# Phase transition of a one-dimensional Ising model with distance-dependent connections

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The critical behavior of the Ising model on a one-dimensional network, which has long-range connections at distances $l>1$ with the probability $\Theta(l)\sim l^{-m}$, is studied by using Monte Carlo simulations. Through analyzing the Ising model on networks with different $m$ values, this paper discusses the impact of the global correlation, which decays with the increase of $m$, on the phase transition of the Ising model. Adding the analysis of the finite-size scaling of the order parameter $[\langle M\rangle]$, it is observed that in the whole range of $0<m<2$, a finite-temperature transition exists, and the critical exponents show consistence with mean-field values, which indicates a mean-field nature of the phase transition.

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## I. INTRODUCTION

Since first proposed by Watts and Strogatz [1] in 1998, the Watts-Strogatz (WS) small world model has been widely and deeply studied. In the WS model, vertices are placed on a ring with each vertex having a finite number of $2k$ nearest regular connections initially. The connections are then rewired with a probability $p_r$ to form long-range connections or short paths. By varying the single parameter $p_r$ from 0 to 1, the WS model displays phase transition from a regular network to a small world, which ends up with a random network at $p_r=1$. Considering that the rewiring of connections may cause isolated vertices, the addition-type WS model–Newman and Watts (NW) model [2,3] was proposed later, in which long-range connections are added with a probability $p$ while keeping the initial regular connections unchanged. The NW model and the WS model are nearly equivalent, especially when the network size $N\to\infty$ [2,3]. Marcelo Kuperman and Guillermo Aberamson generalized the WS model in [4]. In their model, besides connecting with the two nearest neighboring vertices, with a probability $p$ each of the vertices in the model chooses a non-neighboring vertex to construct a long-range connection between them. The choice of the non-neighboring vertex is governed by the distance-dependent probability distribution:

$$\Theta(l) \propto l^{-m}. \tag{1}$$

When $m=0$, the NW model is restored. Through varying the value of $m$, the global correlation of the network is influenced. By analyzing the topological characteristic of the network, it was observed that the network also shows a small world nature, and the network converges to a regular lattice as $m\to\infty$ [4–7].

To analyze the behavior of a small world network from the point of view of statistical physics, researchers focus on its long-range connections or short paths, which may lead to global coherence. In networks without any short path, the transmission of information has to pass a long distance $\sim O(N)$ ($N$ is the network size), so global coherence is difficult to reach. With the increase of the number of short paths, or long-range connections, collective behavior will emerge in the system. This global coherence and the ubiquitous small world phenomenon in real-life make the study of thermodynamic phase transition on networks with small world property significantly popular [8–15], in which the Ising model is one of the most fascinating points.

As a comparatively simple but very important model of statistical physics, the Ising model perfectly shows order-disorder transition of the system. The Ising model on a one-dimensional regular lattice does not show phase transition at any finite temperature and the phase transition on a multidimensional regular lattice is not of mean-field nature. That the Ising model undergoes phase transitions in the addition-type WS model has been studied in [10–13]. It has also been shown that in the small world phase the addition-type WS model has a mean-field nature [11–13]. The critical behavior of the Ising model on a one-dimensional network with distance-dependent connections given by Eq. (1) is expected to reflect the nature of the network at different values of $m$ indirectly [12]. But it remains controversial about the mean-field nature of the phase transition in the range of $1\leq m<2$ [6,7,12].

To detect whether a network has a small world behavior, one can calculate the average shortest-path length and the average clustering coefficient. Sen and Charkrabarti in [6] and Moukarzel and de Menezes in [7] obtained the contradictory results by analyzing the topological structure of networks with connection probability at distance $l$ given by Eq. (1). In Ref. [6], it was found that the average shortest-path length behaves as $\ln N$ on rings of size $N$ for all $m<2$, hence it was concluded that small world behavior occurs for $0<m<2$ while in Ref. [7], it was argued that small world behavior occurs only for $m<1$. In the section $1<m<2$, according to Ref. [7], the average shortest-path length scales as $N^{\delta}$ with the value of the exponent $\delta(0<\delta<\frac{1}{2})$ depending on $m$. Reference [12] studied the critical behavior of the Ising model on such a one-dimensional network, since if the network's behavior is that of small world it should be reflected in the critical exponents of the Ising model, which will assume mean-field values. Their result indicates that there is a mean-field behavior for $m<1$ and a finite-dimensional behavior for $1<m<2$, which is in accordance with the conclusion in Ref. [7].

In this paper, we re-examine the critical behavior of the Ising model on a one-dimensional network with distance-

 

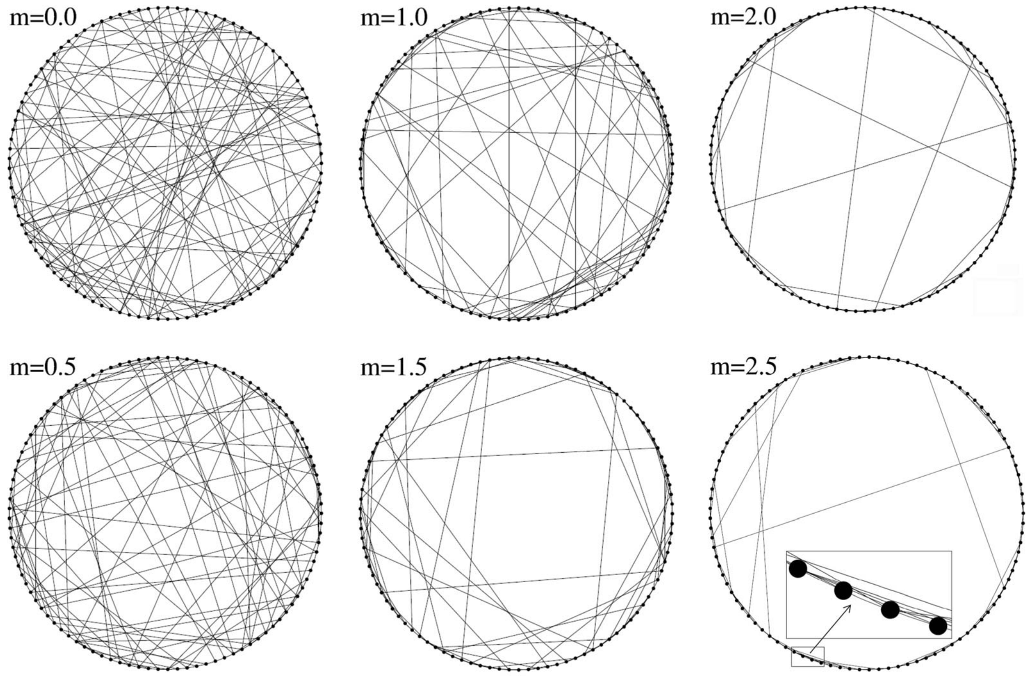


FIG. 1. Examples of generated networks with $N=101$ under the same $p=1$ and different $m$ values. The inner figure inside the bottom right figure shows the magnification of the denoted rectangular region.

dependent connections in Ref. [12] by using Monte Carlo simulations. But we enlarge the underlying network sizes, and also take into account the finite-size scaling analysis of the order parameter $[\langle M\rangle]$ by using the plot of $[\langle M\rangle]N^{1/4}$ vs $T$, which can reflect the mean-field nature indirectly [8]. Here, $[\cdots]$ and $\langle\cdots\rangle$ denote the thermal average taken over Monte Carlo steps for equilibrium at each temperature, and over different network realizations, respectively. Thus more reliable results can be produced from our simulations, since statistical fluctuations are greatly suppressed in these large-scale networks, and the finite-size effects are much more discernible in Binder's cumulant $U_N$ [16,17] and the specific heat $C_v$ than in the order parameter $[\langle M\rangle]$ [8]. The results show that the model has a mean-field nature in the whole range of $0<m<2$.

In the next section, we will introduce the model we use and the Monte Carlo simulation. In Sec. III, the results and analysis are given. Section IV is the conclusion.

## II. ISING MODEL AND MONTE CARLO SIMULATIONS

The underlying network of the one-dimensional Ising model used in this paper is based on the generalized WS model [4]. The distance between vertices is considered in the addition of connections. That means the vertex $j$ to which a connection is attached is not randomly selected, but according to a distribution depending on the distance $l$ from $i$ to $j$: $\Theta(l)\sim l^{-m}$. We start with a one-dimensional regular lattice with $N=2K+1$ vertices and $2k=2$ nearest neighbors. From $i=0$, we select the $i$th vertex and generate a random number $\varphi\in[0,1)$ from a uniform distribution. If $\varphi<p$, we select one vertex from the clockwise $K$ vertices to attach to according to the probability distribution $\Theta(l)\sim l^{-m}$. Self-connections and multiple connections are prohibited. We repeat this process until all the vertices are selected, which results in a network with $N(1+p)$ connections on average.

In order to investigate the influence of $m$ on the critical behavior of Ising model, in this paper we only consider the case $p=1$ for simplicity. Our model with $m=0$ is equivalent to the model used by Andrzej Pekalski [9]. As $m\to\infty$, our model restores to the one-dimensional regular network with $2k=4$ nearest neighbors. Figure 1 shows that with the increase of $m$, long-range connections are more and more difficult to form between vertices with long distances.

Figure 2 shows that the increase of $m$ results in the increase of the average shortest-path length $L$, but the small world property retains within the entire range $0<m<2$ as $L\propto\ln N$. Figure 3(a) shows that with the increase of the $m$

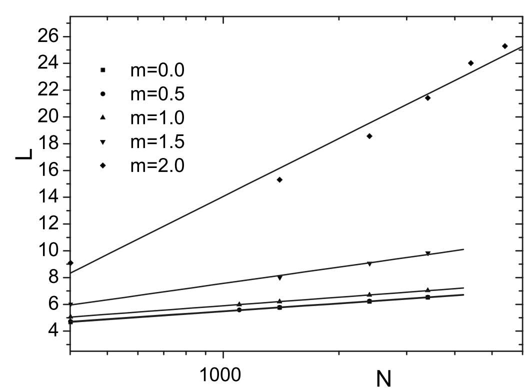


FIG. 2. The relationship of the average shortest-path length $L$ with the size of the network $N$ under different $m$ values (the case $m=0.5$ is almost identical to $m=0$). All the plots have been averaged over 50 realizations.

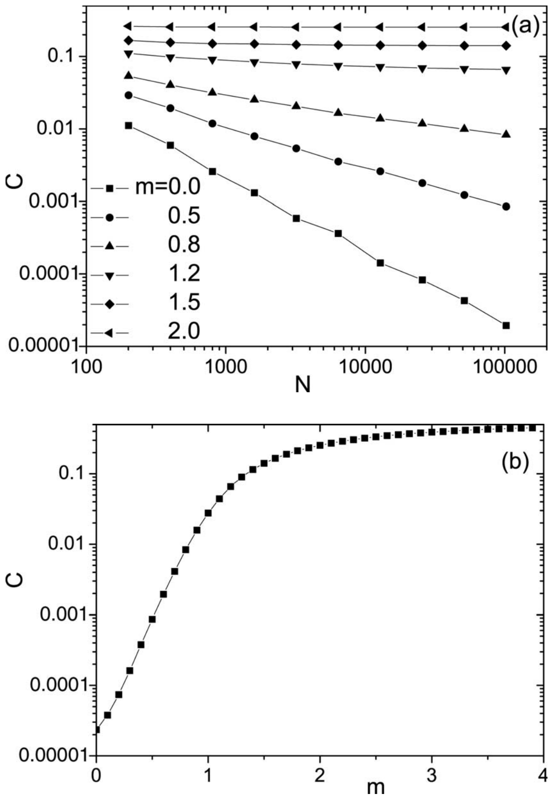


FIG. 3. (a) Network clustering coefficient $C$ vs network size $N$ for different $m$ values. (b) Network clustering coefficient $C$ vs $m$ value for a network with $N=100\,001$ vertices. The shown $C$ values in these two plots have been averaged for 50 different network realizations.

value, especially after $m>1$, the network clustering coefficient becomes independent of system size, which is the characteristic of a regular network. Furthermore, it can be seen from Fig. 3(b) that, for a certain network size, with the increase of $m$, the clustering coefficient converges to $C_{max}=1/2$, which is identical to the clustering coefficient of a one-dimensional regular network with $2k=4$ nearest neighbors.

The Ising model is described by the Hamiltonian

$$H=-J\sum_{(i,j)}\sigma_i\sigma_j, \tag{2}$$

where $J>0$ is the coupling constant between vertex $i$ and $j$ if they are connected, $(i,j)$ is the collection of all the connections in the network.

Our Monte Carlo simulation of the Ising model starts on a periodic one-dimensional ring. At the beginning, all the spins on the ring are given the same value +1, then a random vertex is selected to flip according to the rule of the Metropolis algorithm [9,18] and Glauber dynamics [11] to simulate the evolution of Ising model under different temperature. And finite-size scaling is used to investigate the paramagnetic-ferromagnetic transition temperature and the critical exponents. During the simulations, we mainly use the random number generator in [19], and we also use the drand48( ) function in standard C library as a comparison. It is found that there is no notable influence on the simulation results. In the simulations we measure the Binder cumulant $U_N$, the susceptibility $\chi$, and the specific heat $C_v$ [16,17,20–22]:

$$U_N=1-\frac{[\langle M^4\rangle]}{3[\langle M^2\rangle]^2},$$

$$\chi=\frac{1}{N}\sum_{ij}[\langle\sigma_i\sigma_j\rangle],$$

$$C_v=\frac{[\langle H^2\rangle-\langle H\rangle^2]}{T^2N} \tag{3}$$

with $M=|\frac{1}{N}\Sigma_i\sigma_i|$. Here $[\cdots]$ denotes the thermal average taken over $2.5\times10^4$ Monte Carlo steps after discarding the initial $2.5\times10^4$ ones for equilibrium at each temperature, and $\langle\cdots\rangle$ denotes the average over different network realizations taken over 30–80 configurations.

Besides these three physical quantities, the finite-size scaling analysis of the order parameter $[\langle M\rangle]$, which exhibits the critical behavior $[\langle M\rangle]\sim(T_c-T)^\beta$ in the thermodynamic limit, is used. In a finite-sized system, the order parameter scales as [8]

$$[\langle M\rangle]=N^{-\beta/\bar{\nu}}g((T-T_c)N^{1/\bar{\nu}}), \tag{4}$$

where $g(x)$ is an appropriate scaling function. Equation (4) leads to a unique crossing point at $T_c$ in the plot of $[\langle M\rangle]N^{\beta/\bar{\nu}}$ vs $T$, so Beom *et al.* [8] suggested analyzing the finite-size scaling of the order parameter by $[\langle M\rangle]N^{1/4}$ for large $m$ to overcome the finite-size effects, which are more prominent in other thermodynamic quantities.

## III. RESULTS AND ANALYSIS

As shown in Fig. 4, for $m=0$, Binder's cumulant with different network sizes yields a unique crossing point at $T_c=3.10(5)$ [Fig. 4(a)]. Figure 4(b) results in the critical exponent $\bar{\nu}\approx2$ (the reciprocal of the slope obtained in the linear fit of $\ln\Delta U_N$ vs $\ln N$), which gives the critical exponent $\alpha=2-\bar{\nu}\approx0$ for the specific heat $C_v$. With such a mean-field value, the finite-size scaling of the specific heat can be written as [8,20,23]

$$C_v=f((T-T_c)N^{1/\bar{\nu}}); \tag{5}$$

$f(x)$ is an appropriate scaling function. Thus the specific heat $C_v$ with different network sizes intersect to one unique crossing point and determines $T_c=3.10(7)$ [Fig. 4(c)], which is in accordance with the value obtained from $U_N$ within numerical errors. The expansion of $C_v$ near $T_c$ gives [8,20,23]

$$\Delta C_v\propto N^{1/\bar{\nu}}, \tag{6}$$

the linear fit of $\ln\Delta C_v$ vs $\ln N$ provides an alternative way of determining the critical exponent $\bar{\nu}\approx2$ [the reciprocal of the slope obtained in the fit, Fig. 4(d)]. Combined with the obtained critical exponent $\bar{\nu}\approx2$, finite-size scaling of the susceptibility yields $T_c=3.10(5)$ with the critical exponent $\gamma$

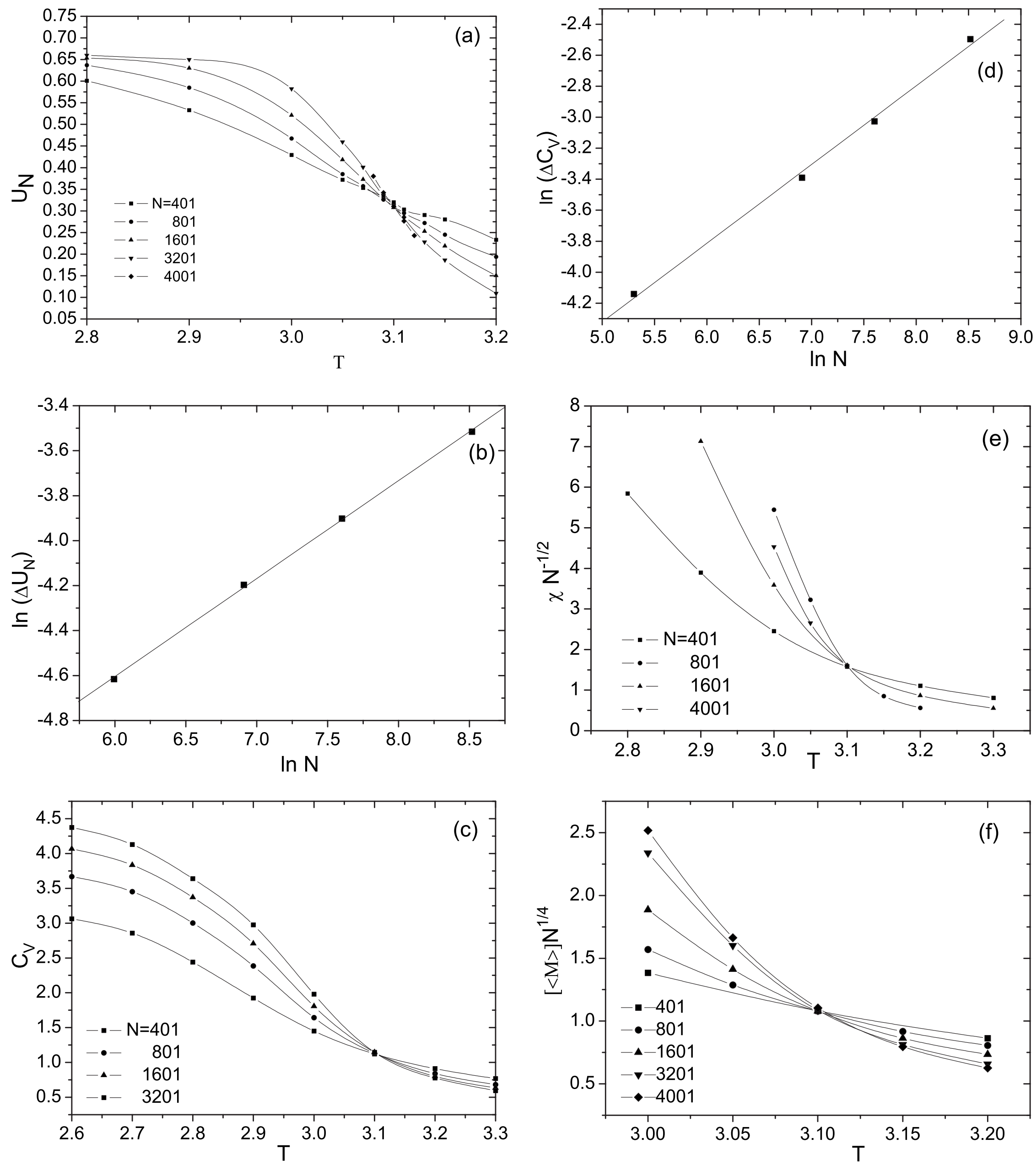


FIG. 4. For $m=0$, (a) Binder's cumulant $U_N$ has a unique crossing point at $T_c=3.10(5)$ (in units of $J/k_B$). (b) The critical exponent $\bar{\nu}=2.08(7)$ is obtained from the linear fit of $\ln \Delta U_N$ vs $\ln N$. (c) Specific heat $C_v$ has a unique crossing point at $T_c=3.10(7)$, suggesting $\alpha \approx 0$. (d) The expansion of $C_v$ near $T_c$ obtains $\bar{\nu}=2.04(6)$. (e) Finite-size scaling of the susceptibility again determines $T_c=3.10(5)$ with the critical exponent $\gamma \approx 1$. (f) Finite-size scaling of the order parameter $[\langle M\rangle]$ leads to a unique crossing point at $T_c=3.10(1)$, and yields $\beta \approx \frac{1}{2}$, $\bar{\nu}\approx 2$ [see Eq. (4) [8]].

$\approx 1$ [Fig. 4(e)]. Finally, finite-size scaling of the order parameter $[\langle M\rangle]$ gives $\beta \approx \frac{1}{2}$, $\bar{\nu}\approx 2$ [Fig. 4(f)]. The value of the obtained critical exponents reveals the mean-field nature of the transition.

Until now, the most controversial property of the phase transition of one-dimensional Ising model with distance-dependent connections given by Eq. (1) is in the range of $1 \leqslant m < 2$ [6,7,12], so we will focus on the critical behavior of the Ising model for $1 \leqslant m < 2$.

Figure 5 presents the finite-size scaling of Binder's cumulant $U_N$, the specific heat $C_v$, and the order parameter $[\langle M\rangle]$ at $m=1.0$. The measured quantities remain a unique crossing point and reveal unanimously $T_c \approx 3.05$, which confirms the presence of a finite-temperature phase transition. The obtained critical exponents indicate the mean-field nature of the transition.

Figure 6 is the finite-size scaling of Binder's cumulant $U_N$, the specific heat $C_v$, and the order parameter $[\langle M\rangle]$ at $m=1.5$. The curves of Binder's cumulant $U_N$ do not intersect to one unique crossing point very well. But the trends of the evolution indicate that it will emerge when the underlying network size is large enough. The finite-size scaling of the

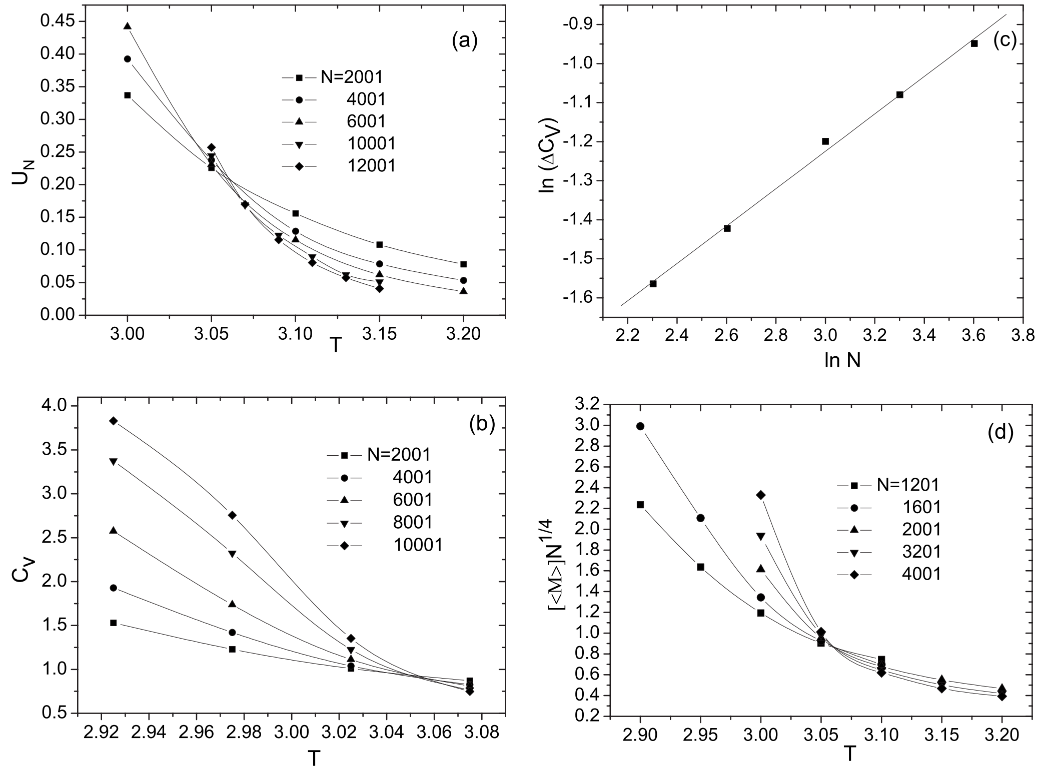


FIG. 5. For $m=1.0$, (a) finite-size scaling of Binder's cumulant $U_N$ has a unique crossing point at $T_c=3.05(2)$. (b) Finite-size scaling of the specific heat $C_v$ determines $T_c=3.05(3)$, which indicates $\alpha \approx 0$. (c) The expansion of $C_v$ near $T_c$ obtains $\bar{\nu}=2.05(8)$. (d) Finite-size scaling of the order parameter $[\langle M \rangle]$ leads to a unique crossing point at $T_c=3.05(3)$, and yields $\beta \approx \frac{1}{2}$, $\bar{\nu} \approx 2$ [see Eq. (4) [8]].

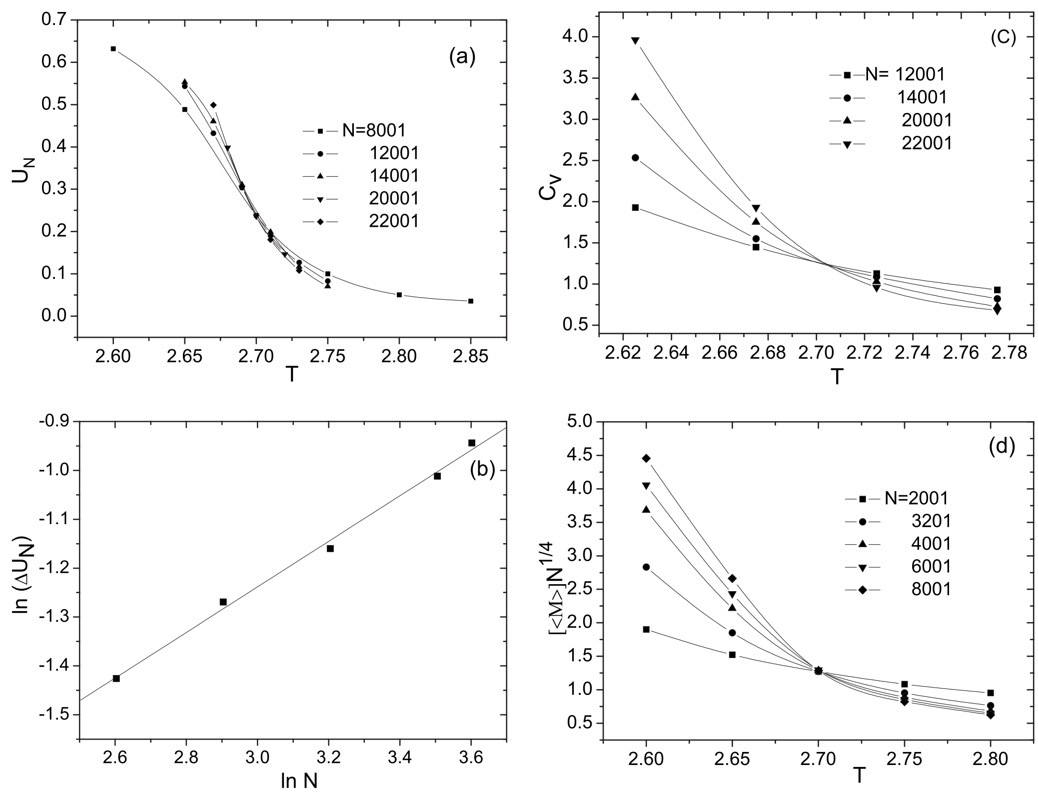


FIG. 6. For $m=1.5$, (a) finite-size scaling of Binder's cumulant $U_N$ has a unique crossing point at $T_c=2.70(2)$. (b) The expansion of $U_N$ near $T_c$ obtains $\bar{\nu}=2.03(3)$. (c) Finite-size scaling of the specific heat $C_v$ determines $T_c=2.70(4)$, which indicates $\alpha \approx 0$. (d) Finite-size scaling of the order parameter $[\langle M \rangle]$ leads to a unique crossing point at $T_c=2.70(2)$, and yields $\beta \approx \frac{1}{2}$, $\bar{\nu} \approx 2$ [see Eq. (4) [8]].

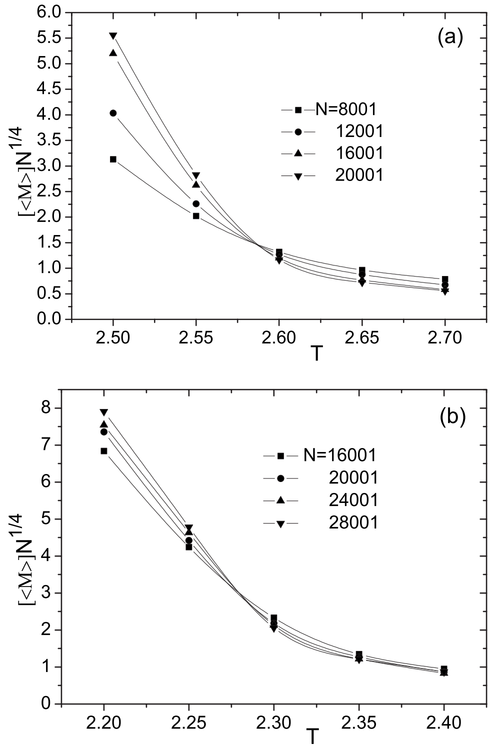


FIG. 7. Finite-size scaling of the order parameter $[\langle M\rangle]$ gives (a) $T_c=2.58(5)$ at $m=1.6$ and (b) $T_c=2.28(2)$ at $m=1.8$.

order parameter $[\langle M\rangle]$ under different network sizes behaves better in intersecting to one unique point, revealing unanimously $T_c \approx 2.70$, which reflects the mean-field nature indirectly. And the obtained critical exponents indicate the presence of a mean-field finite-temperature phase transition also.

With the increase of $m$, the algebra decreasing of $\Theta(l) \sim l^{-m}$ will have a cutoff due to the finite network size. This makes the normalization of the distribution function depend on the size of the network, and makes it inevitable to study systems of very large sizes so as to obtain correct scaling behavior [8,22]. By analyzing the plot of $[\langle M\rangle]N^{1/4}$ vs $T$, we get further progress on the transition temperature of systems with large scale, as shown in Fig. 7 as examples. This also indicates the fact that the phase transition still holds a mean-field nature for $m>1$. The phase transition temperatures here are higher than those found by Hong *et al.* in Ref. [20], which may be due to the presence of more long-range connections in our model which contribute to different phase transition properties.

In Fig. 8, the decrease of $T_c$ in the range of $[0,1)$ is not significant, but this decreasing becomes obvious after $m>1$. This brings difficulties to the phase transition analysis of the Ising model due to the following two factors: (i) with the increase of $m$, the appearance of a convincing unique crossing point needs larger network sizes, as shown in Figs. 6 and 7; (ii) the transition temperature decreases rapidly after $m>1$ (Fig. 8), so we need to do Monte Carlo simulation at

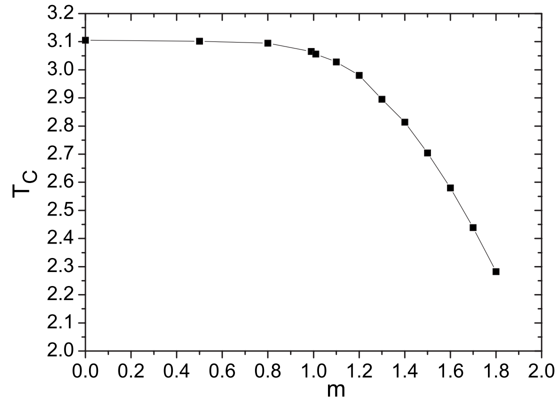


FIG. 8. The evolution of the transition temperature with the increase of $m$.

lower temperature, which means lower probability to flip. What is more, the value of the order parameter $[\langle M\rangle]$ has a strong correlation with its value at the previous step. Thus the system needs more Monte Carlo steps to reach equilibrium, and we need more iterations to get the thermodynamic average. Consequently, the simulations will be more time consuming.

## IV. CONCLUSION

In this paper, we reconsider the phase transition of Ising model on a one-dimensional network with distance-dependent connections given by $\Theta(l) \sim l^{-m}$, by adding the finite-size scaling analysis of the order parameter $[\langle M\rangle]$ to overcome the difficulties at high $m$ values. Reference [12] based their model on networks with hundreds of vertices and concluded that the phase transition has a mean-field nature only in the range of $0<m<1$. Our results, based on larger network sizes and further finite-size scaling analysis of the order parameter $[\langle M\rangle]$ as well as Binder's cumulant $U_N$, the specific heat $C_v$, and the susceptibility $\chi$, show that the phase transition has a mean-field nature in the whole range of $0<m<2$.

Because of the constraint of the long simulation time, we cannot apply the analysis to networks with even larger sizes at present. Thus we cannot make sure whether there is a critical value of $m$ at or beyond $m=2$, above which there will be no mean-field nature or finite-temperature phase transition since the transition point $m=2$ was suggested by the study of the network topological structure [6,7,12]. With the finding of a proper solution to the long simulation time, more interesting results are supposed to come out.

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