \tag{11}$$

where $\mathbf{I}_1 = \mathbf{u}_0$ and

$$\bar{\mathbf{a}}(t) = \int_0^t e^{\Lambda s}\mathbf{a}(s)ds. \tag{12}$$

However, such a solution is not easy to calculate due to the coupling of the resulting equations. To solve this problem we use the same strategy of Ref. [6], where an alternative method of solution has been proposed, which relies upon a transformation of the Langevin equation by means of the change of variables

$$\mathbf{u}' = e^{-\mathbb{W}t}\mathbf{u}. \tag{13}$$

In this new velocity-space the Langevin equation (5) adopts the form

$$\dot{\mathbf{u}}' = -\beta\mathbf{u}' + \mathbf{a}'(t) + \mathbf{A}'(t), \tag{14}$$

such that

$$\mathbf{a}'(t) = \mathbb{R}^{-1}(t)\mathbf{a}(t), \quad \mathbf{A}'(t) = \mathbb{R}^{-1}(t)\mathbf{A}(t). \tag{15}$$

$\mathbb{R}(t) = e^{\mathbb{W}t}$ is an orthogonal rotation matrix given by

$$\mathbb{R}(t) = \begin{pmatrix} \cos\Omega t & \sin\Omega t & 0 \\ -\sin\Omega t & \cos\Omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{16}$$

in such a way that $\mathbb{R}^T(t) = \mathbb{R}^{-1}(t)$, i.e., the transpose is its inverse and therefore $R^{-1}(t) = e^{-\mathbb{W}t}$. The Langevin equation (14) is very similar to that of the ordinary Brownian motion in the presence of an external force $\mathbf{a}'$, which in this case is nothing but a rotation of the original external force $\mathbf{a}(t)$.

Similarly, $\mathbf{A}'(t)$ accounts for a rotation of the original fluctuating force $\mathbf{A}(t)$. In the transformed velocity space, it can be shown immediately from Eq. (14) that the drift and the diffusion coefficients are [6,10]

$$D'_i = -\beta u'_i + \mathbf{a}'_i,$$

$$D'_{ij} = \lambda \delta_{ij}. \tag{17}$$

This is because $(\mathbb{R}^{-1})_{ik}(t)(R^{-1})_{jk}(t)=\delta_{ij}$ and therefore the diffusion coefficient D'_{ij} is the same as that given in Eq. (7). This means that the noise $\mathbf{A}'(t)$ has the same statistical properties as $\mathbf{A}(t)$ if the latter satisfies the property of being a Gaussian white noise. Thus, the FP equation associated with the Langevin equation (14) for the transition probability density of the velocity conditioned by initial data $\mathbf{u}'(0)\equiv\mathbf{u}'_0$ at time $t=0$, i.e., $P'(\mathbf{u}',t|\mathbf{u}'_0)$, is obviously given by

$$\frac{\partial P'}{\partial t} + \mathbf{a}' \cdot \mathrm{grad}_{\mathbf{u}'} P' = \beta\, \mathrm{div}_{\mathbf{u}'}(\mathbf{u}'P') + \lambda \nabla^2_{\mathbf{u}'} P'. \tag{18}$$

This partial differential equation has the same algebraic structure as that associated with the ordinary Brownian motion in the presence of an external field $\mathbf{a}'$, whose solution can be calculated by using the same method developed by Chandrasekhar [7] to solve the FP in the field-free case $\mathbf{a}'=\mathbf{0}$, and subject to the initial condition

$$P'(\mathbf{u}',0|\mathbf{u}'_0) = \delta(\mathbf{u}'-\mathbf{u}'_0). \tag{19}$$

Such a solution is explicitly developed in Appendix A. Briefly stated, it is connected with the solution of the associated first-order equation without the Laplacian term, which involves the three integrals of the Lagrangian subsidiary system

$$\dot{\mathbf{u}}' = -\beta \mathbf{u}' + \mathbf{a}'(t), \tag{20}$$

which are given by

$$e^{\beta t}\mathbf{u}' - \bar{\mathbf{a}}'(t) = \mathbf{I}'_1, \tag{21}$$

where $\mathbf{I}'_1=\mathbf{u}'_0$ and

$$\bar{\mathbf{a}}'(t) = \int_0^t e^{\beta s}\mathbf{a}'(s)ds. \tag{22}$$

If we define the variable

$$\mathbf{S}' \equiv \mathbf{u}' - e^{-\beta t}(\bar{\mathbf{a}}'(t)+\mathbf{u}'_0) \tag{23}$$

such that $P'(\mathbf{S}')\equiv P'(\mathbf{u}',t|\mathbf{u}'_0)$, then the solution of Eq. (18) can be written as

$$P'(\mathbf{S}') = \left(\frac{\beta}{2\pi\lambda(1-e^{-2\beta t})}\right)^{3/2} \exp\left\{-\frac{\beta|\mathbf{S}'|^2}{2\lambda(1-e^{-2\beta t})}\right\}. \tag{24}$$

To return to the original variables, we can observe from Eqs. (12), (13), (22), and (24) that $\mathbf{u}'_0=\mathbf{u}_0$, $\bar{\mathbf{a}}'(t)=\bar{\mathbf{a}}(t)$, and

$$\mathbf{u}' - e^{-\beta t}(\bar{\mathbf{a}}'(t)+\mathbf{u}'_0) = e^{-\mathbb{W}t}[\mathbf{u} - e^{-\Lambda t}(\bar{\mathbf{a}}(t)+\mathbf{u}_0)]. \tag{25}$$

If we now define the $\mathbf{S}$ variable as

$$\mathbf{S} \equiv \mathbf{u} - e^{-\Lambda t}(\bar{\mathbf{a}}(t)+\mathbf{u}_0), \tag{26}$$

then

$$\mathbf{S}' = e^{-\mathbb{W}t}\mathbf{S}, \tag{27}$$

and therefore $|\mathbf{S}'|^2=|\mathbf{S}|^2$. On the other hand, the transformation between P' and P is established by the expression $P\,d\mathbf{S}=P'\,d\mathbf{S}'$, where the volume element transforms as $d\mathbf{S}=J\,d\mathbf{S}'$, J being the Jacobian of the transformation, and thus $JP=P'$ such that [7,10]

$$J \equiv |\mathrm{Det}(\partial S_i/\partial S'_j)| = 1/|\mathrm{Det}(\partial S'_i/\partial S_j)| \equiv 1/J'. \tag{28}$$

It is easy to show, from Eq. (27), that $J'=1$ and therefore $P=P'$. Now, from the initial conditions required by P and P', we conclude that the constant appearing in Eq. (9) must be $C_0=1$. Finally, if $P(\mathbf{S})\equiv P(\mathbf{u},t|\mathbf{u}_0)$, the solution of the FP equation (8) can be written as

$$P(\mathbf{S}) = \left(\frac{\beta}{2\pi\lambda(1-e^{-2\beta t})}\right)^{3/2} \exp\left(-\frac{\beta|\mathbf{S}|^2}{2\lambda(1-e^{-2\beta t})}\right). \tag{29}$$

For a swarm of charged particles the fundamental solution (29) is simply $W=n_{\mathrm{tot}}P$.

If time goes to infinity as $\beta t \gg 1$ this TPD can be approximated by

$$P(\mathbf{S}) = \left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left(-\frac{m|\mathbf{S}_{\mathrm{as}}|^2}{2k_B T}\right), \tag{30}$$

where $\mathbf{S}_{\mathrm{as}}$ is the asymptotic approximation of $\mathbf{S}$. Of course, the explicit evaluation of $\mathbf{S}_{\mathrm{as}}$ depends on the particular expression of the time-dependent external force $\mathbf{a}(t)$, i.e., on the explicit expressions of both mechanical and electric forces.

On the other hand, if we denote the quantity $\hat{\mathbf{x}}$ as any vector on the xy plane and $\hat{P}$ as the TPD describing the diffusion process on the same plane, then, due to the structure of matrix Λ, it can be shown that the above TPD can be written as the product of two independent probability densities, i.e., $P(\mathbf{S})=\hat{P}(\hat{\mathbf{S}})P_z(S_3)$, such that $\hat{P}(\hat{\mathbf{S}})\equiv\hat{P}(\hat{\mathbf{u}},t|\hat{\mathbf{u}}_0)$ and $P_z(S_3)\equiv P_z(u_z,t|u_{0z})$, with $\hat{\mathbf{S}}\equiv(S_1,S_2)$ and $S_z\equiv S_3$. So

$$\hat{P}(\hat{\mathbf{S}}) = \frac{\beta}{2\pi\lambda(1-e^{-2\beta t})} \exp\left(-\frac{\beta|\hat{\mathbf{S}}|^2}{2\lambda(1-e^{-2\beta t})}\right) \tag{31}$$

is the TPD which describes the diffusion process on the xy plane, $\hat{\mathbf{S}}$ being

$$\hat{\mathbf{S}} \equiv \hat{\mathbf{u}} - e^{-\hat{\Lambda}t}(\hat{\bar{\mathbf{a}}}(t)+\hat{\mathbf{u}}_0), \tag{32}$$

with $\hat{\Lambda}=\beta\hat{\mathbb{I}}-\hat{\mathbb{W}}$ a 2×2 matrix given by

$$\hat{\Lambda} = \begin{pmatrix} \beta & -\Omega \\ \Omega & \beta \end{pmatrix} \tag{33}$$

and

$$\hat{\bar{\mathbf{a}}}(t) = \int_0^t e^{\hat{\Lambda}s}\hat{\mathbf{a}}(s)ds. \tag{34}$$

The probability density $P_z(S_3)$ is shown to be

$$P_z(S_3) = \left(\frac{\beta}{2\pi\lambda(1-e^{-2\beta t})}\right)^{1/2} \exp\left(-\frac{\beta S_3^2}{2\lambda(1-e^{-2\beta t})}\right), \tag{35}$$

with $S_3 \equiv [u_z - e^{-\beta t}(\bar{a}_z + u_{0z})]$ and $\bar{a}_z$ being the z component of vector $\hat{\mathbf{a}}(t)$. This TPD describes the diffusion process along the z axis, parallel to the magnetic field. In this case, the diffusion process is not affected by this field.

The probability density for an initially displaced Maxwellian velocity distribution

It is known that a more general probability density can be calculated by assuming an arbitrary initial condition. This can be achieved by taking into account the following facts: the solution $P(\mathbf{u},t|\mathbf{u}_0)$ of the FP equation (8) with the initial condition $P(\mathbf{u},0|\mathbf{u}_0) \equiv \delta(\mathbf{u}-\mathbf{u}_0)$, given in Eq. (29), is referred to as the "fundamental" solution. Once this fundamental solution has been calculated, the solution of Eq. (8) for any other initial charged Brownian particle velocity distribution $f(\mathbf{u},0)$ can be obtained by the following integration:

$$f(\mathbf{u},t) = \int_{\mathbf{u}_0} f(\mathbf{u}_0,0) P(\mathbf{u},t|\mathbf{u}_0) d\mathbf{u}_0. \tag{36}$$

In the particular case of an initial Maxwellian velocity distribution of the charged Brownian particle at a temperature T_0 different from the equilibrium temperature T and with an assigned mean velocity $\langle\mathbf{u}\rangle_0$ given by

$$f(\mathbf{u},0) = \left(\frac{m}{2\pi k_B T_0}\right)^{3/2} \exp\left(-\frac{m|\mathbf{u}-\langle\mathbf{u}\rangle_0|^2}{2k_B T_0}\right), \tag{37}$$

the solution of Eq. (8), after integration of Eq. (36), will be

$$f(\mathbf{u},t) = \left(\frac{m}{2\pi k_B T_t}\right)^{3/2} \exp\left(\frac{m|\mathbf{u}-e^{-\Lambda t}(\bar{\mathbf{a}}(t)+\langle\mathbf{u}\rangle_0)|^2}{2k_B T_t}\right), \tag{38}$$

where

$$T_t = T\left[1 - \left(1 - \frac{T_0}{T}\right)e^{-2\beta t}\right]. \tag{39}$$

From this general solution, we can analyze the following cases.

(a) For $T_0=0$, we can see that $f(\mathbf{u},0)=\delta(\mathbf{u}-\mathbf{u}_0)$ with $\mathbf{u}_0 = \langle\mathbf{u}\rangle_0$ and temperature $T_t = T(1-e^{-2\beta t})$. Therefore $f(\mathbf{u},t)$ reduces to the fundamental solution (29).

(b) For $T_0=T$, we have that the initial heavy-ion velocity distribution is the same as Eq. (37), i.e., a Maxwellian, at the equilibrium temperature T around the initial mean velocity $\langle\mathbf{u}\rangle_0$. The probability density is the same as Eq. (32), but with $T_t = T$.

(c) To compare Eq. (38) with Ferrari's solution, we have to make $\mathbf{F}_{\rm mec}=0$ and to take the electric field on the yz plane; that is, $\mathbf{a}=(0, a_y, a_z)$, where $a_y = a(t)\sin\psi$ and $a_z = a(t)\cos\psi$, with $0 \le \psi \le \pi$. In this case, our general solution (38) reduces to that calculated by Ferrari [8] by a different method. Obviously, Eq. (38) can also be written as the product of two PD's, that is, $f(\mathbf{u},t) = \hat{f}(\hat{\mathbf{u}},t) f_z(u_z,t)$, where $\hat{f}(\hat{\mathbf{u}},t)$ and $f_z(u_z,t)$ are naturally identified, respectively, on the xy plane and along the z axis.

III. THE FPK EQUATION OF A CHARGED PARTICLE IN EXTERNAL FIELDS

In phase space $(\mathbf{r},\mathbf{u})$ the Langevin equation of a heavy ion is now given by

$$\dot{\mathbf{r}} = \mathbf{u}, \tag{40}$$

$$\dot{\mathbf{u}} = -\beta\mathbf{u} + \mathbb{W}\mathbf{u} + \mathbf{a}(t) + \mathbf{A}(t), \tag{41}$$

or

$$\dot{\mathbf{r}} = \mathbf{u}, \tag{42}$$

$$\dot{\mathbf{u}} = -\Lambda\mathbf{u} + \mathbf{a}(t) + \mathbf{A}(t). \tag{43}$$

Its corresponding FPK equation for the transition probability density $P(\mathbf{r},\mathbf{u},t|\mathbf{u}_0,\mathbf{r}_0)$ of the velocity $\mathbf{u}$ and position $\mathbf{r}$ at time t, given that $\mathbf{u}=\mathbf{u}_0$ and $\mathbf{r}=\mathbf{r}_0$ at time $t=0$, is then [10]

$$\frac{\partial P}{\partial t} + \mathbf{u}\cdot \text{grad}_{\mathbf{r}} P + \mathbf{a}\cdot\text{grad}_{\mathbf{u}} P = \text{div}_{\mathbf{u}}(\Lambda\mathbf{u}P) + q\nabla^2_{\mathbf{u}}P, \tag{44}$$

subject to the initial condition

$$P(\mathbf{r},\mathbf{u},0|\mathbf{u}_0,\mathbf{r}_0) = C_1\delta(\mathbf{r}-\mathbf{r}_0)\delta(\mathbf{u}-\mathbf{u}_0), \tag{45}$$

where C_1 is again a constant which will be determined later on. As in the previous section, the first-order solution may be expressed in terms of six integrals of the Lagrangian subsidiary system

$$\dot{\mathbf{r}} = \mathbf{u}, \tag{46}$$

$$\dot{\mathbf{u}} = -\Lambda\mathbf{u} + \mathbf{a}(t). \tag{47}$$

These integrals are

$$e^{\Lambda t}\mathbf{u} - \bar{\mathbf{a}}(t) = \mathbf{I}_1, \tag{48}$$

$$\mathbf{r} + \Lambda^{-1}\left(\mathbf{u} - \int_0^t \mathbf{a}(s)ds\right) = \mathbf{I}_2, \tag{49}$$

$\bar{\mathbf{a}}(t)$ being the same as in Eq. (12), with the constants $\mathbf{I}_1 = \mathbf{u}_0$, $\mathbf{I}_2 = \mathbf{r}_0 + \Lambda^{-1}\mathbf{u}_0$. Λ^{-1} is the inverse of matrix Λ given by

$$\Lambda^{-1} = \begin{pmatrix} \frac{\beta}{\Omega^2+\beta^2} & \frac{\Omega}{\Omega^2+\beta^2} & 0 \\ -\frac{\Omega}{\Omega^2+\beta^2} & \frac{\beta}{\Omega^2+\beta^2} & 0 \\ 0 & 0 & \frac{1}{\beta} \end{pmatrix}. \tag{50}$$

The solution of Eq. (44) is not easy to obtain in terms of the aforementioned integrals. To proceed further, we use the proposal of Ref. [4], which suggests a transformation of the Langevin equations (40) and (41) into another phase space $(\mathbf{r}',\mathbf{u}')$ such that $\dot{\mathbf{r}}'=\mathbf{u}'$ with $\mathbf{u}'$ the same as in Eq. (13). In this new phase space, Eqs. (40) and (41) transform into

$$\dot{\mathbf{r}}' = \mathbf{u}', \tag{51}$$

$$\dot{\mathbf{u}}' = -\beta\mathbf{u} + \mathbf{a}'(t) + \mathbf{A}'(t), \tag{52}$$

where $\mathbf{a}'$ and $\mathbf{A}'$ are the same as those given in Eq. (15). The Langevin equations (51) and (52) display a strong resemblance to those associated with the ordinary Brownian motion in the presence of an external force $\mathbf{a}'$. Therefore the corresponding FPK equation for the transition probability density $P'(\mathbf{r}',\mathbf{u}',t|\mathbf{u}_0',\mathbf{r}_0')$ of the velocity $\mathbf{u}_0'$ and position $\mathbf{r}_0'$ at time t, given that $\mathbf{u}'=\mathbf{u}_0'$ and $\mathbf{r}'=\mathbf{r}_0'$ at time $t=0$, is [7,10]

$$\frac{\partial P'}{\partial t} + \mathbf{u}'\cdot \mathrm{grad}_{\mathbf{r}'}P' + \mathbf{a}'\cdot \mathrm{grad}_{\mathbf{u}'}P' = \beta\,\mathrm{div}_{\mathbf{u}'}(\mathbf{u}'P') + q\nabla^2_{\mathbf{u}'}P', \tag{53}$$

together with the initial condition

$$P'(\mathbf{r}',\mathbf{u}',0|\mathbf{u}_0',\mathbf{r}_0') \equiv \delta(\mathbf{u}'-\mathbf{u}_0')\delta(\mathbf{r}'-\mathbf{r}_0'). \tag{54}$$

Clearly, the solution of Eq. (53) is easier to find than that of Eq. (44). In Appendix B, we give the solution of Eq. (53), making an extension to the Chandrasekhar [7] method by solving the FPK equation in the field-free case. Thus, the Lagrangian subsidiary system associated with the first-order equation is

$$\dot{\mathbf{r}}' = \mathbf{u}', \tag{55}$$

$$\dot{\mathbf{u}}' = -\beta\mathbf{u}' + \mathbf{a}', \tag{56}$$

and their corresponding six integrals are

$$e^{\beta t}\mathbf{u}' - \bar{\mathbf{a}}' = \mathbf{I}_1', \tag{57}$$

$$\mathbf{r}' + \beta^{-1}\left(\mathbf{u}' - \int_0^t \mathbf{a}'(s)ds\right) = \mathbf{I}_2', \tag{58}$$

with $\mathbf{I}_1'=\mathbf{u}_0'$ and $\mathbf{I}_2'=\mathbf{r}_0'+\beta^{-1}\mathbf{u}_0'$. From these equations we can introduce the variables

$$\mathbf{S}' = \mathbf{u}' - (\bar{\mathbf{a}}' + \mathbf{u}_0')e^{-\beta t}, \tag{59}$$

$$\mathbf{R}' = \mathbf{r}' - \mathbf{r}_0' - \Gamma'\mathbf{u}_0' - \bar{\bar{\mathbf{a}}}', \tag{60}$$

where Γ' and $\bar{\bar{\mathbf{a}}}'$ are defined in Appendix B. Note that $\mathbf{S}'$ is the same as Eq. (23). If we now define $P'(\mathbf{R}',\mathbf{S}') \equiv P'(\mathbf{r}',\mathbf{u}',t|\mathbf{u}_0',\mathbf{r}_0')$, then the solution of Eq. (53), with the initial condition given by Eq. (54), can be written as

$$P'(\mathbf{R}',\mathbf{S}') = \frac{1}{8\pi^3(FG-H^2)^{3/2}} \times\exp\left(-\frac{(F|\mathbf{S}'|^2 - 2H\mathbf{R}'\cdot\mathbf{S}' + G|\mathbf{R}'|^2)}{2(FG-H^2)}\right), \tag{61}$$

F, G, and H being

$$F = \frac{\lambda}{\beta^3}(2\beta t - 3 + 4e^{-\beta t} - e^{-2\beta t}), \tag{62}$$

$$G = \frac{\lambda}{\beta}(1 - e^{-2\beta t}), \tag{63}$$

$$H = \frac{\lambda}{\beta^2}(1 - e^{-\beta t})^2. \tag{64}$$

To return to the original phase-space representation, we introduce the variables $\mathbf{S}$ and $\mathbf{R}$, which are obtained from Eqs. (48) and (49), such that

$$\mathbf{S} = \mathbf{u} - e^{-\Lambda t}(\bar{\mathbf{a}} + \mathbf{u}_0), \tag{65}$$

$$\mathbf{R} = \mathbf{r} - \mathbf{r}_0 - \Gamma\mathbf{u}_0 - \bar{\bar{\mathbf{a}}}, \tag{66}$$

$\mathbf{S}$ being the same as in Eq. (26), $\Gamma \equiv \Lambda^{-1}(1-e^{-\Lambda t})$, and

$$\bar{\bar{\mathbf{a}}}(t) = \int_0^t e^{-\Lambda s}\bar{\mathbf{a}}(s)ds. \tag{67}$$

The transformation between $\mathbf{S}'$ and $\mathbf{S}$ has been established in Eq. (27) and the transformation between $\mathbf{R}'$ and $\mathbf{R}$ can be established through the transformation between $\mathbf{r}'$ and $\mathbf{r}$, which can be achieved with the aid of Eqs. (13), (49), and (58), yielding

$$\mathbf{r}' - \beta^{-1}\int_0^t \mathbf{a}'(s)ds - \mathbf{I}_2' = \beta^{-1}e^{-\mathbb{W}t}\Lambda\left(\mathbf{r} - \Lambda^{-1}\int_0^t \mathbf{a}(s)ds - \mathbf{I}_2\right). \tag{68}$$

Therefore

$$\mathbf{S}' = e^{-\mathbb{W}t}\mathbf{S}, \quad \mathbf{R}' = e^{-\mathbb{W}t}\Lambda\mathbf{R}. \tag{69}$$

The transformation between the two TPD's P and P' can be established by taking into account that $P\,d\mathbf{S}\,d\mathbf{R} = P'\,d\mathbf{S}'\,d\mathbf{R}'$. The volume element transforms as $d\mathbf{S}\,d\mathbf{R} = J\,d\mathbf{S}'\,d\mathbf{R}'$; then $JP=P'$. The Jacobian J is shown to be $J=J_SJ_R$ with

$$J_S \equiv |\mathrm{Det}(\partial S_i/\partial S_j')| = 1/|\mathrm{Det}(\partial S_j'/\partial S_j)| \equiv 1/J_S', \tag{70}$$

$$J_R \equiv |\mathrm{Det}(\partial R_i/\partial R_j')| = 1/|\mathrm{Det}(\partial R_j'/\partial R_j)| \equiv 1/J_R'. \tag{71}$$

In this case, a straightforward calculation shows that $J_S'=1$ and $J_R'=(\beta^2+\Omega^2)/\beta^2$, and therefore

$$P = J_S'J_R'P' = \left[1 + \left(\frac{\Omega}{\beta}\right)^2\right]P'. \tag{72}$$

Also, the constant C_1, given in the initial condition (45), will be equal to $C_1=(\beta^2+\Omega^2)/\beta^2$.

Since $\hat{\mathbf{S}}=(S_1,S_2)$ and $\hat{\mathbf{R}}=(R_1,R_2)$ both represent vectors on the xy plane, with S_3 and R_3 being their z components, we can write

$$|\mathbf{S}'|^2 = |\hat{\mathbf{S}}|^2 + S_3^2, \tag{73}$$

$$|\mathbf{R}'|^2 = C_1|\hat{\mathbf{R}}|^2 + R_3^2, \tag{74}$$

$$\mathbf{S}' \cdot \mathbf{R}' = \hat{\mathbf{S}} \cdot \hat{\mathbf{R}} + \frac{\Omega}{\beta}(\hat{\mathbf{S}} \times \hat{\mathbf{R}})_z + S_3 R_3, \tag{75}$$

where $(\hat{\mathbf{S}} \times \hat{\mathbf{R}})_z = (S_1 R_2 - S_2 R_1)$ is the z component of the cross product and

$$\hat{\mathbf{S}} \equiv \hat{\mathbf{u}} - e^{-\hat{\Lambda}t}(\hat{\bar{\mathbf{a}}} + \hat{\mathbf{u}}_0), \tag{76}$$

$$\hat{\mathbf{R}} \equiv \hat{\mathbf{r}} - \hat{\mathbf{r}}_0 - \hat{\Gamma}\hat{\mathbf{u}}_0 - \hat{\bar{\bar{\mathbf{a}}}}, \tag{77}$$

$$S_3 \equiv u_z - e^{-\beta t}(\bar{a}_z + u_{0z}), \tag{78}$$

$$R_3 \equiv z - z_0 - \beta^{-1}(1 - e^{-\beta t})u_{0z} - \bar{\bar{a}}_z, \tag{79}$$

with $\hat{\Gamma} \equiv \hat{\Lambda}^{-1}(1 - e^{-\hat{\Lambda}t})$ and

$$\hat{\Lambda} = \begin{pmatrix} \beta & -\Omega \\ \Omega & \beta \end{pmatrix}, \quad \hat{\Lambda}^{-1} = \begin{pmatrix} \dfrac{\beta}{\Omega^2 + \beta^2} & \dfrac{\Omega}{\Omega^2 + \beta^2} \\ -\dfrac{\Omega}{\Omega^2 + \beta^2} & \dfrac{\beta}{\Omega^2 + \beta^2} \end{pmatrix}. \tag{80}$$

Accordingly, if Eqs. (76)–(79) are substituted into Eq. (61), it can be shown that the solution of the FPK equation (44) can be written as the product of two independent TPD's, that is,

$$P(\mathbf{R},\mathbf{S}) = \hat{P}(\hat{\mathbf{R}},\hat{\mathbf{S}})P_z(R_3,S_3), \tag{81}$$

such that

$$\hat{P}(\hat{\mathbf{R}},\hat{\mathbf{S}}) \equiv \hat{P}(\hat{\mathbf{r}},\hat{\mathbf{u}},t|\hat{\mathbf{r}}\,,\hat{\mathbf{u}}_0), \tag{82}$$

$$P_z(R_3,S_3) \equiv P(z,u_z,t|z_0,u_{0z}\,). \tag{83}$$

where

$$\hat{P}(\hat{\mathbf{R}},\hat{\mathbf{S}}) = \frac{C_1}{4\pi^2(FG - H^2)}\exp\left\{-\left[F|\hat{\mathbf{S}}|^2 - 2H\hat{\mathbf{R}} \cdot \hat{\mathbf{S}} - 2\frac{\Omega}{\beta}H(\hat{\mathbf{S}} \times \hat{\mathbf{R}})_z + C_1 G|\hat{\mathbf{R}}|^2\right] \Big/ 2(FG - H^2)\right\}, \tag{84}$$

is the planar TPD describing the diffusion process of a Brownian charged particle across the magnetic field and under the action of a planar external force $\hat{\mathbf{a}}(t)$, with the initial condition $\hat{P}(\hat{\mathbf{r}},\hat{\mathbf{u}},0|\hat{\mathbf{u}}_0\,) = C_1\delta(\hat{\mathbf{r}} - \hat{\mathbf{r}}_0)\delta(\hat{\mathbf{u}} - \hat{\mathbf{u}}_0)$. The TPD P_z is equal to

$$P_z(R_3,S_3) = \frac{1}{[4\pi^2(FG - H^2)]^{1/2}} \times \exp\left(-\frac{(FS_3^2 - 2HR_3S_3 + GR_3^2)}{2(FG - H^2)}\right). \tag{85}$$

It describes the diffusion process along the z axis, that is, parallel to the magnetic field, in the presence of an external force $a_z(t)$, and satisfying the initial condition $P_z(z,u_z,0|z_0,u_{0z}\,) = \delta(z - z_0)\delta(u_z - u_{0z})$. This TPD is the same as that of the ordinary Brownian motion in the presence of an external force $a_z(t)$, without the influence of the magnetic field, as expected. For a swarm of independent particles, we simply have $W = n_{\rm tot}P$.

Our solution given in Eq. (81) is similar to that obtained by SL [1] but not identical. The reasons are the following: SL's strategy starts with the same FPK equation (44), which is transformed into a field-free case by means of Ferrari's [3] gauge. The solution to the transformed FPK equation is thus established via CG's ansatz. This solution is a Gaussian TPD subject to the initial condition $G(\mathbf{x},\mathbf{v},0|\mathbf{v}_0,\mathbf{x}_0\,) = \delta(\mathbf{x} - \mathbf{x}_0)\delta(\mathbf{x} - \mathbf{x}_0)$, as given in that reference. On the other hand, in our proposal we transform Eq. (44) to the form given by Eq. (53), which is quite similar to that satisfied by the ordinary Brownian motion in the presence of an external field $\mathbf{a}'(t)$. We show in this case that the solution to the FPK equation (44) is also a Gaussian TPD, but with the initial condition given by Eq. (45), where the multiplicative constant is shown to be $C_1 = \beta^2 + \Omega^2/\beta^2$. In this sense, it is thus clear that our solution (81) and that obtained by SL [Eq. (7) of Ref. [1]] are similar but not identical. However, a common and expected result for both of them is that, for zero magnetic field ($\Omega = 0$ in our case and $\omega = 0$ for SL), both solutions reduce, as can easily be checked, to that of the ordinary Brownian motion in the presence of the external field $\mathbf{a}(t)$ only. In this case, $C_1 = 1$, and therefore both initial conditions coincide, as expected. A similar situation was studied in Ref. [6] by comparing the solution calculated in that reference to that calculated by CG [2] without the presence of the external field $\mathbf{a}(t)$. In fact, when this external field is zero ($\mathbf{a} = \mathbf{0}$), our solution (81) reduces to that given in Ref. [6] and SL's solution reduces, in the planar case, to that calculated by CG.

The immediate consequences of Eqs. (84) and (85) are the following.

(a) In the absence of magnetic ($\Omega = 0$) and mechanical $\mathbf{F}_{\rm mec} = \mathbf{0}$) forces, it is shown that $C_1 = 1$ and therefore the product given in Eq. (81) reduces exactly to the same fundamental solution as that obtained by Ferrari [3] when the diffusion process is under the action of an electric field only.

(b) Both $\hat{P}(\hat{\mathbf{S}})$ and P_z calculated in Sec. II can also be obtained from the integrals

$$\hat{P}(\hat{\mathbf{S}}) = \int \hat{P}(\hat{\mathbf{S}},\hat{\mathbf{R}})d\hat{\mathbf{R}}, \tag{86}$$

$$P_z(S_3) = \int P_z(R_3,S_3)dR_3, \tag{87}$$

respectively, yielding

$$\hat{P}(\hat{\mathbf{S}}) = \frac{\beta}{2\pi\lambda(1 - e^{-2\beta t})}\exp\left(-\frac{\beta|\hat{\mathbf{S}}|^2}{2\lambda(1 - e^{-2\beta t})}\right) \tag{88}$$

and

$$P_z(S_3) = \left(\frac{\beta}{2\pi\lambda(1-e^{-2\beta t})}\right)^{1/2} \exp\left(-\frac{\beta S_3^2}{2\lambda(1-e^{-2\beta t})}\right), \tag{89}$$

consistently with Eqs. (31) and (35). These results clearly show the consistency of the solutions (84) and (85).

(c) Similarly, in the configuration space $\mathbf{r}$ the planar-spatial TPD $\hat{P}(\hat{\mathbf{R}})$ and spatial TPD $P_z(R_3)$ defined by

$$\hat{P}(\hat{\mathbf{R}}) \equiv \hat{P}(\hat{\mathbf{r}},t|\hat{\mathbf{r}}_0,\hat{\mathbf{u}}_0), \tag{90}$$

$$P_z(R_3) \equiv P(z,t|z_0,u_{0z}), \tag{91}$$

can be calculated through the integrals

$$\hat{P}(\hat{\mathbf{R}}) = \int \hat{P}(\hat{\mathbf{S}},\hat{\mathbf{R}})d\hat{\mathbf{S}}, \tag{92}$$

$$P_z(R_3) = \int P_z(R_3,S_3)dS_3. \tag{93}$$

After long but straightforward algebra, they reduce to

$$\hat{P}(\hat{\mathbf{r}},t|\hat{\mathbf{r}}_0,\hat{\mathbf{u}}_0) = \frac{\beta}{2\pi D_e(2\beta t-3+4e^{-\beta t}-e^{-2\beta t})} \times\exp\left(-\frac{\beta|\hat{\mathbf{r}}-\hat{\mathbf{r}}_0-\hat{\Lambda}^{-1}(1-e^{-\hat{\Lambda}t})\hat{\mathbf{u}}_0-\bar{\bar{\hat{\mathbf{a}}}}|^2}{2D_e(2\beta t-3+4e^{-\beta t}-e^{-2\beta t})}\right), \tag{94}$$

where $D_e = D\beta^2/(\Omega^2+\beta^2)$ represents a rescaling of the Einstein diffusion constant ($D=\lambda/\beta^2=k_BT/m\beta$), and

$$P_z(R_3) = \frac{1}{[2\pi D(2\beta t-3+4e^{-\beta t}-e^{-2\beta t})/\beta]^{1/2}} \times\exp\left(-\frac{\beta(z-z_0-\beta^{-1}(1-e^{-\beta t})u_{0z}-\bar{\bar{a}}_z)^2}{2D(2\beta t-3+4e^{-\beta t}-e^{-2\beta t})}\right). \tag{95}$$

Clearly, the configuration fundamental solution is $P(\mathbf{r},t|\mathbf{r}_0,\mathbf{u}_0) = \hat{P}(\hat{\mathbf{R}})P_z(R_3)$.

(d) A more general PD $f(\mathbf{r},\mathbf{u},t)$ satisfying an arbitrary initial condition $f(\mathbf{r},\mathbf{u},0)$ can in principle be obtained by integration such that

$$f(\mathbf{r},\mathbf{u},t) = \int_{\mathbf{r}_0} d\mathbf{r}_0 \int_{\mathbf{u}_0} d\mathbf{u}_0 f(\mathbf{r}_0,\mathbf{u}_0,0)P(\mathbf{r},\mathbf{u},t|\mathbf{r}_0,\mathbf{u}_0), \tag{96}$$

$P(\mathbf{r},\mathbf{u},t|\mathbf{r}_0,\mathbf{v}_0)$ being the fundamental solution (81).

(e) In the large-time limit $\beta t \gg 1$, Eqs. (94) and (95) reduce, respectively, to

$$\hat{P}(\hat{\mathbf{r}},t|\hat{\mathbf{r}}_0,\hat{\mathbf{u}}_0) \simeq \frac{1}{(4\pi D_e t)}\exp\left(-\frac{|\hat{\mathbf{R}}_{\mathrm{as}}|^2}{4D_e t}\right) \tag{97}$$

and

$$P(z,t|z_0,u_{0z}) \simeq \frac{1}{(4\pi Dt)}\exp\left(-\frac{R_{3\mathrm{as}}^2}{4Dt}\right). \tag{98}$$

When the external fields are taken into account, we can estimate the equilibrium mean square displacements (MSD's) along and across the magnetic field, although the expression of the external force $\mathbf{a}(t)$ is not explicitly given. We can do this in the following way: in the usual field-free Brownian motion it is well known that the equilibrium MSD reads $\langle|\mathbf{r}-\mathbf{r}_0|^2\rangle = 2Dt$. Therefore, in the presence of an external field, and according to Eqs. (77) and (79), we could in principle neglect the term $\hat{\Gamma}\hat{\mathbf{u}}_0$ with respect to $\hat{\mathbf{r}}-\hat{\mathbf{r}}_0-\bar{\bar{\hat{\mathbf{a}}}}$, and the term $\beta^{-1}(1-e^{-\beta t})u_{0z}$ with respect to $z-z_0-\bar{\bar{a}}_z$. However, according to Ferrari [3], there is no reason to neglect the terms $\hat{\Lambda}^{-1}\mathbf{u}_0$ and $\beta^{-1}u_{0z}$, since they must be taken into account in the integration process over the whole $\mathbf{u}_0$ space in Eq. (96) for a general probability distribution calculation. In this case the fundamental solutions (97) and (98) can be approximated by

$$\hat{P}(\hat{\mathbf{r}},t|\hat{\mathbf{r}}_0,\hat{\mathbf{u}}_0) \simeq \frac{1}{(4\pi D_e t)}\exp\left(-\frac{|\hat{\mathbf{r}}-\hat{\mathbf{r}}_0-\hat{\Lambda}^{-1}\mathbf{u}_0-\bar{\bar{\hat{\mathbf{a}}}}|^2}{4D_e t}\right) \tag{99}$$

and

$$P_z(z,t|z_0,u_{0z}) \simeq \frac{1}{(4\pi Dt)}\exp\left(-\frac{(z-z_0-\beta^{-1}u_{oz}-\bar{\bar{a}}_z)^2}{4Dt}\right). \tag{100}$$

Therefore, the equilibrium MSD's for each variable x, y, and z, defined as $\langle(\Delta x)^2\rangle \equiv \langle(x-x_0)^2\rangle$, $\langle(\Delta y)^2\rangle \equiv \langle(y-y_0)^2\rangle$, and $\langle(\Delta z)^2\rangle \equiv \langle(z-z_0)^2\rangle$ are easily calculated from Eqs. (99) and (100), thus giving

$$\langle(\Delta x)^2\rangle = 2D_e t + \left(\frac{\beta u_{0x}}{\beta^2+\Omega^2}+\frac{\Omega u_{0y}}{\beta^2+\Omega^2}+\bar{\bar{a}}_x\right)^2, \tag{101}$$

$$\langle(\Delta y)^2\rangle = 2D_e t + \left(\frac{\beta u_{0y}}{\beta^2+\Omega^2}-\frac{\Omega u_{0x}}{\beta^2+\Omega^2}+\bar{\bar{a}}_y\right)^2, \tag{102}$$

$$\langle(\Delta z)^2\rangle = 2Dt + \left(\frac{u_{0z}}{\beta}+\bar{\bar{a}}_z\right)^2. \tag{103}$$

In the absence of all external forces, these MSD's can be written in shorthand as $\langle|\mathbf{r}-\mathbf{r}_0|^2\rangle = 2Dt + (|\mathbf{u}_0|/\beta)^2$, which is the same as that of the ordinary Brownian motion if the term $\mathbf{u}_0/\beta$ is not neglected. However, if $\mathbf{u}_0=\mathbf{0}$ and also $\bar{\bar{\mathbf{a}}}=\mathbf{0}$, then they reduce to $\langle(\Delta x)^2\rangle = \langle(\Delta y)^2\rangle = 2D_e t$ and $\langle(\Delta z)^2\rangle = 2Dt$, corresponding to the MSD's when the diffusion process occurs under the action of a constant external magnetic field only [2,6].

IV. CONCLUSIONS

When a magnetic field $\mathbf{B}$ is allowed to point along the z axis of a Cartesian reference frame, then it influences the diffusion process only across the magnetic field, i.e., on the *xy* plane. Along the magnetic direction, a magnetic-field-free diffusion process is obtained, as expected. These statements have been well characterized through the fundamental solutions given in Eqs. (31), (35), (84), and (85). Formally, these solutions have been obtained not from the FP (8) and FPK (44) equations, but from their respective transformations (18) and (53). In contrast with SL's proposal, where a Gaussian probability distribution associated with the correlation functions is assumed as an ansatz, our solution method is markedly different and allows for a solution free of extra assumptions. Simões and Lagos's PD satisfies the same initial condition as that given by Eq. (9), except for the constant C_1. These are the main reasons why both fundamental solutions are similar, but not exactly the same.

In the velocity space, we have obtained the more general PD (38) with an initial Maxwellian velocity distribution (37). As can be readily seen, our theory is an alternative one to that proposed by Ferrari [8]. The conditions under which our result reduces to that obtained by Ferrari are given in point (c) of Sec. II.

The consistency of the phase-space fundamental solution (81), or equivalently (84) and (85), have been well established through the different cases studied in the points (a)–(e) of Sec. III. A particular and interesting point we want to stress here is that, when both mechanical and magnetic fields are absent, our fundamental solution (81) reduces exactly to Ferrari's phase-space solution [3]. As a simple calculation, we obtain the MSD across [Eqs. (101) and (102)] and along [Eq. (103)] the magnetic field, assuming that the initial condition $\mathbf{u}_0$ is not necessarily neglected.

Once the fundamental solution (81) is obtained, a more general phase-space PD can be calculated for an arbitrary initial PD. However, this task is still under investigation. Finally, another interesting problem to which our proposal can be extended is that of anisotropic diffusion, as studied by Holod *et al.* [4]. This work is in progress.

ACKNOWLEDGMENTS

The authors wish to thank R. M. Velasco and F. Uribe for their valuable comments. Financial support from Consejo Nacional de Ciencia y Tecnología (CONACYT) México is also acknowledged.

APPENDIX A: SOLUTION OF THE FP EQUATION IN VELOCITY SPACE u′

To solve the Fokker-Planck equation (18), which is quite similar to that of the ordinary Brownian motion in the presence of a general time-varying external field $\mathbf{a}'(t)$, we follow Chandrasekhar's methodology to solve the FP equation in the field-free case [7]. Accordingly, the solution of Eq. (18) is connected to the solution of the associated first-order equation without the Laplacian term, that is,

$$\frac{\partial P'}{\partial t} + \mathbf{a}' \cdot \mathrm{grad}_{\mathbf{u}'} P' = \beta \, \mathrm{div}_{\mathbf{u}'}(\mathbf{u}' P'). \tag{A1}$$

The solution of Eq. (A1) involves the first three integrals of the Lagrangian subsidiary system

$$\dot{\mathbf{u}}' = -\beta \mathbf{u}' + \mathbf{a}'(t), \tag{A2}$$

which are

$$e^{\beta t}\mathbf{u}' - \bar{\mathbf{a}}' = \mathbf{I}_1'. \tag{A3}$$

In this last expression $\mathbf{I}_1' = \mathbf{u}_0'$, and

$$\bar{\mathbf{a}}'(t) = \int_0^t e^{\beta t}\mathbf{a}'(s)ds. \tag{A4}$$

By defining a new vector

$$\mathbf{p}' = (\xi', \eta', \zeta') = e^{\beta t}\mathbf{u}' - \bar{\mathbf{a}}', \tag{A5}$$

it can be shown that each term of Eq. (18) reduces in this space to

$$\frac{\partial P'}{\partial t} = \frac{\partial P'(\mathbf{p}', t)}{\partial t} + \beta \mathbf{p}' \cdot \mathrm{grad}_{\mathbf{p}'} P' + \beta \bar{\mathbf{K}}' \cdot \mathrm{grad}_{\mathbf{p}'} P' - e^{-\beta t}\mathbf{a}' \cdot \mathrm{grad}_{\mathbf{p}'} P', \tag{A6}$$

$$\mathbf{a}' \cdot \mathrm{grad}_{\mathbf{u}'} P' = e^{-\beta t}\mathbf{a}' \cdot \mathrm{grad}_{\mathbf{p}'} P', \tag{A7}$$

$$\beta \mathbf{u}' \cdot \mathrm{grad}_{\mathbf{u}'} P' = \beta \mathbf{p}' \cdot \mathrm{grad}_{\mathbf{p}'} P' + \beta \bar{\mathbf{a}}' \cdot \mathrm{grad}_{\mathbf{p}'} P', \tag{A8}$$

$$\nabla^2_{\mathbf{u}'} P' = e^{2\beta t}\nabla^2_{\mathbf{p}'} P'. \tag{A9}$$

Then Eq. (18) becomes

$$\frac{\partial P'}{\partial t} = 3\beta P' + \lambda \nabla^2_{\mathbf{p}'} P'. \tag{A10}$$

This equation can be further simplified by introducing the variable

$$\chi' = e^{-3\beta t}P', \tag{A11}$$

giving as a result

$$\frac{\partial \chi'}{\partial t} = \lambda e^{2\beta t}\left(\frac{\partial^2 \chi'}{\partial \xi'^2} + \frac{\partial^2 \chi'}{\partial \eta'^2} + \frac{\partial^2 \chi'}{\partial \zeta'^2}\right). \tag{A12}$$

Thus, following Chandrasekhar's proposal, if $\phi(t)$ is an arbitrary function of time, the solution of the equation $\partial \chi'/\partial t = \phi^2(t)\nabla^2_{\mathbf{p}'}\chi'$ which has a source in $\mathbf{p}' = \mathbf{p}_0'$ at time $t=0$ is given by

$$\chi' = \frac{1}{\left(4\pi \int_0^t \phi^2(t)dt\right)^{3/2}} \exp\left(-\frac{|\mathbf{p}' - \mathbf{p}_0'|^2}{4\int_0^t \phi^2(t)dt}\right). \tag{A13}$$

Finally, for $\phi(t) = \lambda e^{\beta t}$, the solution of the Fokker-Planck equation (18) will be

$$P'(\mathbf{u}', t|\mathbf{u}_0') = \frac{1}{[2\pi\lambda(1 - e^{-2\beta t})/\beta]^{3/2}} \times \exp\left(-\frac{\beta|\mathbf{u}' - e^{-\beta t}(\bar{\mathbf{a}}' + \mathbf{u}_0')|^2}{2\lambda(1 - e^{-2\beta t})}\right). \tag{A14}$$

APPENDIX B: SOLUTION OF THE FPK EQUATION IN PHASE SPACE $(\mathbf{r}',\mathbf{u}')$

Again, according to Chandrasekhar, the solution of Eq. (53) is connected with the solution of the associated first-order equation without the Laplacian term, that is,

$$\frac{\partial P'}{\partial t} + \mathbf{u}' \cdot \mathrm{grad}_{\mathbf{r}'} P' + \mathbf{a}' \cdot \mathrm{grad}_{\mathbf{u}'} P' = \beta\, \mathrm{div}_{\mathbf{u}'}(\mathbf{u}' P'). \tag{B1}$$

The solution of Eq. (B1) involves six integrals of the Lagrangian subsidiary system given by

$$\dot{\mathbf{u}}' = -\beta \mathbf{u}' + \mathbf{a}'(t), \quad \dot{\mathbf{r}}' = \mathbf{u}'. \tag{B2}$$

Thus, their corresponding six integrals are

$$e^{\beta t}\mathbf{u}' - \bar{\mathbf{a}}' = \mathbf{I}_1', \tag{B3}$$

$$\mathbf{r}' + \beta^{-1}\mathbf{u}' - \beta^{-1}\int_0^t \mathbf{a}'(s)ds = \mathbf{I}_2', \tag{B4}$$

where $\bar{\mathbf{a}}'$ is the same as in (A4), $\mathbf{I}_1' = \mathbf{u}_0'$, and $\mathbf{I}_2' = \mathbf{r}_0' + \beta^{-1}\mathbf{u}_0'$. Accordingly, to solve Eq. (53), we introduce the following change of variables:

$$\mathbf{p}' \equiv (\xi', \eta', \chi') = e^{\beta t}\mathbf{u}' - \bar{\mathbf{a}}', \tag{B5}$$

$$\mathbf{Q}' \equiv (X', Y', Z') = \mathbf{r}' + \beta^{-1}\mathbf{u}' - \beta^{-1}\int_0^t \mathbf{a}'(s)ds. \tag{B6}$$

For these transformations, we can show that

$$\frac{\partial P'}{\partial t} = \frac{\partial P'(\mathbf{p}',\mathbf{Q}',t)}{\partial t} + \beta \mathbf{p}' \cdot \mathrm{grad}_{\mathbf{p}'} P' + \beta \bar{\mathbf{a}}' \cdot \mathrm{grad}_{\mathbf{p}'} P' - e^{-\beta t}\mathbf{a}' \cdot \mathrm{grad}_{\mathbf{p}'} P' - \beta^{-1}\mathbf{a}' \cdot \mathrm{grad}_{\mathbf{Q}'} P', \tag{B7}$$

$$\mathrm{grad}_{\mathbf{r}'} P' = \mathrm{grad}_{\mathbf{Q}'} P', \tag{B8}$$

$$\mathrm{grad}_{\mathbf{u}'} P' = e^{\beta t}\mathrm{grad}_{\mathbf{p}'} P' + \beta^{-1}\, \mathrm{grad}_{\mathbf{Q}'} P', \tag{B9}$$

and

$$\nabla^2_{\mathbf{u}'} P' = e^{2\beta t}\nabla^2_{\mathbf{p}'} P' + 2\beta^{-1}e^{\beta t}\nabla_{\mathbf{p}'} \cdot \nabla_{\mathbf{Q}'} P' + \beta^{-2}\nabla^2_{\mathbf{Q}'} P'. \tag{B10}$$

Substituting Eqs. (B7)–(B10) into Eq. (53) we get

$$\frac{\partial P'}{\partial t} = 3\beta P' + \lambda(e^{2\beta t}\nabla^2_{\mathbf{p}'} P' + 2\beta^{-1}e^{\beta t}\nabla_{\mathbf{p}'} \cdot \nabla_{\mathbf{Q}'} P' + \beta^{-2}\nabla^2_{\mathbf{Q}'} P'). \tag{B11}$$

Again, by introducing the variable

$$\chi' = e^{-3\beta t}P', \tag{B12}$$

Eq. (B11) reduces to

$$\frac{\partial \chi'}{\partial t} = \lambda(e^{2\beta t}\nabla^2_{\mathbf{p}'}\chi' + 2\beta^{-1}e^{\beta t}\nabla_{\mathbf{p}'} \cdot \nabla_{\mathbf{Q}'}\chi' + \beta^{-2}\nabla^2_{\mathbf{Q}'}\chi'). \tag{B13}$$

This equation has the same structure as that obtained by Chandrasekhar to solve the FPK equation of a Brownian particle in the field-free case [7]. Following his method, and by defining the variables

$$\mathbf{S}' = \mathbf{u}' - (\bar{\mathbf{a}}' + \mathbf{u}_0')e^{-\beta t}, \tag{B14}$$

$$\mathbf{R}' = \mathbf{r}' - \mathbf{r}_0' - \Gamma'\mathbf{u}_0' - \bar{\bar{\mathbf{a}}}', \tag{B15}$$

where $\Gamma' = \beta^{-1}(1 - e^{-\beta t})$ and

$$\bar{\bar{\mathbf{a}}}' = \int_0^t e^{-\beta s}\bar{\mathbf{a}}'(s)ds, \tag{B16}$$

we finally show that the solution of the FPK equation (53) is

$$P'(\mathbf{R}',\mathbf{S}') = \frac{1}{8\pi^3(FG - H^2)^{3/2}} \times \exp\left(-\frac{(F|\mathbf{S}'|^2 - 2H\mathbf{R}' \cdot \mathbf{S}' + G|\mathbf{R}'|^2)}{2(FG - H^2)}\right), \tag{B17}$$

where the parameters F, G, and H are given by

$$F = \frac{\lambda}{\beta^3}(2\beta t - 3 + 4e^{-\beta t} - e^{-2\beta t}), \tag{B18}$$

$$G = \frac{\lambda}{\beta}(1 - e^{-2\beta t}), \tag{B19}$$

$$H = \frac{\lambda}{\beta^2}(1 - e^{-\beta t})^2. \tag{B20}$$

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